Multivariate Change Point Control Chart Based on Data Depth for Phase I Analysis

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Abstract

A multivariate change point control chart based on data depth (CPDP) is considered for detecting shifts in either the mean vector, the covariance matrix, or both of the process for Phase I. The proposed chart is preferable from a robustness point of view, has attractive detection performance and can be especially useful in Phase I analysis setting where there is limited information about the underlying process. Comparison results and an illustrative example show that our CPDP chart has great potential for Phase I analysis of multivariate individual observations. The application of CPDP chart is illustrated in a real data example.

Key words: False Alarm Probability; Individual Observations; Nonparametric Test; Robustness.

1 Introduction

In modern statistical process control (SPC) applications, it is common to monitor several correlated quality characteristics of a process simultaneously. This challenge motivates attempts to extend the univariate Shewhart (Shewhart , 1931) chart, cumulative sum (CUSUM) chart (Page , 1954) and exponentially weighted moving average (EWMA) (Roberts , 1959) chart to multivariate data, such as Hotelling's T^2 chart (Hotelling , 1947), multivariate CUSUM (MCUSUM) chart (Crosier , 1988) and multivariate EWMA (MEWMA) chart (Lowry et al. , 1992).

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In the literature, there are Phase I and Phase II control charts that need to be distinguished. In Phase II, the process distribution is assumed to be completely known. Most of the classical application of control schemes require the assumption that the process being inspected follows a multivariate normal distribution. Bersimis et al. (2007) gave an overview on multivariate SPC charts before 2007 and some recent work include Huwang et al. (2007), Reynolds and Stoumbos (2008), Hawkins and Maboudou-Tchao (2008), Hawkins and Deng (2009) and Zhang et al. (2010). In most applications, the normality assumption is, however, not valid, especially for multivariate observations. Stoumbos and Sullivan (2002) studied the robustness of MEWMA control chart under non-normality. The results showed that if the true distribution is quite different from the assumed form, the properties of the control procedure can be considerably different. From another perspective, in most industrial and service applications, even the process follows multivariate normal distribution, the mean vector and covariance matrix of the process are replaced with estimates from an in-control reference sample. However, using estimated parameters usually leads to significantly deteriorated chart performance. Interested readers are referred to an excellent literature review paper by Jensen et al. (2006) and references therein. Under such circumstances, it is pragmatic to consider a nonparametric control procedure, which is less influenced by the underlying distribution.

Nonparametric control schemes enjoy the advantage of greater robustness over parametric control schemes. Chakraborti et al. (2001) gave an overview of the development of the univariate nonparametric control schemes, where the statistics used are mostly ranks. Recently, Qiu and Li (2011a,b) proposed nonparametric control charts from an aspect different from ranks and they constructed nonparametric control charts by categorizing and transforming the observations, respectively. Few nonparametric multivariate control schemes have been proposed. Liu (1995) proposed three nonparametric control charts based on simplicial data depth, the r chart, Q chart and S chart. These three charts can be viewed as simplicial-depth-based multivariate generalizations of the univariate X, $\overline{X}$ and CUSUM chart. Qiu and Hawkins (2001, 2003) and Qiu (2008) studied nonparametric CUSUM chart based on anti-ranks. However, all of these charts need large Phase I sample. Woodall and Montgomery (1999) and Stoumbos et al. (2000) showed that nonparametric procedure for multivariate problems is an open area with great potential.

All the literature mentioned above focus on Phase II charts. In Phase I, however, the presumption is that the process distribution, and by extension process parameters, is unknown. Therefore, it is necessary to establish that a process is statistically in control and estimate the process parameters, referred to as Phase I or retrospective analysis. In Phase I analysis, the finite historical data are used to decide if the process is statistically in control and to estimate process parameters. Sometimes, the nature of the process may suggest rational subgroups within which the quality measurements are relatively homogeneous. On the other hand, there are many situations where it is reasonable to analyze individual observations, possibly because measurement is automated and every unit is measured, the rate of production is slow, or for other reasons (Montgomery , 2005). The Phase I analysis with individual multivariate observations is the focus of this paper.

In the literature, there are some multivariate control charts used in Phase I analysis for individual or grouped observations, such as the usual T^2 control charts (Wierda , 1994; Lowry , 1995; Mason et al. , 1997), the developed T^2 control charts (T^2D) (Holmes and Mergen , 1993), the control chart based on likelihood ratio test (LRT) (Sullivan and Woodall , 2000; Srivastava and Worsley , 1986). Sullivan and Woodall (2000) pointed out the LRT control chart is more powerful than the T^2 and T^2D control charts in detecting a shift in either the mean vector, the covariance matrix, or both for Phase I analysis. The LRT chart of Srivastava and Worsley (1986) has better properties to detect shift in mean vector, but it cannot detect the shift of covariance matrix.

Statistical depth functions have become increasingly researched as a useful tool in nonparametric inference for multivariate data. They can be used as quality index in quality control schemes. Data depth is one of the efficient methods dealing with multivariate robustness for Phase I. Several definitions of data depth are given in the literature, such as Mahalanobis depth (M_hD) (Mahalanobis , 1936), Tukey depth (TD) (Tukey , 1975), simplicial depth (SD) (Liu , 1990), majority depth (M_jD) (Singh , 1991), projection depth (PD) (Zuo and Serfling , 2000) and spatial rank depth (SRD) (Gao , 2003). Based on data depth, a change point control chart (called CPDP chart) is proposed in this paper for detecting shifts in either the mean vector, the covariance matrix, or both of the process. For comparisons, the LRT-based control chart of Sullivan and Woodall (2000) is taken as a standard alternative. However, in fact, there are no standard alternatives because LRT methods rely on the assumption that the observations follow multivariate normal distribution. Therefore, we choose some perhaps imperfect comparisons to show the effectiveness of our approach.

The rest of this paper is organized as follows. The motivation, description, design of our proposed control chart and estimate of the change point are given in Section 2. The performance comparisons with LRT chart are discussed in Section 3. An illustrative example and a real data example are considered in Section 4 and Section 5, respectively. In Section 6, some computing aspects are shown and the conclusion and discussion of the proposed chart are given in Section 7.

2 A New Nonparametric Control Chart for Phase I Analysis

2.1 The Effect of Non-Normality on LRT Chart

The design of LRT chart (Sullivan and Woodall , 2000) is based on the assumption that the distribution of the observations is multivariate normal, denoted as $N_p(\mu, \Sigma)$ and as N_p for short. Stoumbos and Sullivan (2002) studied the robustness to non-normality of the MEWMA control chart. They mainly studied the robustness of the MEWMA chart to non-normality when the underlying distribution is multivariate t with ζ degrees of freedom, denoted as $T_{p,\zeta}$ and multivariate gamma with shape parameter ζ and scale parameter 1, denoted as $\Gamma_{p,\zeta}$. Details on the multivariate t and gamma distributions can be found in the Appendix to Stoumbos and Sullivan (2002). In addition, the following two distributions are involved, i.e., measurement components are independent and identically distributed (i.i.d.) from t distributions with ζ degrees of freedom, denoted as $t_{p,\zeta}$ and measurement components are i.i.d. from χ^2 distributions with ζ degrees of freedom, denoted as $\chi^2_{p,\zeta}$. We will study the robustness to non-normality of the LRT chart.

Table 1 gives the false alarm probability (FAP) α for LRT chart for $T_{p,\zeta}$ distribution, $\Gamma_{p,\zeta}$ distribution, $t_{p,\zeta}$ distribution and $\chi^2_{p,\zeta}$ distribution, respectively. Here, for simplicity, only the case when $p = 2$ and $\zeta = 5$ is listed.

Insert Table 1 Here.

From Table 1, we see that the actual FAP is much larger than the calculation based on a multivariate normal distribution. The difference between the actual FAP and the FAP for the multivariate normal distribution is particularly pronounced when the distribution is non-normal. This means that, false signal will occur much more frequently than expected when the distribution is non-normal, even when the process is operating properly. For example, when the observations have multivariate t distribution $T_{2,5}$, the FAP will be 0.52, which is almost 10 times as much as 0.05. For the multivariate gamma distribution $\Gamma_{2,5}$, the FAP can even be as large as 0.79. Therefore, a robust multivariate control chart is highly urgent to be constructed, which motivates our proposed control chart.

2.2 Design of the proposed scheme

In this section, we propose a Phase I control chart based on data depth, which is used in detecting the shifts in either the mean vector, the covariance matrix, or both of the process.

Suppose we have n independent observations from a multivariate distribution of dimensionality p, i.e.,

$$x_i \sim F_{(p)}(\mu_i, \Sigma_i),\ i = 1, \cdots, n. \tag{1}$$

If the process is in control, then $\mu_i = \mu$ and $\Sigma_i = \Sigma$ for all i. Assume that a step shift in the mean or variance or both occurs after τth observations, i.e. the mean and variance of the first τ observations is (μ, Σ), and the last $n - \tau$ observations have different mean and variance (μ^*, Σ^*), where $\mu^* \neq \mu$ or $\Sigma^* \neq \Sigma$. Note that some non-normal distributions may not have a specific covariance structure, such as $t_{p,\zeta}$ distribution and $\chi^2_{p,\zeta}$ distribution mentioned in the previous subsection. Our proposed control chart may not be applicable for multivariate non-normal distributions whose covariance is not well defined.

When the dimensionality $p = 1$, a straightforward nonparametric test to detect a mean change would be to use the Mann-Whitney two-sample test or the Wilcoxon rank-sum test. For any $1 \leq n_1 < n$, the Mann-Whitney statistic for testing whether two samples $x_1, \cdots, x_{n_1}$ and $x_{n_1+1}, \cdots, x_n$ come from the same distribution is defined as

$$MW_{n_1} = \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n} I(x_j < x_i), \tag{2}$$

where

$$I(x_j < x_i) = \begin{cases} 1,\ x_i > x_j, \\ 0,\ x_i \leq x_j. \end{cases}$$

The exact distributions of these statistics for different n_1, n are tabulated, and the asymptotic distribution are known (Lehmann , 1975; Hettmansperger , 1984).

In the high dimension ($p > 1$), we will propose a new test statistic, which can be regarded as a generalized Mann-Whitney statistic. Consider a p-variate quality vector, whose distribution is $F_{(p)}$ given by equation (1) when the process is in control. Assume that an assignable cause occurs, then any resulting change in the process will be reflected by a location change and/or a scale increase and characterized as a departure from $F_{(p)}(\mu, \Sigma)$ to an out-of-control distribution $F_{(p)}(\mu^*, \Sigma^*)$, and the departure will be reflected by a change in the data depth of the observations.

We define the statistic

$$Q(n_1) = \sum_{j=n_1+1}^{n} R_{n_1}(j)$$

where

$$\begin{aligned} R_{n_1}(j) =& \#\{x_i | D_{F_{n_1}}(x_i) < D_{F_{n_1}}(x_j), i = 1, 2 \cdots\cdots n_1\} \\ &+ \frac{1}{2} \#\{x_i | D_{F_{n_1}}(x_i) = D_{F_{n_1}}(x_j), i = 1, 2 \cdots\cdots n_1\} \end{aligned}$$

and $D_{F_{n_1}}(x_i)$ denotes the data depth of x_i according to the empirical distribution of $x_1, \cdots, x_{n_1}$.

Although our $Q(n_1)$ is the generation of MW_{n_1} to multivariate distributions, there is difference. In the definition of $R_{n_1}(j)$, we consider the case $D_{F_{n_1}}(x_i) = D_{F_{n_1}}(x_j)$, while the case $x_i = x_j$ is not involved in the definition of $I(x_j < x_i)$. Qiu and Hawkins (2001) pointed out when all or $p-1$ measurements are continuous, the chance of ties in p measurements is negligible for all practical purposes. When two or more measurements are discrete and these discrete measurements can take the same values, however, ties among the p measurements are possible. We overcome the difficulty caused by ties by allocating probability 1/2 to each two observations that share the same data depth. By using this definition, no information about the data depth is lost. One may also define

$$R'_{n_1}(j) = \#\{x_i | D_{F_{n_1}}(x_i) \leq D_{F_{n_1}}(x_j), i = 1, 2 \cdots\cdots n_1\}$$

to consider the ties. However, under this definition, in the two extreme cases, i.e., if any two of the observations have different depth, the expectation of $R'_{n_1}(j)$ will be $\frac{n_1}{2}$ and if all the observations have the same depth, the expectation of $R'_{n_1}(j)$ will be n_1, which seems unreasonable. But the expectation of $R_{n_1}(j)$ under our definition will be $\frac{n_1}{2}$ for these two extreme cases.

Note that when n_1 is small, the information on which $Q(n_1)$ is constructed is relatively scare. Take $n_1 = 1$ for example, the data depths of observations 2 to n are calculated based on the empirical distribution of just one observation x_1. An immediate alternative method is to construct $Q(n_1)$ only when n_1 is relatively large (Loader , 1996). However, it is advisable to start the control with small n_1, so that the first sample is immediately considered after the process is started.

The standardized statistic $Q(n_1)$ is defined by

$$SQ(n_1) = \frac{\mathrm{E}(Q(n_1)) - Q(n_1)}{\sqrt{\mathrm{Var}(Q(n_1))}}, \tag{3}$$

where

$$E(Q(n_1)) = \frac{n_1(n-n_1)}{2} \quad \text{and} \quad \mathrm{Var}(Q(n_1)) = \frac{n_1(n-n_1)(n+1)}{12}$$

are the mean and variance of $Q(n_1)$, respectively, when the process is in control. Note that we use $\mathrm{E}(Q(n_1)) - Q(n_1)$ rather than $Q(n_1) - \mathrm{E}(Q(n_1))$ for the reason that if there exist some shifts in the observations, the depth of these observations will become smaller so that $\mathrm{E}(Q(n_1)) - Q(n_1)$ will be positive with high probability. We prefer monitoring statistics getting larger if the process is out-of-control. One may also use $|Q(n_1) - \mathrm{E}(Q(n_1))|$, which we donot recommend because $|Q(n_1) - \mathrm{E}(Q(n_1))|$ are always non-negative even all the observations are from the same distribution.

So, in the rest of this paper, the standardized likelihood ratio is defined by

$$SQ(n_1) = \frac{\frac{n_1(n-n_1)}{2} - Q(n_1)}{\sqrt{\frac{n_1(n-n_1)(n+1)}{12}}}. \tag{4}$$

Our proposed change point control chart based on data depth (CPDP) is constructed by plotting the statistics $SQ(i)$ versus $i \quad (1 \le i < n)$. An out-of-control signal is triggered if $\max_{1 \le i < n} SQ(i)$ exceeds the given decision interval (or control limit) $h_{n,\alpha}$, which depends on the desired in-control FAP. Note that we start plotting from $i = 1$, although $SQ(1)$ will always be 0. One might as well start from $i = 2$, which has no effect on the performance of the CPDP chart.

It can be shown that, for fixed p,

$$SQ(n_1) \xrightarrow{\mathcal{D}} N(0,1), \quad \text{as } n_1 \to \infty, n - n_1 \to \infty. \tag{5}$$

However, for a given sample size n, our proposed CPDP chart calls for calculating $Q(n_1)$ as n_1 varies from small number to $n-1$. That is, regardless of how big n might be, one still has to calculate $SQ(n_1)$ for small n_1 values. Moreover, even if equation (5) holds, the charting statistic $\max_{1 \le i < n} SQ(i)$ does not converge to $N(0,1)$. The distribution of $\max_{1 \le i < n} SQ(i)$, which essentially is the maximum of $n-1$ dependent random variables, is very difficult to derive even in asymptotic sense. We do not recommend to use the asymptotic distribution $N(0,1)$ in practice to find the decision interval $h_{n,\alpha}$. Instead, we search for the $h_{n,\alpha}$ through simulation.

For given $p = 2$ and various combinations of FAP α and n, the $h_{n,\alpha}$ for our CPDP chart based on 10,000 Monte Carlo simulations are shown in Table 2. In the simulations, the observations are generated from standard multivariate normal distribution and the simplicial depth of Liu (1990) is used. However, the $h_{n,\alpha}$ can also be used for other multivariate distributions and data depth and the reasons are discussed below.

Insert Table 2 Here.

From Table 2, we observe that $h_{n,\alpha}$ generally increases as n increases and nearly stabilized when n is large. We also listed the upper α percentile of $N(0, 1)$ in the bottom row of Table 2. We can see that the simulated $h_{n,\alpha}$ are lager than the corresponding percentile of $N(0, 1)$, which, again, indicates that the convergence in equation (5) does not hold well because small n_1 values are involved.

The data depth $D_{F_{n_1}}(x_i)$ can be any kind of sample data depth introduced in the Introduction. Suppose that, given a sample of n observations from a multivariate distribution, one wants to calculate the data depth of a new observation with respect to this sample. If the new observation comes from the same distribution, then the distribution of the data depth is approximately $U(0, 1)$. This conclusion holds regardless of the underlying multivariate distribution (under very mild restrictions) and the type of data depth used. This proposition is particularly useful in determining the control limit, $h_{n,\alpha}$, because, for any multivariate distribution, it is the same as achieving the desired in-control FAP.

Note also that if the new observation comes from a different distribution, the result of a change in mean or covariance matrix or both of the original distribution, then the distribution of the data depth is not $U(0, 1)$ any more. The problem of detecting changes from a distribution to another different distribution is interesting and warrants further research.

2.3 Estimate of the Change Point

In Phase I analysis, when a special cause produces a change in one or more process parameters, it is important to detect this change quickly, and it is also necessary to give an estimate for the position of shift if the process parameters have been shifted. Such an estimate of the change point is particularly important for our CPDP chart, which is used for Phase I analysis where the information we can obtain is not so much and data must be made "clean" for Phase II analysis. The estimate of the change point in the process will help one to identify and eliminate the special cause of a problem quickly and easily.

We propose an estimate of the change point based on the maximum likelihood estimator of the change point τ, i.e., the change occurs at the time $\tau + 1$. For our proposed CPDP chart, the estimate of the position of shift, under the assumption that there is only one sustained shift in the process parameter(s), is given by

$$\hat{\tau} = \underset{1\le t<n}{arg} \max\{SQ(t)\}, \tag{6}$$

which is consistent with Pettitt (1979). If there are multiple sustained shifts in the process parameter(s), we can use the binary segmentation method recursively, i.e., split the sample into two sets, $x_1, \cdots, x_{\hat{\tau}}$ and $x_{\hat{\tau}+1}, \cdots, x_n$, and find possible change points from these two separate sets until there is no evidence for change points (Yao , 1988; Zou et al. , 2008).

3 Performance Comparisons

As Sullivan and Woodall (1996) pointed out, the average run length (ARL) can not be used as the criterion of performance in Phase I analysis. Therefore, as Sullivan and Woodall (1996) and Koning and Does (2000), the FAP and the true signal probability (TSP) are used to compare the performance of control charts for Phase I analysis. A control chart is said to be better than another one if its TSP is larger than the other's when the process is out of control, while they have the same FAP when the process is in control.

In this section, we compared our proposed CPDP chart with the LRT chart of Sullivan and Woodall (2000) only, because there is no corresponding nonparametric multivariate detecting scheme in Phase I analysis as far as we know and the LRT chart has shown to be quite competitive among all the existing control charts for location change and/or a scale increase in parametric settings. Note that the LRT chart of Sullivan and Woodall (2000) is constructed under the assumption that the observations are from multivariate normal distribution.

Following the robustness analysis in Stoumbos and Sullivan (2002), we consider multivariate normal distribution (N_p), multivariate t distribution with ζ degrees of freedom ($T_{p,\zeta}$), multivariate gamma distribution with shape parameter ζ and scale parameter 1 ($\Gamma_{p,\zeta}$), measurement components i.i.d. from t distributions with ζ degrees of freedom ($t_{p,\zeta}$) and measurement components i.i.d. from χ^2 distributions with ζ degrees of freedom ($\chi^2_{p,\zeta}$). For simplicity, the case for 30 observations, $p = 2$, $\zeta = 5$ and $FAP = 0.05$ is presented only in this paper. The results in this section are evaluated by 10,000 simulations.

Table 3 compares the TSP when the mean vector is shifted after 15 of 30 observations. The squared length of the difference in the mean vectors $\delta^* = (\mu^* - \mu)^T \Sigma^{-1} (\mu^* - \mu)$ is shown in the top row. As an anonymous referee pointed out, although the LRT chart is developed specifically for multivariate normal distribution, it is possible, as Jones-Farmer et al. (2009), to adjust the control limit of the LRT chart, so that it will have the desired $FAP = 0.05$ under a multivariate non-normal distribution, and then compare the chart performance of the CPDP chart against the LRT chart under this multivariate non-normal distribution. Therefore, we adjusted the control limit of the LRT chart such that the FAP is maintained at 0.05 for all distributions considered.

From Table 3, we can have the following conclusions. First, when the process is multivariate normal N_2, the LRT chart performs much better for $\delta^* \leq 3$, (e.g., the TSP could be twice that of the CPDP chart for $\delta^* = 1.5$), slightly better or almost the same for $\delta^* \geq 4.5$. Second, when the process is multivariate t distribution $T_{2,5}$, the LRT chart has scarcely detection power for $\delta^* \leq 4.5$, while our CPDP chart has much better performance (e.g., the TSP of CPDP chart is nearly 8 times that of the LRT chart for $\delta^* = 4.5$). Although the detection ability of LRT chart gets better as the shift δ^* gets larger, it still has quite lower TSP than our CPDP chart. Third, when the process is multivariate gamma distribution $\Gamma_{2,5}$, the LRT chart has scarcely detection power for all the shift considered here (e.g., even for $\delta^* = 12.0$, the TSP of the LRT chart is only as 0.12). Compared with LRT chart, our CPDP chart has quite satisfactory performance. Fourth, for the measurement components non-normal distributions $t_{2,5}$ and $\chi^2_{2,5}$, our CPDP chart is uniformly much better than the LRT chart.

Insert Table 3 Here.

Note, when $p = 1$, that the LRT chart is designed under the condition that the process variance is stable. In this case, the LRT chart is equivalent to the well known two-sample t test between the left and right part of the sequence, maximized across all possible change-points (Hawkins et al. , 2003). The two-sample t test is a direct competitor to the Mann-Whitney test. Remarkably, even when the underlying distributions are normal, the Mann-Whitney test is about 0.96 as efficient (Gibbons , 2003) as two-sample t test for moderately large sample sizes, and yet, unlike the two-sample t test, it does not require normality to be valid. Moreover, for some skewed or heavy-tailed distributions, the Mann-Whitney test is known to be more efficient than the two-sample t test.

Table 4 shows the TSP when the covariance matrix shifts after 15 of 30 observations. The shifted covariance matrix is a scalar multiple of the first element and the scalar multiple σ^* appears in the top row. Note that there is no specific covariance structure for the $t_{p,\zeta}$ and $\chi^2_{p,\zeta}$ distributions, therefore, covariance shift is not considered for these two distributions. From Table 4, when the process is multivariate normal, the LRT chart is better than the CPDP chart, except for $\sigma^* = 2.5$ and 3.0. When the process is multivariate non-normal, our CPDP chart is uniformly much better than the LRT chart, especially for $T_{2,5}$.

Insert Table 4 Here.

From Tables 3 and 4, for multivariate normal distribution, the chart performance of the LRT chart, as expected, is generally better than our proposed CPDP chart, even when the process shift is small. As the shift size grows, the performance of our CPDP chart gets better and is comparable with LRT chart. Therefore, our CPDP chart does not lose much efficiency even though the underlying distribution is normal. Our CPDP chart can be used for any multivariate distributions, while maintaining the same FAP. This is one advantage of our CPDP chart compared with LRT chart, which can only be used for multivariate normal distributions. When the process is multivariate nonnormal, our CPDP chart is uniformly much better than the LRT chart. This is another advantage of our CPDP chart. Moreover, from the simulation results, it seems that our CPDP chart is more effective for symmetric distributions with heavy tails than skewed distributions.

4 Illustrative Example

In this section, an illustrative example is given to introduce the implementation of CPDP control chart.

The observations of the example are total $n = 30$ observations, which are normally distributed with mean vector μ and covariance matrix I_2. Note that the mean vector μ has been shifted from $(0, 0)'$ to $(1.5, 1.5)'$ after sample 20. As the LRT chart of Sullivan and Woodall (2000) is constructed under the assumption that the observations are from multivariate normal distribution, we also incorporated the monitoring statistics of LRT chart for this example.

Table 5 presents the observations $x_{i,1}$, $x_{i,2}$, the expectation and variance of $Q(i)$, the statistic $Q(i)$ and the standardized statistic $SQ(i)$ based on SD of Liu (1990), and the monitoring statistics of $LRT(i)$, respectively. Given FAP $\alpha = 0.05$, the control limit for our CPDP chart is 2.280 from Table 2. It can be clearly seen from Table 5 that our CPDP control chart can detect the change. Moreover, note that the maximum value of $SQ(i)$ in column 7 is $SQ(20)$, which indicates exactly the shift after observation 20. At the same time, noting the control limit for LRT chart is 1, the maximum value of $LRT(i)$ in the last column is $LRT(20)$ and also indicates exactly the shift. From Table 5, for multivariate distribution and step mean vector shift, our CPDP chart has as good as performance of LRT chart.

Insert Table 5 Here.

As an anonymous referee pointed out, for any Phase I control charting scheme, it is typically assessed against three possible out-of-control scenarios, (i) a step-change in the process; (ii) the presence of out-of-control observations at fixed or random sampling periods, and (iii) a gradual shift in the process. A Phase I control chart is not expected to perform well under all three out-of-control scenarios. However, it will provide better understanding of our proposed CPDP chart if its performance can also be evaluated under scenarios (ii) and (iii).

We assumed a mean shift of 3 for the original samples 20, 25 and 30 and a gradual mean increase of $1.5 \times \frac{i-20}{10}$ for the original samples $i, 20 \leq i \leq 30$ of this illustrative example, to study the performance of our CPDP chart under scenarios (ii) and (iii), and the results are shown in Tables 6 and 7, respectively. From Table 6, we can see that our CPDP chart can not detect out-of-control observations at fixed or random sampling periods, even the mean shift of these out-of-control observations is as much as 3. Although in this case, the LRT chart can give an out-of-control signal, the maximum value of LRT chart is $LRT(28) = 1.72$, which can not divide these observations into two groups as the LRT chart can not be constructed for samples 29 and 30. From Table 7, we can see that our CPDP chart can detect gradual shift. Note that the gradual shift begins at sample 21 and our CPDP chart gives a signal at sample 23. The 2-sample-delay may be caused by the fact that the mean shift is still small for the initial stage of gradual shift. The LRT chart, however, can not give an out-of-control signal in this case. From Tables 5-7, we believe our CPDP chart is a good alternative of LRT chart when the observations are from multivariate normal distributions.

Insert Table 6 Here.

Insert Table 7 Here.

Note that our proposed CPDP chart is aimed at location change and/or a scale increase. In practice, the process mean and variability can vary simultaneously during the monitoring period and it may be desirable to construct a control chart that not only can detect changes in the process variability but also is insensitive to shifts in the process mean. Huwang et al. (2007) proposed a control chart that can serve the purpose above by subtracting an estimate of the mean. At first thought, for our CPDP chart, if the first n_1 and the last $n_2 = n - n_1$ observations were each centered first by subtracting the mean before calculating the $Q(n_1)$ and thus $SQ(n_1)$, an out-of-control signal seems more likely to indicate a change in the process covariance matrix. However, this is not true based on our simulation results below.

For this example, we simulated mean shift ($\mu^* = \mu + 1.5, \Sigma^* = \Sigma$), variance shift ($\mu^* = \mu, \Sigma^* = 2\Sigma$) and simultaneous shift ($\mu^* = \mu + 1.5, \Sigma^* = 2\Sigma$) for the original samples 20–30 and the $SQ(i)$ values are listed in Table 8. From Table 8, it is expected our CPDP chart can not give a signal if the observations were each centered and there is only mean shift. Although our CPDP chart can give a signal when there is only variance shift or simultaneous shift, even if the observations were each centered first, the signal points are before the true change point, which means the signals are false. If the observations were not each centered first, our CPDP chart can give out-of-control signals and indicate the exact change point when there is only mean shift or simultaneous shift and 1-sample-delay when there is only variance shift. For the unsatisfactory performance of our CPDP chart with observations centered first, a possible explanation is that the estimates of mean for the first n_1 and the last n_2 observations are far from the true mean if n_1 is far from the true change point.

Insert Table 8 Here.

We also compared the results calculated using both SD of Liu (1990) and M_jD of Singh (1991). Judging from the results, they both produced nearly the same results. Therefore, the results based on M_jD of Singh (1991) are not listed here, but available from the authors upon request.

5 Real Data Example

In this section, the application of our proposed CPDP chart chart is illustrated in a real data example, i.e., the gravel data, which was also used by Sullivan and Woodall (2000) to show the implementation of their LRT chart for change-point detection of mean vector or covariance matrix shifts. The data set contains 56 individual observations from a European plant producing grit, or gravel, giving the percent of the particles (by weight) that are large and medium in size and is shown in Table 2 of Sullivan and Woodall (2000), thus omitted here. Interested readers are referred to Sullivan and Woodall (2000) for deeper background.

In Sullivan and Woodall (2000), the FAP α is set to 0.05. Although we have made a comparative study with LRT chart in Sections 3 and 4, we set the same FAP α with Sullivan and Woodall (2000) to show the application of our CPDP chart more clearly. Note that, for our chart, the decision interval $h_{56,0.05}$ is about 2.519 by linear interpolation of $h_{50,0.05} = 2.463$ and $h_{60,0.05} = 2.557$ from Table 2. Figure 1 (a) shows the $SQ(i)$ values (solid curve connecting the dots) along with $h_{56,0.05} = 2.519$ (the horizontal dashed line). From Figure 1 (a), we see that our CPDP chart gives an out-of-control signal at observation 24, identifying the first 24 observations as group 1 and the rest as group 2, which is consistent with the result of Sullivan and Woodall (2000).

Insert Figure 1 Here.

Having divided the observations into two groups, it is generally useful to see if there is evidence of other shifts within these groups. With the binary segmentation method in Section 2.3, the first group shows no evidence of a shift within it, based on analysis not shown here. The analysis of the second group is shown in Figure 1 (b) and $h_{32,0.05} = 2.306$ also obtained by interpolation. From Figure 1 (b), we see that our CPDP chart gives another out-of-control signal at observation 42, which is a little different from the result of Sullivan and Woodall (2000), while the LRT chart of Sullivan and Woodall (2000) gives an out-of-control signal at observation 43.

Holmes and Mergen (1993) also analyzed the gravel data, also concluding that there was evidence of special causes of variation, associated specially with observations 26 and 45. Compared with the analysis results of Sullivan and Woodall (2000) and Holmes and Mergen (1993), our CPDP chart gives reasonable analysis for the gravel data, which shows that our CPDP chart is quite a useful tool for practitioners.

6 Computing the Depth

Kim et al. (2003) described an optimal algorithm which computes all bivariate depth contours in $O(n^2)$ time and space, using topological sweep of the dual arrangement of lines. Once these contours are known, the location depth of any point can be computed in $O(\log^2 n)$ time with no additional preprocessing or in $O(\log n)$ time after $O(n^2)$ preprocessing. We implement this algorithm to compute TD (Tukey , 1975), SD (Liu , 1990), M_jD (Singh , 1991), PD (Zuo and Serfing , 2000) and SRD (Gao , 2003) of a point. We compared the performance of the control charts using these kinds of values and our simulation results show that there is negligible difference between the simulated values.

Rousseeuw and Ruts (1992) developed a highly efficient Fortran algorithm to compute the data depth of a point in a bivariate distribution. It requires only $O(n \log n)$ times instead of $O(n^4)$ as required by direct computation based on solving systems of equations. As to higher dimensions, Rousseeuw and Struyg (1998) gave an algorithm for the computation of the location depth.

7 Conclusions and Considerations

Based on the Mann-Whitney test and data depth, a new Phase I change point control chart (denoted as CPDP) was introduced to detect location change and/or scale increase in Phase I analysis. The simulation results show that our proposed CPDP chart can match the performance of the LRT chart in the normal distribution setting and give much better performance in non-normal setting, as it has much greater robustness of good performance. The major advantage of our multivariate CPDP control charts is their attractive applicability when there is little information about the underlying distribution. Therefore, our proposed CPDP chart seems to offer an attractive alternative to the normal-based charts for cases where normality can not reasonably be assumed.

As shown in this paper, there exist many algorithms for computing the depth in the literature. However, the effort in computation the depth is still high, especially when the dimension is high. We believe that finding more efficient algorithms is quite an interesting topic. The problem of detecting changes from a distribution to another different distribution is also interesting and warrants further research.

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Table 1
Simulated FAP α for LRT chart.

Dist.	N_2	$T_{2,5}$	$\Gamma_{2,5}$	$t_{2,5}$	$\chi^2_{2,5}$
α	0.05	0.52	0.79	0.53	0.33

Table 2
Simulated control limit $h_{n,\alpha}$ for CPDP chart.

	α	0.10	0.05	0.03	0.02	0.01
	30	2.058	2.280	2.406	2.532	2.679
	40	2.175	2.413	2.557	2.637	2.818
	50	2.218	2.463	2.652	2.779	2.980
	60	2.305	2.557	2.697	2.844	3.057
n	70	2.326	2.588	2.754	2.881	3.084
	80	2.387	2.658	2.835	2.966	3.150
	90	2.391	2.671	2.848	2.980	3.196
	100	2.408	2.680	2.850	2.989	3.202
$N(0,1)$		1.282	1.645	1.881	2.054	2.326

Table 3
Simulated TSP for mean shift.

	δ^*	0.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5	12.0
N_2	LRT	0.05	0.53	0.60	0.72	0.84	0.92	0.97	0.99	0.99
	CPDP	0.05	0.22	0.47	0.69	0.83	0.91	0.95	0.98	0.99
$T_{2,5}$	LRT	0.05	0.06	0.07	0.08	0.11	0.15	0.20	0.27	0.35
	CPDP	0.05	0.24	0.44	0.62	0.75	0.83	0.89	0.92	0.94
$\Gamma_{2,5}$	LRT	0.05	0.06	0.06	0.06	0.07	0.07	0.08	0.10	0.12
	CPDP	0.05	0.10	0.30	0.52	0.68	0.80	0.87	0.91	0.94
$t_{2,5}$	LRT	0.05	0.06	0.07	0.08	0.09	0.13	0.17	0.24	0.31
	CPDP	0.05	0.23	0.42	0.60	0.74	0.82	0.88	0.92	0.95
$\chi^2_{2,5}$	LRT	0.05	0.06	0.08	0.10	0.16	0.25	0.36	0.47	0.59
	CPDP	0.05	0.11	0.32	0.54	0.70	0.82	0.88	0.92	0.94

Table 4
Simulated TSP for covariance shift.

	σ^*	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
N_2	LRT	0.05	0.50	0.51	0.53	0.80	1.00	1.00	1.00	1.00
	CPDP	0.05	0.22	0.47	0.69	0.83	0.91	0.95	0.98	0.99
$T_{2,5}$	LRT	0.05	0.05	0.06	0.07	0.08	0.10	0.13	0.16	0.21
	CPDP	0.05	0.24	0.44	0.62	0.75	0.83	0.89	0.92	0.94
$\Gamma_{2,5}$	LRT	0.05	0.05	0.07	0.11	0.19	0.32	0.47	0.61	0.73
	CPDP	0.05	0.10	0.30	0.52	0.68	0.80	0.87	0.91	0.94

Table 5
Data for illustrative example with a step shift in the mean.

i	$x_{i,1}$	$x_{i,2}$	$E(Q(i))$	Var(Q(i))	$Q(i)$	$SQ(i)$	$LRT(i)$
1	-0.09	-1.32	14.50	8.66	14.50	0.00	na
2	1.84	0.58	28.00	12.03	28.00	0.00	na
3	0.58	0.71	40.50	14.47	43.00	-0.17	0.10
4	-0.36	-0.97	52.00	16.39	53.50	-0.09	0.23
5	0.86	-0.65	62.50	17.97	64.00	-0.08	0.35
6	-0.42	0.12	72.00	19.29	72.00	0.00	0.34
7	1.15	0.54	80.50	20.39	68.00	0.61	0.42
8	-1.81	-0.74	88.00	21.32	74.50	0.63	0.43
9	0.62	-0.38	94.50	22.10	66.50	1.27	0.55
10	-0.94	0.89	100.00	22.73	82.50	0.77	0.47
11	-0.50	-0.41	104.50	23.24	72.50	1.38	0.62
12	0.18	0.51	108.00	23.62	67.00	1.74	0.69
13	2.62	1.82	110.50	23.89	78.50	1.34	0.50
14	-0.64	0.12	112.00	24.06	72.50	1.64	0.61
15	0.75	-1.61	112.50	24.11	68.00	1.85	0.58
16	-1.16	0.05	112.00	24.06	63.00	2.04	0.73
17	1.43	0.98	110.50	23.89	56.00	2.28	0.70
18	1.75	-1.08	108.00	23.62	63.00	1.91	0.71
19	-1.64	0.67	104.50	23.24	62.50	1.81	0.90
20	-0.51	-0.27	100.00	22.73	45.50	**2.40**	**1.33**
21	1.74	1.69	94.50	22.10	45.50	2.22	1.10
22	1.13	2.11	88.00	21.32	47.50	1.90	0.90
23	2.12	1.40	80.50	20.39	38.00	2.08	0.76
24	3.12	1.14	72.00	19.29	39.00	1.71	0.73
25	1.87	3.43	62.50	17.97	50.00	0.70	0.42
26	0.85	-0.26	52.00	16.39	31.00	1.28	0.72
27	1.13	2.42	40.50	14.47	22.00	1.28	0.42
28	2.54	0.17	28.00	12.03	21.50	0.54	0.48
29	1.40	1.93	14.50	8.66	3.50	1.27	na
30	1.12	3.48	na	na	na	na	na

Table 6
Data for illustrative example with 3 increase for samples 20, 25 and 30.

i	$x_{i,1}$	$x_{i,2}$	$E(Q(i))$	Var(Q(i))	$Q(i)$	$SQ(i)$	$LRT(i)$
1	-0.09	-1.32	14.50	8.66	14.50	0.00	na
2	1.84	0.58	28.00	12.03	28.00	0.00	na
3	0.58	0.71	40.50	14.47	44.00	-0.24	0.13
4	-0.36	-0.97	52.00	16.39	55.50	-0.21	0.25
5	0.86	-0.65	62.50	17.97	65.00	-0.14	0.40
6	-0.42	0.12	72.00	19.29	75.00	-0.16	0.41
7	1.15	0.54	80.50	20.39	72.00	0.42	0.52
8	-1.81	-0.74	88.00	21.32	75.00	0.61	0.51
9	0.62	-0.38	94.50	22.10	68.00	1.20	0.65
10	-0.94	0.89	100.00	22.73	93.50	0.29	0.60
11	-0.50	-0.41	104.50	23.24	85.50	0.82	0.74
12	0.18	0.51	108.00	23.62	80.00	1.19	0.85
13	2.62	1.82	110.50	23.89	89.00	0.90	0.68
14	-0.64	0.12	112.00	24.06	85.50	1.10	0.80
15	0.75	-1.61	112.50	24.11	82.00	1.27	0.73
16	-1.16	0.05	112.00	24.06	80.50	1.31	0.85
17	1.43	0.98	110.50	23.89	75.50	1.46	0.88
18	1.75	-1.08	108.00	23.62	87.00	0.89	0.93
19	-1.64	0.67	104.50	23.24	97.50	0.30	1.09
20	2.49	2.73	100.00	22.73	102.50	-0.11	0.77
21	0.24	0.19	94.50	22.10	82.50	0.54	0.85
22	-0.37	0.61	88.00	21.32	72.50	0.73	0.96
23	0.62	-0.10	80.50	20.39	52.50	1.37	1.03
24	1.62	-0.36	72.00	19.29	43.50	1.48	1.09
25	4.87	6.43	62.50	17.97	42.50	1.11	0.21
26	-0.65	-1.76	52.00	16.39	41.00	0.67	0.20
27	-0.37	0.92	40.50	14.47	33.00	0.52	0.20
28	1.04	-1.33	28.00	12.03	27.00	0.08	**1.72**
29	-0.10	0.43	14.50	8.66	3.00	1.33	na
30	4.12	6.48	na	na	na	na	na

Table 7
Data for illustrative example with gradual increase $1.5 \times \frac{i-20}{10}$ for sample $i, 20 \leq i \leq 30$.

i	$x_{i,1}$	$x_{i,2}$	$E(Q(i))$	Var(Q(i))	$Q(i)$	$SQ(i)$	$LRT(i)$
1	-0.09	-1.32	14.50	8.66	14.50	0.00	na
2	1.84	0.58	28.00	12.03	28.00	0.00	na
3	0.58	0.71	40.50	14.47	44.00	-0.24	0.09
4	-0.36	-0.97	52.00	16.39	57.50	-0.34	0.19
5	0.86	-0.65	62.50	17.97	68.50	-0.33	0.30
6	-0.42	0.12	72.00	19.29	78.00	-0.31	0.26
7	1.15	0.54	80.50	20.39	76.00	0.22	0.34
8	-1.81	-0.74	88.00	21.32	80.50	0.35	0.31
9	0.62	-0.38	94.50	22.10	72.50	1.00	0.41
10	-0.94	0.89	100.00	22.73	89.00	0.48	0.29
11	-0.50	-0.41	104.50	23.24	80.00	1.05	0.40
12	0.18	0.51	108.00	23.62	75.00	1.40	0.45
13	2.62	1.82	110.50	23.89	87.00	0.98	0.31
14	-0.64	0.12	112.00	24.06	80.50	1.31	0.38
15	0.75	-1.61	112.50	24.11	79.00	1.39	0.30
16	-1.16	0.05	112.00	24.06	75.50	1.52	0.36
17	1.43	0.98	110.50	23.89	70.00	1.69	0.35
18	1.75	-1.08	108.00	23.62	77.50	1.29	0.34
19	-1.64	0.67	104.50	23.24	76.50	1.21	0.38
20	-2.01	-1.77	100.00	22.73	77.50	0.99	0.68
21	0.39	0.34	94.50	22.10	57.50	1.67	0.71
22	-0.07	0.91	88.00	21.32	48.00	1.88	0.78
23	1.07	0.35	80.50	20.39	31.00	**2.43**	0.75
24	2.22	0.24	72.00	19.29	31.00	2.13	0.72
25	1.12	2.68	62.50	17.97	44.00	1.03	0.39
26	0.25	-0.86	52.00	16.39	28.50	1.43	0.60
27	0.68	1.97	40.50	14.47	17.00	1.62	0.46
28	2.24	-0.13	28.00	12.03	16.00	1.00	0.56
29	1.25	1.78	14.50	8.66	3.00	1.33	na
30	1.12	3.48	na	na	na	na	na

Table 8
Data for illustrative example with mean shift, variance shift and simultaneous shift.

	with subtracting mean			without subtracting mean		
i	mean	variance	simultaneous	mean	variance	simultaneous
1	0.00	0.00	0.00	0.00	0.00	0.00
2	0.00	0.00	0.00	0.00	0.00	0.00
3	0.07	-0.07	0.03	-0.17	-0.07	-0.03
4	0.12	-0.12	0.46	-0.09	-0.03	0.12
5	0.17	-0.14	0.58	-0.08	0.03	0.19
6	0.08	-0.29	0.52	0.00	0.16	0.29
7	0.69	0.25	0.98	0.61	0.76	0.93
8	0.63	0.19	1.20	0.63	0.89	0.96
9	0.97	0.52	1.52	1.27	1.52	1.58
10	1.28	0.97	1.89	0.77	1.19	1.23
11	1.48	1.40	2.35	1.38	1.74	1.83
12	1.88	1.88	2.48	1.74	2.14	2.24
13	1.30	1.49	2.51	1.34	1.76	2.05
14	1.31	2.02	2.72	1.64	2.04	2.41
15	1.47	2.22	3.24	1.85	2.24	2.74
16	1.33	2.22	**3.26**	2.04	2.35	2.97
17	1.80	**2.66**	2.99	2.28	2.57	2.80
18	1.10	2.16	2.52	1.91	2.24	2.92
19	0.30	2.00	2.00	1.81	1.98	2.80
20	-0.20	1.25	1.28	**2.40**	1.96	**3.12**
21	0.36	1.95	1.97	2.22	**2.72**	3.05
22	0.26	2.11	2.11	1.90	2.44	2.84
23	1.10	2.35	2.28	2.08	2.65	2.65
24	0.52	1.37	1.53	1.71	2.51	2.20
25	0.47	1.61	1.59	0.70	1.89	1.28
26	-0.21	0.18	0.37	1.28	1.56	1.56
27	0.52	0.69	1.00	1.28	1.11	1.38
28	-1.16	0.21	-0.50	0.54	0.50	0.75
29	-1.68	-1.68	-1.56	1.27	1.33	1.33

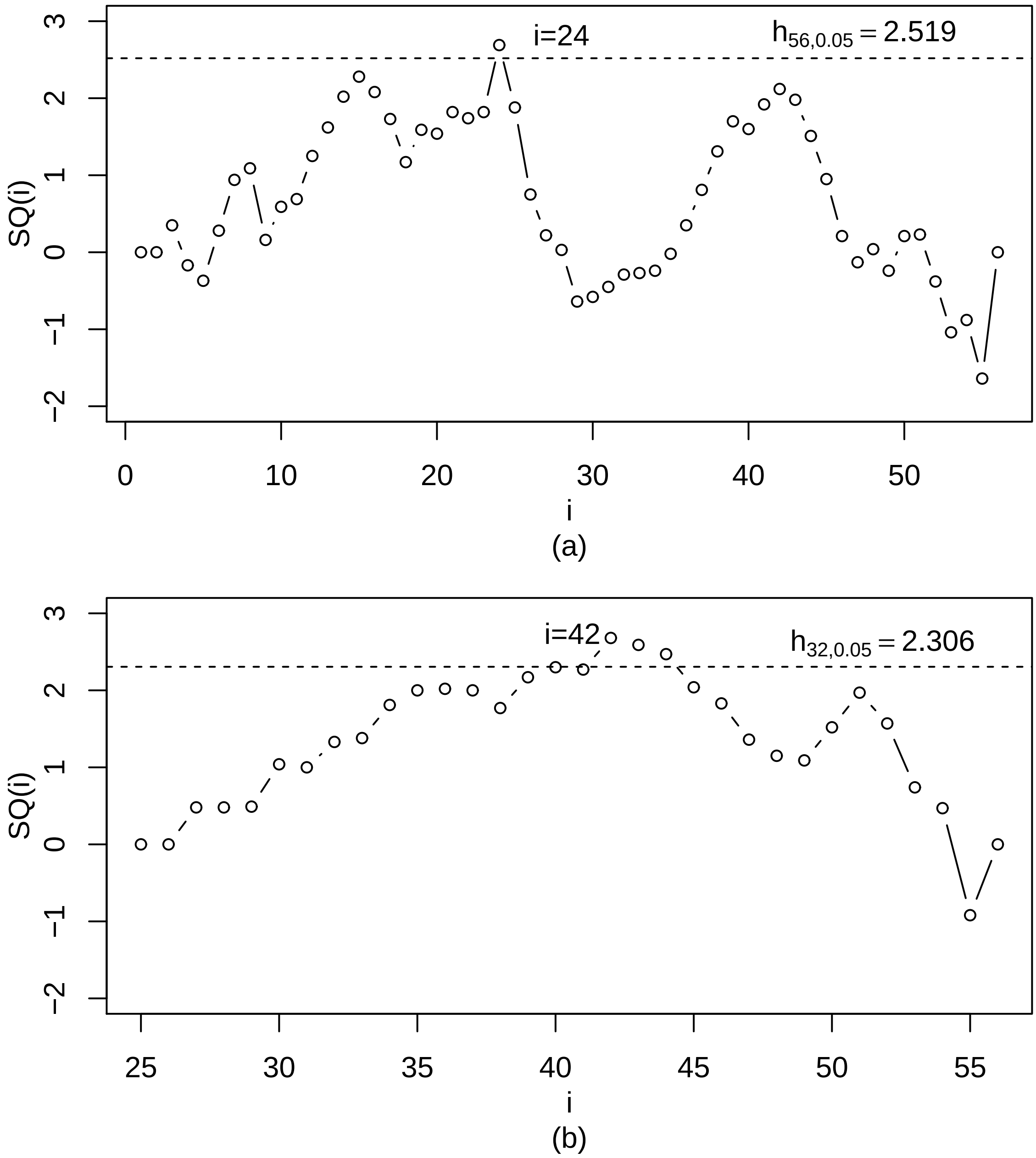

Fig. 1. $SQ(i)$ values for the gravel data.