Multivariate Exponentially Weighted Moving Average Chart for Monitoring Poisson Observations*

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Abstract

In many practical situations, multiple variables often need to be monitored simultaneously to ensure the process is in control. In this article, we develop a feasible multivariate monitoring procedure based on the general Multivariate Exponentially Weighted Moving Average (MEWMA) to monitor the multivariate count data. The multivariate count data is modeled using Poisson-Lognormal distribution to characterize their inter-relations. We systematically investigate the effects of different charting parameters, and propose an optimization procedure to identify the optimal charting parameters. In particular, we provide a design table to the quality engineers as a simple tool to design the optimal MEWMA chart. To further improve the efficiency, we integrate the variable sampling intervals (VSI) in the monitoring scheme. We use simulation studies and an example to elicit the application of the proposed scheme. The results are encouraging and demonstrate effectiveness of the proposed methods well.

Key words: Count Data; Individual Observation; Multivariate Poisson; Statistical Process Control

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1 Introduction

Control charts, since the first introduction by Shewhart (1926), have been proven effective in statistical process control (SPC) to monitor and improve the performance of products or manufacturing processes. They have been widely used in many quality control applications and developed with many variants (e.g., Jiang et al., 2012; Zou et al., 2012; Zou and Tsung, 2010). They also play an important role in the success of enterprise in today's globally competitive marketplace (e.g., Wu et al., 2007; Morgan and Dewhurst, 2008).

In recent decades, advances in modern data acquisition techniques and computing power have enabled the collection and analysis of many quality characteristics simultaneously. And it has been noted that if these quality characteristics are monitored separately as individuals, it might not be very effective in detecting process changes (Lowry et al., 1992). As a result, many multivariate control charts (e.g., Reynolds and Cho, 2006; Zou and Tsung, 2011; Li et al., 2013b) have been proposed to better utilize the abundant data for process monitoring. In fact, Woodall and Montgomery (1999), Stoumbos et al. (2000), Bersimis et al. (2007) and Woodall and Montgomery (2014) point out that multivariate control charts are one of the most rapidly developing areas of SPC and suggest that basic and applied research is still needed.

Although multivariate control charts have been receiving a well-deserved attention in the literature, most of the works assumed that the data follow multivariate normal distributions. While this assumption often holds in manufacturing processes where quality characteristics are represented as continuous measurements, it might not be valid in many service industries or social sciences where discrete data are more common (Li et al., 2013a). For example, multivariate count data can be found useful in epidemiology (e.g., incidences of different types of illness), marketing (purchases of different products), industrial control (different types of faults) (Brijs et al., 2004; Karlis and Meligkotsidou, 2005), and the number of vacations, career interruptions, scores of soccer games, number of children, etc (Berkhout and Plug, 2004). In all these circumstances, traditional analysis using multivariate normal approximation can be misleading because the data might have a lot of zero counts when the marginal mean is small.

Due to its increasing popularity and critical importance, many researchers have proposed to generalize the Poisson distribution to multivariate case to model multi-dimensional count data. Among them, the first type approximates the multivariate count data by multivariate normal distributions through transformation (Niaki and Abbasi, 2009). However, when the expectation of the data are small and many zeros are present, the approximation could be misleading. Another method, proposed by Tsionas (2001) and Karlis (2003), considers a special case of the multivariate Poisson model, which assumes all the pairs of variables have the same covariance. This assumption is rather restrictive in practice. Karlis and Meligkotsidou (2005) propose a multivariate Poisson distribution with general covariance structures. Although it relaxes the assumptions in Tsionas (2001) and Karlis (2003), the inference becomes quite difficult and computationally demanding, especially when the dimension or the sample size is large.

In addition, a common drawback of the aforementioned works is that they do not allow for zero or negative correlation, and thus lack generality. A few other studies recognized this limitation. For example, Chib and Winkelmann (2001) and Munkin and Trivedi (1999) propose an alternative bivariate count model that allows for more general correlation structures by considering dependence among counts through correlated random effects. Van Ophem (1999) models dependence through known univariate distribution functions up to some parameters. Aitchison and Ho (1989) propose the multivariate log normal mixture of independent Poisson distributions, which allows for zero or negative correlation for data with any dimensions (not limited to bivariate data).

Despite these model developments, there lacks a systematic methodology to monitor the multivariate count data. To fill in the research gap, we propose an MEWMA scheme to monitor the multivariate Poisson count data. The remainder of the article is organized as follows. Section 2 introduces the proposed MEWMA scheme in details. Section 3 studies the numerical performance of the proposed MEWMA control chart using extensive simulation studies. Section 4 presents a data example about household purchase amounts for four different products. Section 5 discusses the extension to Variable Sampling Intervals (VSI). Section 6 concludes this article with some remarks and future works.

2 Multivariate Count Data Monitoring

In this section, we first present a brief introduction to multivariate Poisson-log normal model proposed in Aitchison and Ho (1989). Then we develop a multivariate count data monitoring scheme based on this model using the general MEWMA method.

2.1 Multivariate Poisson-Log Normal Distribution

Aitchison and Ho (1989) propose a multivariate Poisson-log normal distribution to model the multivariate count data. The model is composed of two parts. Given the parameter $\boldsymbol{\theta} \equiv [\theta_1, \theta_2, \cdots, \theta_d]$, which is a d dimensional vector, each element of $\mathbf{X} \equiv [X_1, X_2, \cdots, X_d]$ follows a Poisson distribution $X_i \sim \text{Poiss}(\theta_i), i = 1, 2, \cdots, d$. And X_i, X_j $(i \neq j)$ are independent from each other conditional on $\boldsymbol{\theta}$. Here we use $\text{Poiss}(\theta_i)$ to denote the univariate Poisson distribution with mean θ_i.

To model the correlation among different components of $\mathbf{X}$, we let $\boldsymbol{\theta}$ be random, and follow a multivariate log-normal distribution, with density function $g(\boldsymbol{\theta}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$g(\boldsymbol{\theta}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = (2\pi)^{-d/2} \prod_{i=1}^{d} \theta_i^{-1} \cdot |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \cdot \exp\left\{ -\frac{(\ln \boldsymbol{\theta} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\ln \boldsymbol{\theta} - \boldsymbol{\mu})}{2} \right\}, \tag{1}$$

where $\ln \boldsymbol{\theta}$ shall be interpreted as $[\ln \theta_1, \ln \theta_2, \cdots, \ln \theta_d]$. Equivalently, we have $\ln \boldsymbol{\theta}$ follows the multivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. As a result, the distribution of $\mathbf{X}$ can be considered as a (continuous) mixture of independent Poisson distribution with mixture probability specified according to a log-normal distribution. By integrating $\boldsymbol{\theta}$ out, the marginal distribution of $\mathbf{X}$ is

$$\text{P}(\mathbf{X}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \int_{\mathbb{R}_+^d} \prod_{i=1}^{d} \frac{\exp(-\theta_i)\theta_i^{X_i}}{X_i!} \cdot g(\boldsymbol{\theta}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) \mathrm{d}\boldsymbol{\theta}, \tag{2}$$

where $\mathbb{R}_+^d$ denotes the positive orthant of d-dimensional real space $\mathbb{R}^d$, and $\mathbf{X}$ can only take non-negative integers. Although model (2) is complicated, it may be found useful in many cases, e.g., the count of butterflies of different species (Aitchison and Ho, 1989), where the count data are naturally obtained in two-step procedures.

When the parameters $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ are unknown in model (2), they can be estimated from historical collected data $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$. Even though no analytical form of the maximum likelihood estimator (MLE) can be obtained, efficient numerical methods can be used to find the solution, as detailed in Aitchison and Ho (1989). Alternatively, we can use the moment based estimation, where a more concise representation is available. In particular, through the formula of conditional expectation

$$\text{E}(\mathbf{X}) = \text{E}[\text{E}(\mathbf{X}|\boldsymbol{\theta})]$$

and the conditional variance

$$\mathrm{Var}(\mathbf{X}) = \mathrm{E}[\mathrm{Var}(\mathbf{X}|\boldsymbol{\theta})] + \mathrm{Var}[\mathrm{E}(\mathbf{X}|\boldsymbol{\theta})],$$

we have

$$\begin{aligned} \mathrm{E}(X_i) &\equiv \alpha_i = \exp(\mu_i + \sigma_{ii}/2), \\ \mathrm{Var}(X_i) &= \alpha_i + \alpha_i^2 \cdot [\exp(\sigma_{ii}) - 1], \\ \mathrm{Cov}(X_i, X_j) &= \alpha_i \alpha_j \cdot [\exp(\sigma_{ij}) - 1], \end{aligned} \tag{3}$$

where σ_{ij} denotes the (i, j) element of $\boldsymbol{\Sigma}$. We can observe from (3) that, the expectation of $\mathbf{X}$ only depends on $\boldsymbol{\mu}$ and the diagonal elements of $\boldsymbol{\Sigma}$, while the variance/covariance of $\mathbf{X}$ depends on both parameters.

2.2 MEWMA Scheme for Multivariate Count Data

In this section, we further develop a monitoring scheme to monitor the multivariate count data. The statistical monitoring method developed here can be used to detect changes in the distribution, and to provide support for further analysis and decision making. In this article, we focus on the Phase II monitoring problem, i.e., we assume the in-control parameters $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are known exactly or has been accurately estimated from historical in-control samples. In practice, these parameters can be estimated from a group of measurements. Provided that the sample size is sufficiently large, it has minimal impact to treat the estimated parameters as known (Dai et al., 2011).

In the proposed approach, we directly construct the monitoring statistics based on the individual observations $\mathbf{X}_k, k = 1, 2, \cdots, n$. Note that the sampling interval κ is set to 1 here for clear exposition, and the generalization to VSI will be discussed in Section 5. We use the general MEWMA (Hawkins et al., 2007) method to detect small to moderate changes effectively. When the process is in control, $\mathbf{X}_k$ has mean $\mathbf{m}_0$ and covariance matrix $\boldsymbol{\Omega}_0$, both of which depending on $\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0$ implicitly according to (3). Upon the collection of each sample $\mathbf{X}_k$, it iteratively calculates

$$\mathbf{Y}_k = \mathbf{R} \cdot (\mathbf{X}_k - \mathbf{m}_0) + (\mathbf{I} - \mathbf{R})\mathbf{Y}_{k-1}, \tag{4}$$

where $\mathbf{Y}_0 = \mathbf{0}$, $\mathbf{I}$ is the identity matrix, and $\mathbf{R}$ is the smoothing matrix. As suggested by Hawkins et al. (2007), we use equal diagonal elements and equal off-diagonal elements in $\mathbf{R}$

$$r_{ii} = \frac{\lambda}{1+(d-1)c}, \qquad r_{ij} = \frac{c\lambda}{1+(d-1)c}, \quad i \neq j \tag{5}$$

where r_{ij} are the $(i,j)^{th}$ element of $\mathbf{R}$, and λ, c are design parameters of the MEWMA chart. In particular, when $c = 0$, the smoothing matrix $\mathbf{R}$ becomes the conventional form in Lowry et al. (1992).

Based on the MEWMA statistic $\mathbf{Y}_k$, we can detect the changes using the T^2 statistic

$$T_k^2 = \mathbf{Y}_k' \mathbf{W}_k^{-1} \mathbf{Y}_k, \tag{6}$$

where

$$\begin{aligned} \mathbf{W}_k \equiv \mathrm{Var}(\mathbf{Y}_k) &= \sum_{j=0}^{k-1} (\mathbf{I}-\mathbf{R})^j \mathbf{R}\boldsymbol{\Omega}_0 \mathbf{R} (\mathbf{I}-\mathbf{R})^j \\ &= \mathbf{W}_{k-1} + (\mathbf{I}-\mathbf{R})^{k-1} \mathbf{R}\boldsymbol{\Omega}_0 \mathbf{R} (\mathbf{I}-\mathbf{R})^{k-1} \\ &= \mathbf{R}\boldsymbol{\Omega}_0\mathbf{R} + (\mathbf{I}-\mathbf{R})\mathbf{W}_{k-1}(\mathbf{I}-\mathbf{R}) \end{aligned} \tag{7}$$

is the covariance matrix of $\mathbf{Y}_k$. When $T_k^2 > h$, the control limit, we generate an alarm and declare the process is out of control. The h values are determined such that the average time to signal (ATS) of the MEWMA when the process is in control meets the specification ATS_0.

2.3 Performance Measures

The Mahalanobis distance s, which is used to measure the magnitude of change from the in-control mean vector $\boldsymbol{\mu}_0$ to the out-of-control mean vector $\boldsymbol{\mu}_1$, is defined as

$$s = [(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)]^{1/2}. \tag{8}$$

When the process is in-control, $s = 0$; on the contrary, when there is a shift in the mean vector, $s > 0$. Despite the T^2 type statistic, the MEWMA scheme is not directionally invariant. Because of the continuous mixture, both expectation and covariance matrix of $\mathbf{X}$ depend on $\boldsymbol{\mu}$. As a result, the distribution of the test statistic T_k^2 generally depends on the direction of $\boldsymbol{\mu} - \boldsymbol{\mu}_0$ in addition to s. The design of our procedure will be inevitably quite complicated because it is not invariant. As Hawkins et al. (2007) point out, "the ARL performance of the nondiagonal smoothing scheme is affected by the direction of the shift and by the correlation structure, thereby complicating the chart design", and at the same time, they also show that "using nondiagonal components for the smoothing matrix creates additional computational requirements but offers a practical advantage of improving the performance in detecting a shift in the process mean vector for many quality control environments." Considering the improvement of the performance, we still suggest the general smoothing method of Hawkins et al. (2007), and the design codes for the proposed MEWMA procedure are available from the authors upon request.

The in-control and out-of-control performance of a univariate or multivariate control chart is usually measured by the Average Time to Signal (ATS), which indicates the average time required to signal a process shift (for out-of-control cases) or to produce a false alarm from the beginning of the process (for in-control status). When the process is in control, we want the ATS, denoted by ATS_0, to be large enough so that false alarms occur infrequently. On the other hand, when the process is out-of-control, the ATS should be as short as possible in order to minimize the delay in detecting the process shifts. In this article, the out-of-control ATS will be calculated in the steady-state mode, assuming that the process has reached its steady state when the shift occurs. In contrast, the ATS_0 is calculated under zero-state. Besides, the out-of-control shift is assumed to occur in the middle of the time between taking two samples.

Since it is quite difficult to predict the sizes of process shifts in most scenarios, we want to design the chart to have satisfactory performance over a wide range of possible process shifts rather than one particular shift (Sparks, 2000). Average Extra Quadratic Loss (AEQL) is a widely used design criterion in the literature to measure the general detecting ability over the entire range of shifts (Taguchi and Wu, 1980; Serel and Moskowitz, 2008). As the name indicates, AEQL is based on the quadratic loss function. The index AEQL can be calculated as

$$\text{AEQL} = \frac{1}{s_{max}} \int_0^{s_{max}} s^2 \cdot \text{ATS}(s)\text{d}s, \tag{9}$$

where $\text{ATS}(s)$ is the ATS when the process mean shifts with magnitude s, and s_{max} is the maximum range of shift that is possible or meaningful. Here, we use the Mahalanobis distance as the shift magnitude s to account for the scale differences in different quality characteristics. It is noted that AEQL is a weighted average of ATS using the squared shift magnitude (s^2) as the weight. This weight can be justified as quality is inversely proportional to variability (Montgomery, 2009). This reflects the fact that loss in quality per unit time increases quadratically with an increase in s (Taguchi and Wu, 1980). The bigger s is, the more it affects the production, and consequently the larger effect it has on AEQL. If a chart has a small AEQL value, its out-of-control ATS value over the entire shift range is expected to be small on average, subsequently reducing the loss in quality incurred in the *unknown* out-of-control cases.

Besides AEQL, there are several other criteria in the literature. The standard deviation of the run length (SDRL) or median run length (MRL) considers the shape of the run length distribution, which changes with the magnitude of the shift (Gan, 1993). However, they only consider the detecting performance for one particular shift instead of a wide range of shifts. In addition, neither the MRL nor SDRL considers the sampling interval, so their usage is limited. Another heuristic measure of the overall performance is the Average Ratio of ATS (ARATS) (Wu et al., 2009; Ou et al., 2011a). It directly calculates the average of the ratios between the out-of-control ATS(s) of a chart to be evaluated and the ATS(s) of a benchmark chart. In this article, AEQL will be used as the objective function for the designs of the control charts, because the computation of AEQL does not require a predetermined benchmark chart and therefore is relatively more tractable.

To find the parameters of the MEWMA chart leading to the smallest AEQL, we can solve the following optimization problem:

$$\begin{cases} \text{Objective:} \quad \min \text{AEQL}, \\ \text{Constraint:} \quad \text{ATS}_0 = \tau, \\ \text{Independent design variables: } c, \lambda, \\ \text{Dependent design variables:} \quad h. \end{cases} \tag{10}$$

The optimal values of the charting parameters c, λ, h can be determined by minimizing AEQL. In determining the parameters of the MEWMA chart, different combinations of independent design variables (c and λ) are searched. Correspondingly, the dependent design variable (control limit h) is adjusted simultaneously such that the constraint in Eq. (10) is satisfied.

3 Simulation Study

3.1 Design Table

This section provides a design table (Table 1) for various specifications of d, τ and s_{max} (the specification of in-control mean and variance is consistent with Jiang et al. (2012)). For each case, the charting parameters are provided as well as the optimal AEQL. In practice, the SPC practitioners can select a chart for which the tabulated values of d, τ

and s_{max} are closest to the desired values for their application. This design table should cover the most common used occasions in industrial applications.

We want to highlight that the optimal combinations of (c, λ) have the same values across different settings. This reflects the relative robustness of the smoothing matrix. In fact, our numerical results show that if other values are used, the performance is not optimal, but close to optimal, in terms of AEQL. For example, when $d = 3, \tau = 370, s_{max} = 3$ and $(c, \lambda) = (0.8, 0.05)$, the AEQL is 12.235; when $(c, \lambda) = (0.7, 0.1)$, the AEQL is 12.418. As Ou et al. (2011a) point out, most of the design strategies used in SPC are heuristic. They make no attempt to secure the global optimal solution. Instead, they focus on deriving a relatively convenient procedure for approximating the optimum that could be adopted in practice.

Table 1: Design table of the MEWMA chart

d	τ	s_{max}	c	λ	h	AEQL
2	200	3	0.9	0.1	8.107	10.356
		4	0.9	0.1	8.109	10.868
	370	3	0.9	0.1	11.357	12.597
		4	0.9	0.1	11.213	12.795
	500	3	0.9	0.1	13.168	13.794
		4	0.9	0.1	13.230	13.927
3	200	3	0.9	0.1	9.504	10.040
		4	0.9	0.1	9.621	10.631
	370	3	0.9	0.1	13.037	12.139
		4	0.9	0.1	12.900	12.282
	500	3	0.9	0.1	14.489	13.028
		4	0.9	0.1	14.800	13.284
4	200	3	0.9	0.1	10.882	9.856
		4	0.9	0.1	10.882	10.378
	370	3	0.9	0.1	14.339	11.769
		4	0.9	0.1	14.298	11.968
	500	3	0.9	0.1	16.289	12.924
		4	0.9	0.1	16.273	12.899

Table 2: A factorial experiment for specifications d, τ and s_{max}. Their effects on AEQL are estimated as -0.722, 3.021, 0.286, respectively.

	$d=2$				$d=4$			
	τ=200		τ=500		τ=200		τ=500	
s_{max}	3	4	3	4	3	4	3	4
c	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
λ	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1
AEQL	10.356	10.868	13.794	13.927	9.856	10.378	12.924	12.899

3.2 Factorial Experiment

In this section, a 2^3 experiment is carried out to evaluate the performance of the MEWMA control chart. The effects of dimension d, allowable minimum in-control average time to signal τ, and the maximum shift s_{max} are discussed thoroughly. The specifications vary at two levels (d = 2, 4), (τ = 200, 500) and (s_{max} = 3, 4) (see Table 2 and Figure 1). The selection of d, τ and s_{max} are referring to Wu et al. (2007) and Ou et al. (2012). However, when it is necessary, more case studies for any particular values of d, τ and s_{max} can be easily applied through the same algorithm. In this factorial experiment, there are totally $2 \times 2 \times 2 = 8$ cases. For these eight combinations, with the constraint $\text{ATS}_0 = \tau$, the MEWMA chart is optimized.

Table 2 and Figure 1 show the general effect of d, τ and s_{max} on AEQL. It can be observed that the specification τ has a significant impact on AEQL. The AEQL raises along with the increase of τ, which coincides with the observations in univariate control chart (Ou et al., 2012). However, the effects of the other two specifications d and s_{max} are negligible. Table 2 shows that for all eight cases, the combination $(c, \lambda) = (0.9, 0.1)$ are always the best selection. In another word, to improve the detection speed among the shift range as well as to simplify the design, it is strongly suggested to adopt $(c, \lambda) = (0.9, 0.1)$ under different specifications, which is reliable and convenient for quality engineers to implement.

3.3 An Illustrative Example

In this part, an illustrative example is provided to demonstrate the implementation of the proposed MEWMA chart. The values of the zero-state ATS_0 and steady-state ATS

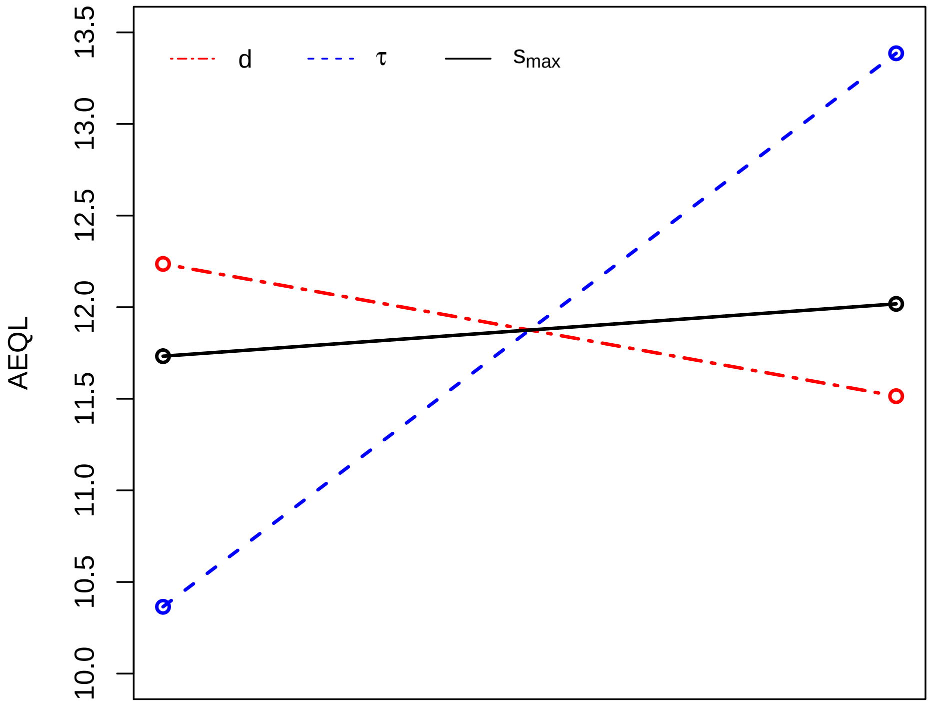

Figure 1: Marginal curves of specifications with levels d = 2, 4, τ = 200, 500, and s_{max} = 3, 4.

of the MEWMA chart are simulated using R. The parameters are selected such that $s_{max} = 3$, and $\text{ATS}_0 \approx 370$ (see Table 3). The performance of the chart is then evaluated under different shift magnitudes, with results summarized in Table 4. It is interesting to observe the following from Tables 3 and 4:

Table 3: Design parameters for the illustrative example

Specifications			Charting Parameters			Outcome
d	τ	s_{max}	c	λ	h	AEQL
3	370	3	0.9	0.1	13.037	12.139

1. As shown in Table 4, the real ATS_0 (= 371.185) obtained using simulation is very close to τ (= 370) when the process is in-control (s = 0). It guarantees that the false alarm rate will not be too high which would lead to the over control in practice.

2. The bigger the shift s, the smaller the ATS, which indicates fast response in de-

Table 4: ATS vs. s in the illustrative example

s	ATS
0.00	371.185
0.30	93.395
0.60	30.945
0.90	14.678
1.20	8.606
1.50	5.678
1.80	4.004
2.10	2.912
2.40	2.260
2.70	1.796
3.00	1.430

tecting the changes. When the shift s is as large as 3, ATS will be as small as 1.430.

3. In this illustrative example, since it is known that the AEQL equals to 12.139, it can be compared with other charts in terms of AEQL. Normally, the smaller the AEQL, the better its detecting performance. Therefore, practitioners can know which chart has better monitoring performance for a wide range of unknown shifts.

4 Data Example

The approach developed in this article is applied to an example provided in Brijs et al. (2004). The data is collected in a large grocery store in the western United States and it contains the purchase rates of 155 households over a period of 26 weeks in four product categories, i.e. cake mix (C), cake frosting (F), fabric detergent (D) and fabric softener (S). Table 5 lists the mean $\mathbf{m}_0$ of the purchase for each product category and Figure 2 shows the distribution of purchase rates. The purchases of the above commodities follow the Poisson distribution. It is known that there exists a strong positive relation between cake mix and cake frosting, and between fabric softener and fabric detergent, but not between other combinations of these products. More concretely, only two correlations are significantly larger than zero, i.e., $r(\mathrm{C, F}) = 0.66$, and $r(\mathrm{D, S}) = 0.48$, where $r(\mathrm{A,B})$

denotes the correlation between the variables A and B. The correlation matrix $\mathbf{\Omega}_0$ is

$$\begin{array}{c|cccc} & C & F & D & S \\ \hline C & 1 & 0.66 & 0 & 0 \\ F & 0.66 & 1 & 0 & 0 \\ D & 0 & 0 & 1 & 0.48 \\ S & 0 & 0 & 0.48 & 1 \end{array} . \tag{11}$$

Table 5: Average purchase counts in each product category

	Cakemix	Frosting	Detergent	Softener
Mean	2.07742	1.54839	3.15484	2.20000

If the purchase pattern is regular, it can be considered to be in-control. However, in case there is a sustained increase or decrease in purchase volume, the chart would trigger an alarm, and further diagnosis and control can be pursued. According to domain or expert knowledge, the shift is believed to be ranging $0 < s \leqslant 3$, i.e., $s_{max} = 3$. In this study, the benefit of making quick response to the out-of-control state overcomes the negative impact raised by the minor increase of the false alarm rate and due to confidential reasons, a set of simulated data is used for illustration. In case there is a sudden increase of demand but the chart fails to detect it, the market might suffer a great loss. Similarly, if demands decrease, it needs to be detected early as well and proper actions should be carried out. Therefore, we consider the design with small τ = 200. The optimal design parameters can be selected from the design table (Table 1) within the catalogs of $d = 4$ and $s_{max} = 3$. The optimal parameters for this problem are $c = 0.9, \lambda = 0.1, h = 10.882$.

Figure 3 shows the statistic T^2 calculated with mean $\mathbf{m}_0$ and correlation matrix $\mathbf{\Omega}_0$ (Brijs et al., 2004). In the first 155 samples, the process is in-control ($s = 0$), however, after that a shift occurs at $s = 1.65$. The MEWMA chart signals an alarm at the 159^{th} sample with $T^2_{159} = 11.024$. From the time the shift occurred to the time the system detected it, there are 4 samples taken. Therefore the Run Length (RL) equals to 4. This demonstrates that the MEWMA chart is very sensitive and is able to detect the shift in a very short time. With the application of this MEWMA chart, this grocery store can manage their product categories in a more scientific and efficient perspective.

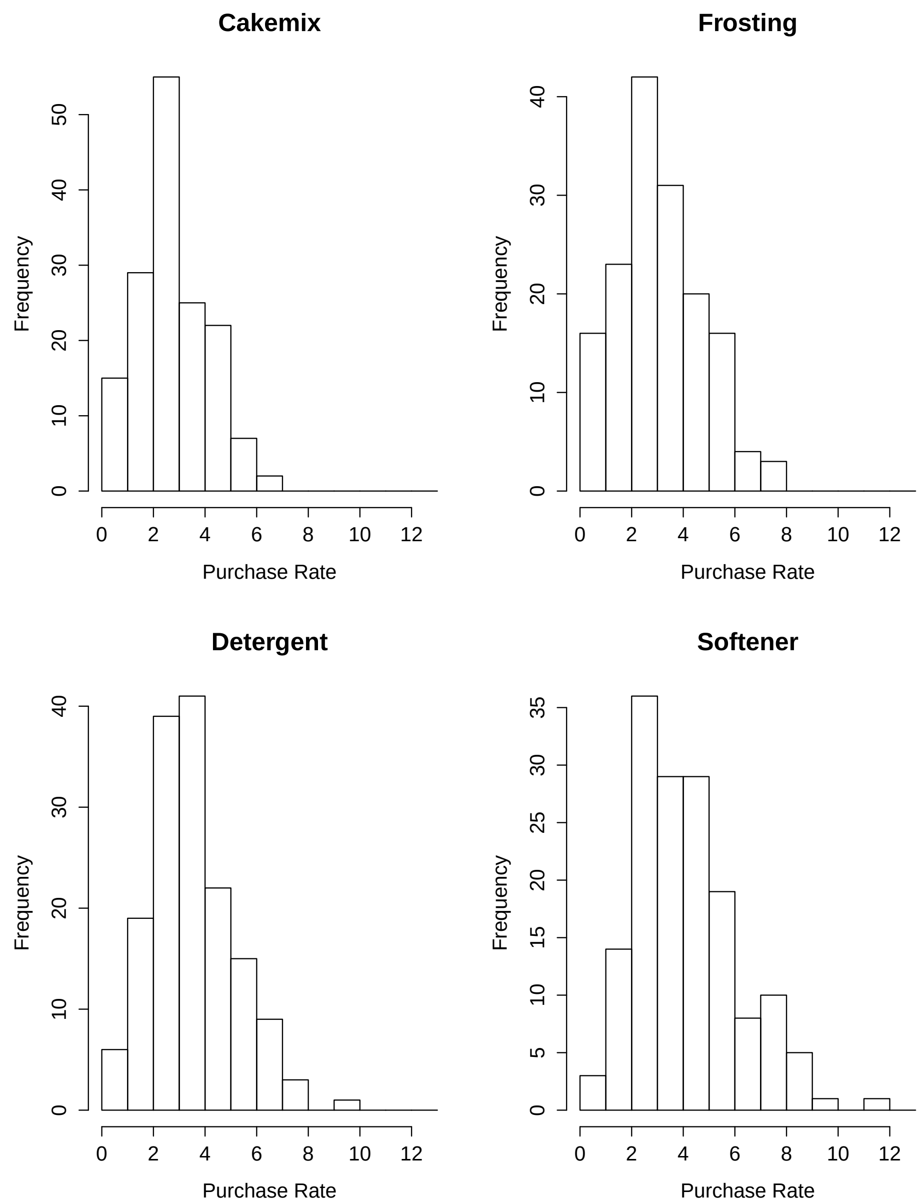

Figure 2: Distribution of Purchase Rates in the Example

5 Implementation of the VSI MEWMA Chart

Traditional control charts are operated by taking samples of fixed size (n) from the process using a fixed sampling interval (κ). Conversely, the variable charts, such as the VSI chart, vary the sampling rate as a function of the observed data from the process (Arnold and Reynolds, 2001). Compared with the traditional control charts with fixed sampling rate, the variable charts can detect process changes faster by sampling at a higher rate when there is an indication of a process change. Recent developments in

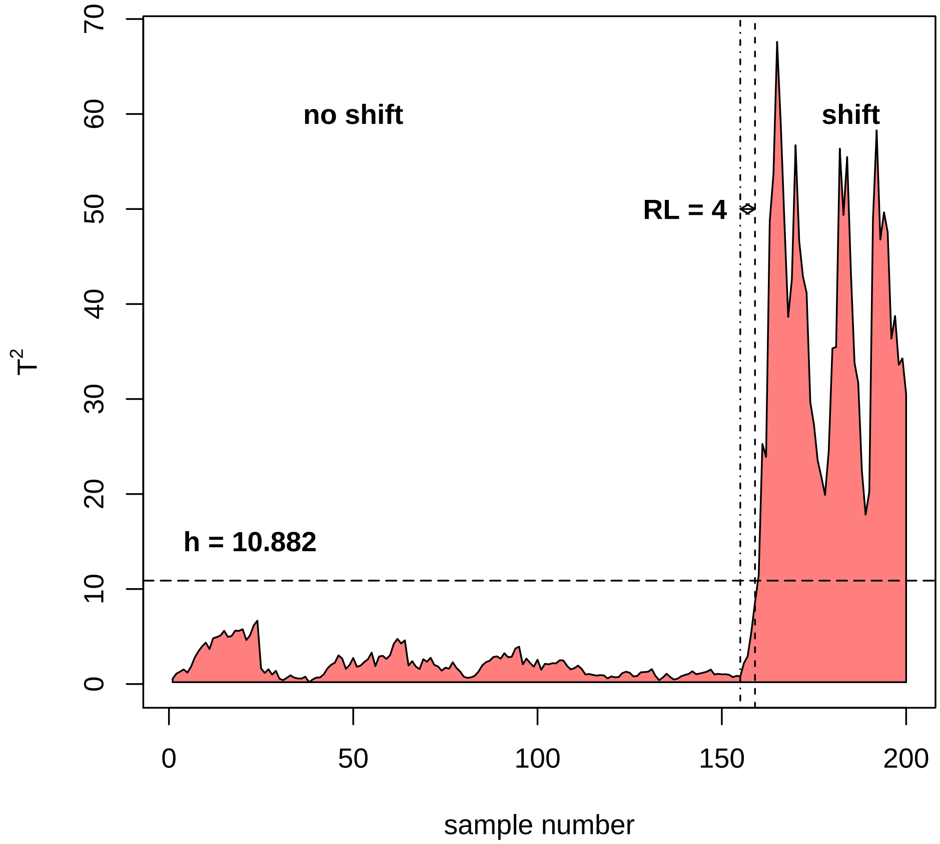

Figure 3: The charting example in monitoring grocery purchase data.

variable charts include the variable sample size and sampling intervals (VSSI) $\bar{X}$ chart using two sampling intervals and three sample sizes (Mahadik and Shirke, 2009), the VSI cumulative sum (CUSUM) of Q chart for monitoring the process mean (Li et al., 2010), the VSI EWMA chart for monitoring linear profiles (Li and Wang, 2010), the VSI SPRT chart with super high detecting ability (Ou et al., 2011b) and the VSI and VSSI CUSUM chart for monitoring process mean and/or variance (Ou et al., 2013). In this section, a VSI MEWMA chart for detecting the multivariate Poisson process will be investigated as a tentative exploration. This VSI MEWMA chart will adapt the sampling interval between samples according to the on-line observed data and it is expected to have an outstanding performance. For facilitation, the previous proposed MEWMA chart is denoted as the Fixed Sampling Intervals (FSI) MEWMA chart.

For the proposed VSI MEWMA chart, the sample interval at the current sample point depends on the data obtained in the last sample. The detection effectiveness of the MEWMA chart should be further enhanced by adopting the VSI feature. This feature allows the sampling interval between two sample groups to be changed based on the values of the sample statistics that provide information about the current state of the process. The VSI charts using two sampling intervals are recommended by most researchers (Daudin, 1992), because the dual scheme gains most of the benefits that can be reached by the VSI charts and, meanwhile, is relatively easier to implement. The VSI MEWMA chart proposed in this article also uses two different sampling intervals alternatively depending on the current process status. When the process seems close to an out-of-control condition, a short sampling interval κ_1 will be used. Conversely, when the process is likely to be in control, a long sampling interval κ_2 is employed. In the actual implementation of a VSI MEWMA chart for detecting mean shifts, the statistic T_k^2 for each sample is checked against a warning limit w. If T_k^2 is larger than w, it is considered as a warning of a shift in process mean and leads to the use of the short sampling interval κ_1 for the next sample. Otherwise (i.e. $T_k^2 \leq w$), the process is thought in control and the long sampling interval κ_2 will be used next. The first sample taken from the process when it just starts could be chosen arbitrarily because it has little influence on the in-control and steady-state out-of-control performance of the VSI MEWMA chart. However, as recommended by most researchers (e.g., Li et al., 2010; Li and Wang, 2010), the short sampling interval κ_1 should be used as a safeguard to provide additional protection against problems that may occur during the start-up.

A VSI MEWMA chart has six parameters: c, λ, h, κ_1, κ_2 and w. It is implemented as follows (see Figure 4).

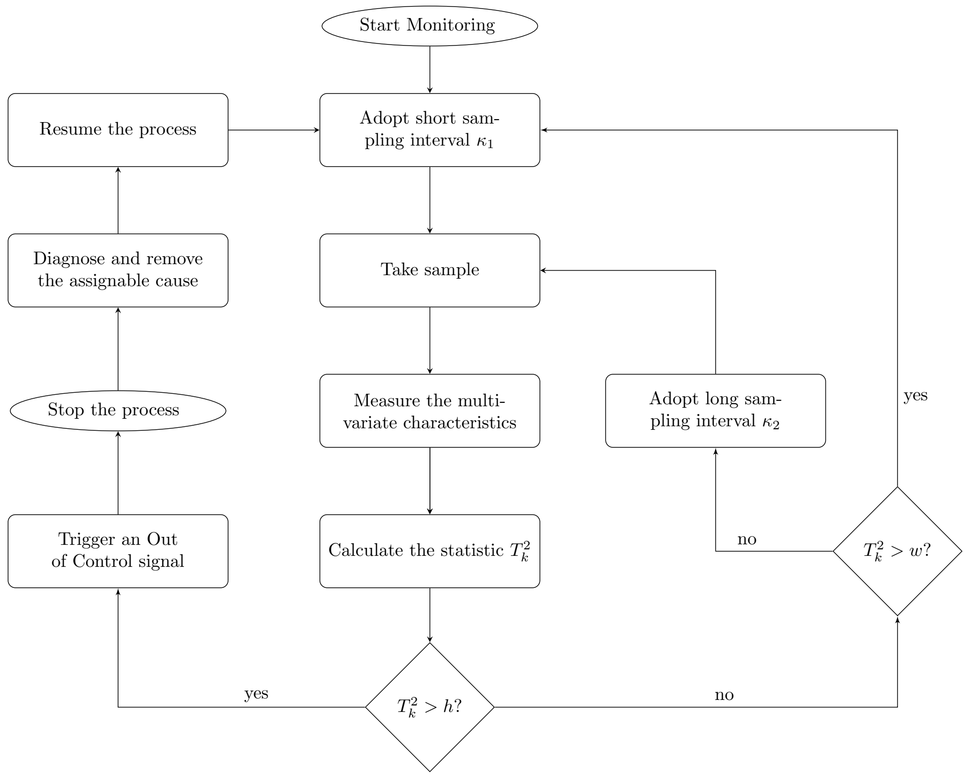

Figure 4: Flowchart of the Implementation of the VSI MEWMA Chart

Table 6 enumerates the charting parameters for the VSI MEWMA chart while Table 7 demonstrates the values of ATS. The charting parameters are set referring to Ou et al. (e.g. 2011b). Compared with the ATS values for the FSI MEWMA chart in Table 4, while the process is in control ($s = 0.00$), both of these two MEWMA charts generate an ATS_0 very close to τ ($= 370$). The superior of the VSI MEWMA chart becomes obvious when the mean shift is large ($s \geq 1.80$). In general, the VSI MEWMA chart outperforms the FSI version by 8.05% in terms of AEQL.

Table 6: Case Study for VSI MEWMA Chart

Specifications			Charting Parameters						Outcome
d	τ	s_{max}	c	λ	h	κ_1	κ_2	w	AEQL
3	370	3	0.9	0.1	10.782	0.800	1.500	5.391	11.235

It is noted that, the FSI MEWMA chart is just a special case of the VSI MEWMA chart with $\kappa_1 = \kappa_2$. It means that, under any circumstances or specifications, one can design a VSI MEWMA chart that is more, or at least equally, effective compared with a

Table 7: ATS vs. s in the Case Study for VSI MEWMA Chart

s	ATS
0.00	368.923
0.30	98.191
0.60	33.274
0.90	15.684
1.20	9.199
1.50	5.722
1.80	3.897
2.10	2.661
2.40	1.881
2.70	1.288
3.00	0.903

counterpart FSI MEWMA chart. However, there are a few limitations to the applications of the VSI MEWMA chart. Firstly, since two different sampling intervals (κ_1 and κ_2) are to be used alternatively, the implementation of this chart is more complicated than the FSI counterparts from a managerial and operational viewpoint. It may be impractical in many production lines. Secondly, the determination of the parameters of the VSI MEWMA chart is more difficult. Finally, the superiority of the VSI MEWMA chart over the FSI MEWMA chart diminishes when the mean shift is small.

6 Conclusion

This article mainly conducts the investigation of the statistic based on a general MEWMA control chart monitoring the mean of multivariate Poisson process with individual observations. More specifically, the rationale behind each setting of the control chart has been discussed at length. It is strongly recommended to adopt $c = 0.9$ and $\lambda = 0.1$, as it can improve the overall performance of the charts and meanwhile does not increase the difficulty in implementation. Besides, a comprehensive investigation is carried out for the effects of the specifications like dimension d and τ, et al. The AEQL is applied as the general assessment to evaluate the overall effectiveness of the control charts. Practitioners may get some general idea about the relative performance of the MEWMA charts in different conditions under the multivariate Poisson-log Normal distribution.

This article provides a design table containing 18 cases for different design specifications. It will aid the practitioners to select a chart conveniently and make contribution to promote the application of multivariate control chart. Finally, an example about the purchase frequency in a set of product categories is presented to illustrate how the design table can facilitate the quality practitioners to employ the MEWMA chart for their applications in practice. The result shows that the proposed procedure succeed to help the retail category managers to devise customized merchandizing strategies.

Further, this article also proposes a VSI MEWMA chart, which uses the adaptive sampling intervals to further increase the effectiveness of the MEWMA chart for detecting process mean shifts. The procedures for the implementation of the VSI MEWMA chart have been presented at full length. This VSI MEWMA chart employs a long sampling interval κ_2 when the process is likely to be in control and adopts a short sampling interval κ_1 when the process seems close to an out-of-control condition. The results of the case study show that the adaptive feature has the potential to further increase the overall detection effectiveness of the MEWMA chart compared with the FSI MEWMA chart and, therefore, to reduce the manufacturing cost. This VSI chart is particularly effective when the shifts are moderate to large.

The proposed methodology can be extended to some other topics in the field of multivariate SPC. First, the current version of the proposed chart is designed for detecting mean shifts only. By using certain proper monitoring statistics, the proposed method may be able to handle cases in which monitoring both the mean and covariance structure is of interest. Second, much future research is also needed to construct a control chart for multinomial observations, other than multivariate Poisson. Third, this article focuses on Phase II monitoring only and assumes that the in-control parameters $\boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma}_0$ are known exactly or has been well estimated from historical in-control samples. Therefore, much future work is needed to extend our method to Phase I analysis, in which detection of outliers in a historical data set would be of interest.

Appendix

To explain the parameters optimization in Eq. (10) clearly, we now provide a flow chart (Figure A1) to illustrate the searching at full length.

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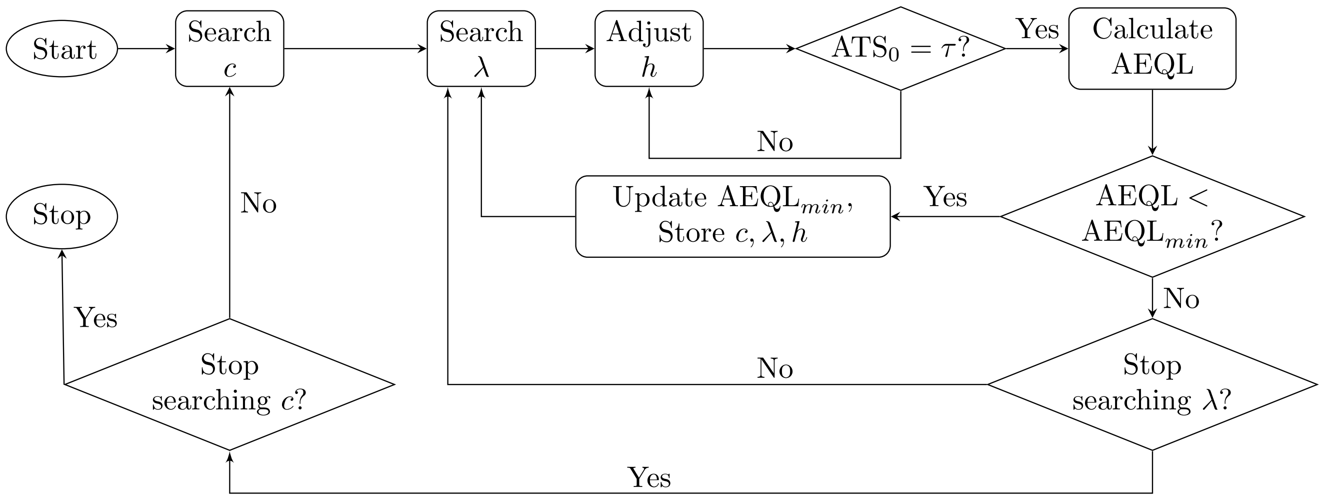

Figure A1: Searching of the independent parameters (c, λ) and dependent parameter h

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