A New Adaptive CUSUM Control Chart for Detecting the Multivariate Process Mean[*]

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Abstract

We propose a new multivariate CUSUM control chart, which is based on self-adaption of its reference value according to the information of current process readings, to quickly detect the multivariate process mean shifts. By specifying the minimum magnitude of the process mean shift in terms of its non-centrality parameter, our proposed control chart can achieve an overall performance for detecting a particular range of shifts. This adaptive feature of our method is based on two EWMA operators to estimate the current process mean level and make the detection at each step be approximately optimal. Moreover, we compare our chart to the conventional multivariate CUSUM chart. The advantages of our control chart detection for range shifts over the existing charts are greatly improved. The Markovian chain method, through which the average run length can be computed, is also presented.

KEY WORDS: Average Run Length; Exponentially Weighted Moving Average; Multivariate CUSUM; Multivariate Mean; Statistical Process Control.

1 Introduction

We assume that a process consists of p ($p > 1$) quality characteristics denoted by vector $\mathbf{X}$, where $\mathbf{X} \sim N_p(\mu, \Sigma)$. The process is said to be in-control (IC) if $\mu = \mu_0$ and $\Sigma = \Sigma_0$. We assume that the mean vector μ_0 and the covariance matrix Σ_0 are known exactly. In practice, these parameters are estimated from a sample of measurements on the process. Provided that this sample was sufficiently large, little harm is done by treating the parameters as known. To facilitate the later discussion, assume that all components of vector $\mathbf{X}$ obtained over time are mutually independent and have been re-scaled to unit standard deviation, i.e., the diagonal elements of Σ_0 are all equal to 1. Let the in-control mean vector μ_0 be $\mathbf{0}$. In this paper, we only focus on the detection of the shifts in the process means, so hereafter, we assume that the covariance matrix remains Σ_0 over time no matter the process is in-control or out-of-control.

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The average run length (ARL) performance of the multivariate Shewhart chart, which is also known as a Hotelling T^2 chart, depends on the mean vector μ and covariance matrix Σ_0 only through the non-centrality parameter $\lambda = \sqrt{\mu'\Sigma_0^{-1}\mu}$, where $\lambda = 0$ if and only if the process is in-control. The multivariate Shewhart chart signals when

$$T_t = \sqrt{\mathbf{X}_t'\Sigma_0^{-1}\mathbf{X}_t} > H_s,$$

where H_s is the control limit. Similar to its univariate counterpart, Shewhart $\bar{X}$ chart is more sensitive to large mean shifts but less to small and moderate mean shifts. For detecting small and moderate mean shifts, a natural method to develop a multivariate CUSUM procedure which has ARL's dependency only on the non-centrality parameter λ, is a test for any change in λ suggested by [1], which can be derived as

$$c_t = \sqrt{(\mathbf{S}_{t-1} + \mathbf{X}_t)'\Sigma_0^{-1}(\mathbf{S}_{t-1} + \mathbf{X}_t)},$$

where

$$\mathbf{S}_t = \begin{cases} \mathbf{0}, & \text{if } \ c_t \leq k, \\ (1 - k/c_t)(\mathbf{S}_{t-1} + \mathbf{X}_t), & \text{if } \ c_t > k, \end{cases}$$

with $t = 1, 2, ...,$ and $\mathbf{S}_0 = \mathbf{0}$. This multivariate CUSUM chart, denoted by MCUSUM, signals when

$$y_t = \sqrt{\mathbf{S}_t'\Sigma_0^{-1}\mathbf{S}_t} > H_c, \tag{1}$$

where $H_c > 0$ is called the decision interval and $k > 0$ is the reference value.

It is well known that, in univariate case, CUSUM chart gives the optimal detection for any particular mean shift δ if one sets the corresponding reference value ([2, 3, 4, 5, 6]). However, in the multivariate case, the optimal design for any particular mean vector shift in terms of λ, can not be simply the case. Searching the optimal value of k is too complicated in that the MCUSUM procedure given by Equation (1) is no longer equivalent to a series of sequential tests as the univariate one is, so [1] suggested to use the similar form of selecting k to that of the univariate CUSUM to approximate the optimal performance for that particular shift λ, i.e., $k = \lambda/2$. For more detailed information about MCUSUM, see [1] and [7]. From the discussion above, we know that the detecting performance of MCUSUM chart greatly depends on the exact pre-knowledge about the magnitude of process shift λ, but the shift λ can not be always known in real practice. If the true magnitude of λ is not the same as expected, the detection performance of MCUSUM chart could be badly destroyed. This is also true for univariate CUSUM ([8]). For univariate CUSUM, [9] proposed an adaptive CUSUM control chart (called ACUSUM) when the true magnitude of the future mean shift is unknown. Moreover, a two-dimensional Markov chain model was developed by [10], which can greatly simplify the evaluation of the ARL performance of ACUSUM charts instead of simulation method.

In this paper, we recommend a more generalized MCUSUM control chart, called Adaptive MCUSUM (AMCUSUM) control chart, which does not only operate without any pre-knowledge about the process shift, but also achieve an overall approximately optimal performance at each point in a broader range of mean shifts.

The rest of this paper is organized as follows. In Section 2, our proposed control chart is presented. Section 3 and Section 4 are devoted to design of our proposed AMCUSUM charts and performance comparison between AMCUSUM charts and MCUSUM charts, respectively. In Section 5, we conclude by addressing some relevant issues and discussing some problems. The Markov-Chain representation of the ARL is deferred to Appendix.

2 The description of our proposed control chart

We now use the MCUSUM chart to introduce the idea of our proposed AMCUSUM chart. In practice, the magnitude of the future shift λ is often unknown and needs to be estimated. If λ can be efficiently estimated at time t, say λ_t, then the estimated value is used to optimize the MCUSUM statistic in Equation (1) by selecting the reference value $k_t = \lambda_t/2$ at time t, i.e. let

$$\mathbf{S}_t = \begin{cases} \mathbf{0}, & \text{if } \; c_t \le k_t, \\ (1 - k_t/c_t)(\mathbf{S}_{t-1} + \mathbf{X}_t), & \text{if } \; c_t > k_t, \end{cases}$$

where

$$c_t = \sqrt{(\mathbf{S}_{t-1} + \mathbf{X}_t)'\Sigma_0^{-1}(\mathbf{S}_{t-1} + \mathbf{X}_t)}.$$

The control chart signals if

$$y_t = \sqrt{\mathbf{S}_t'\Sigma_0^{-1}\mathbf{S}_t} > h(k_t), \tag{2}$$

where $h(k_t)$ is the control limit at time t. However, just as the univariate case mentioned by [9], the statistic y_t is inconvenient to use in practice because the control limit $h(k_t)$ always changes with changes in k_t. Moreover, if one wants to have an equal false alarm rate at every step, this procedure could be hard to operate in practice because $h(k_t)$ is a decreasing function of k_t for a fixed IC ARL (denoted by ARL_0, hereafter). Thus, monitoring the MCUSUM statistic with a fixed threshold implies relatively tight control for small shifts and relatively loose control for large shifts. Clearly, this results in different sensitivities to different levels of mean shifts. To balance the detection sensitivity to both small and larger shifts, a natural idea is to adjust the statistic y_t by dividing the corresponding value of $h(k_t)$ to standardize the control limit over time, that is:

$$y_t^* = \sqrt{\mathbf{S}_t'\Sigma_0^{-1}\mathbf{S}_t}/h(k_t) > H^*, \tag{3}$$

where H^* is a threshold to maintain a desired ARL_0 and its value is close to 1 but not exactly due to estimating errors of $h(k_t)$, which we will discuss later. Given $h(k_t)$, H^*

can be searched by golden section method or bisection method using the Markov-Chain representation of ARL derived in the Appendix.

By far, along with its continuously updating its reference value k according to the sample readings, the conventional MCUSUM is now presented in a form that makes its extension to a self−adaptive one. Just as mentioned by [9] and [10], in real practice, process products may perform fairly well for small shifts in the process characteristics. Therefore, there is often a minimum magnitude of process shifts with high importance for early detection, denoted by $\lambda_{min} > 0$. We wish to design a control procedure to be more efficient at signalling the shifts $\lambda \geq \lambda_{min}$, but it may reduce the efficiency in signalling any shifts $0 < \lambda < \lambda_{min}$. As such, a more appropriate estimate of the true magnitude of λ, at time t, denoted by λ_t^* should be required as:

$$\lambda_t^* = \max\left(\lambda_{min}, \lambda_t\right). \tag{4}$$

Therefore, the statistic y_t^* in Equation (3) with the restriction (4) reduces the efficiency for signalling the shifts $\lambda < \lambda_{min}$ as compared to the one only in Equation (3) but increases the efficiency for signalling the shifts $\lambda \geq \lambda_{min}$.

Different schemes can be used to estimate λ. EWMA is one of the most popular scheme to most SPC practitioners due to its simplicity and efficiency ([11, 12]). To include the purpose in Equation (4) for increasing the efficiencies for signalling the shifts larger than λ_{min}, we prose our AMCUSUM control chart as follows.

First, we use a vector-type EWMA statistic, denoted by $\{\mathbf{e}_t\}$, to cumulate the information from the sample readings, which is derived by the following recursive form

$$\mathbf{e}_t = (1-r)\mathbf{e}_{t-1} + r\mathbf{X}_t,$$

where $\mathbf{e}_0 = \mathbf{0}$ and $r \in (0,1)$ is a smoothing parameter. This vector-type EWMA gives a good estimation of the direction and magnitude of the current mean vector μ, thus, it can be used to estimate the magnitude of λ by computing the out-of-control mean of its quadratic form, that is

$$\mathrm{E}\left(\mathbf{e}_t'\Sigma_0^{-1}\mathbf{e}_t\right) = \lambda^2\left[1-(1-r)^t\right]^2 + \left[1-(1-r)^{2t}\right]\tfrac{rp}{2-r}. \tag{5}$$

From Equation (5), we could get an unbiased estimate of the squared value of λ at time t, denoted by $(\lambda_t)^2$, that is

$$(\lambda_t)^2 = \tfrac{1}{(1-(1-r)^t)^2} \times \left\{\mathbf{e}_t'\Sigma_0^{-1}\mathbf{e}_t - \left[1-(1-r)^{2t}\right]\tfrac{rp}{2-r}\right\}. \tag{6}$$

Second, based on Equation (6) and the Equation (4), another EWMA operator $\{\lambda_t^*\}$ is used to estimate true mean shift λ with the restriction of only detecting shifts larger than λ_{min}, that is:

$$(\lambda_t^*)^2 = \max\left\{\lambda_{min}^2, (1-r)(\lambda_{t-1}^*)^2 + r(\lambda_t)^2\right\}. \tag{7}$$

Finally, our proposed AMCUSUM statistics can be derived as follows

$$c_t^* = \left\{(\mathbf{S}_{t-1}^* + \mathbf{X}_t)'\Sigma_0^{-1}(\mathbf{S}_{t-1}^* + \mathbf{X}_t)\right\}^{1/2},$$

and

$$\mathbf{S}_t^* = \begin{cases} 0, & \text{if } c_t^* \leq k_t^*, \\ (1 - k_t^*/c_t^*)(\mathbf{S}_{t-1}^* + \mathbf{X}_t), & \text{if } c_t^* > k_t^*. \end{cases}$$

where

$$k_t^* = \lambda_t^*/2.$$

The AMCUSUM chart signals when

$$y_t^* = \sqrt{\mathbf{S}_t^{*\prime}\Sigma_0^{-1}\mathbf{S}_t^*}/h(k_t^*) > H^*, \tag{8}$$

where H^* is the control limit to obtain a pre-defined ARL_0.

3 The Design of AMCUSUM chart

3.1 The estimate of $h(k)$

As mentioned in last section, the value of k varies over time, so the control limit of MCUSUM statistic needs to be changed to maintain the same ARL_0 at every step. Therefore, we give an operating model of $h(k)$ to standardize the control limit over time for easy implementation. However, three parameters are related to it, they are: p, k and ARL_0. Getting a full form including all these parameters is not easy, so here, we give a list of $h(k, \text{ARL}_0)$ with different values of p by both simulation and regression methods.

Given the value of p, the operating model $h(k, \text{ARL}_0)$ were established by first estimating the $h(k_i, \text{ARL}_0^j)$ for 55 different values of $\{k_i\}$ ranging from 0.2 to 3 and 16 different values of $\{\text{ARL}_0^j\}$ ranging from 200 to 1000 using Markov Chain Model, which is given in Appendix of this paper. Then empirical models were fitted to points $(k_i, \text{ARL}_0^j, h(k_i, \text{ARL}_0^j))$, for $i = 1, 2, ..., 55$ and $j = 1, 2, ..., 16$. Note that this method estimates $h(k, \text{ARL}_0)$ accurately when $0.2 \leq k \leq 3$, and extrapolating $h(k, \text{ARL}_0)$ beyond this range may be inaccurate.

The operating models $h(k, \text{ARL}_0)$ for the MCUSUM statistic are recorded in Table 1 for $p = 2, 3, ..., 10$. The results for the cases when ($p \geq 11$) are not listed for shorting the paper. Hereafter, we assume that all the sample readings are obtained under the bivariate normal case, that is $p = 2$, and the conclusions for $p > 2$ are similar. Also note that all the results are implemented in Fortran 95 program with IMSL package. The multivariate normal vector $\mathbf{X}$ is generated by routine "rnmvn". Routine "blinf" is used to obtain the ARL by the expressions in Appendix. For a given λ, we generate multivariate p-dimensional normal vectors $\mathbf{X}$ such that $\lambda = \sqrt{\mathbf{X}'\Sigma_0^{-1}\mathbf{X}}$ in each run. Note that [1] has proved that ARL depends only on the non-centrality parameter λ (see the Appendix of [1]). So the selection of normal vectors $\mathbf{X}$ is reasonable.

Insert Table 1 here.

3.2 The Effect of λ_{min}

As discussed above in Equation (4), the value of λ_{min} improves the detection performance of AMCUSUM charts for shifts $\lambda \geq \lambda_{min}$, but reduces the efficiency for shifts $\lambda < \lambda_{min}$. Table 2 presents the ARL values of AMCUSUMs with different values of λ_{min} by 100,000 simulated runs, where $r = 0.1$, $\lambda_0^* = \lambda_{min}$ and ARL_0 is set to 200. We set $\lambda_0^* = \lambda_{min}$ here because we want to study the performance of our AMCUSUM charts under the condition that $\lambda \geq \lambda_{min}$, although we can have other alternatives if we have different purposes (see the last subsection of this section). The numerical results show that the AMCUSUM charts with $\lambda_{min} = 0.25$, 0.50, 0.75, 1.00, 1.25 and 1.50 have the minimum ARL values for signalling shifts of sizes 0.25, 0.50, 0.75, 1.00, 1.25 and 1.50, respectively. For balancing the performance for both large and small shifts, here, we recommend to chose $\lambda_{min} = 0.5$ for practical use.

Insert Table 2 here.

3.3 The Effect of r

It can be seen that the EWMA operator affects the adaptive feature of AMCUSUM statistic only through its smoothing parameter r. As most literature focusing on the EWMA operator, larger values of r could improve the efficiency of control chart for large shifts, and vice versa. Table 3 gives the numerical results of the ARL performance of AMCUSUM charts with different values of $r = 0.05$, 0.1, 0.2, 0.3 and 0.5, where all these charts are mainly focused on detecting the range shifts $(0.5, 4.0)$, and the initial value of EWMA operator $\lambda_0^* = \lambda_{min} = 0.5$. Conclusions are similar for other ranges of detection interests.

Insert Table 3 here.

Theoretically, once λ_{min} and λ_0^* are fixed, r can be tuned to minimize the out-of-control ARL for any particular shift. However, choosing r to satisfy some optimal criterion for one or two shifts is not acceptable when the main purpose is to achieve an overall efficient performance over a particular range of shifts ([13]). This is because the optimal value of r, which optimize the ARL performance at some specified magnitude of mean shift, decrease the sensitivity to other shifts. From Table 3, it is shown that with different choice of r, our proposed control chart performs differently over the range shifts. The larger values of r, the more sensitive to larger shifts. However, these differences are very minor. Therefore, we recommend that one chooses $r = 0.2$ to balance the performance for both small and large shifts, as usually suggested in [14].

3.4 The Effect of the initial values of EWMA operator λ_0^*

Suppose $(\lambda_{min}, \lambda_{max})$ is the range of potential shift for future detection. To investigate effects of the initial value of λ_0^*, five models are carried out for choosing λ_0^*: (a) the A1 design, $\lambda_0^* = \lambda_{min}$; (b) the A2 design, λ_0^* is set at the lower quarter of the range $(\lambda_{min}, \lambda_{max})$; (c) the A3 design, λ_0^* is set in the middle point of the range $(\lambda_{min}, \lambda_{max})$; (d) the A4 design, λ_0^* is set at the upper quarter of the range $(\lambda_{min}, \lambda_{max})$; (e) the A5 design, $\lambda_0^* = \lambda_{max}$. Numerical results of ARL performances for all these 5 design models are given in Table 4, where the detecting range is $(0.5, 4.0)$, $r = 0.2$. Therefore, the corresponding values of λ_0^* for these 5 models are 0.5, 1.375, 2.25, 3.125, 4.0, respectively.

Insert Table 4 here.

From Table 4, it can be seen that the larger the value of λ_0^*, the more sensitive our proposed control chart to larger shifts. For example, the A1 design, outperforms other design schemes in detecting the small shifts ranging from 0.5 to 1.0. On the other side, the A5 schemes, performs the best for shifts larger than 3.0. Among these five designs, the A3 design, can be viewed as a overall balanced design over other four designs, therefore, we recommend to use A3 for practical use, i.e $\lambda_0^* = (\lambda_{min} + \lambda_{max})/2$.

3.5 The Guideline for designing AMCUSUM chart

Based on the discussion above about the properties of our AMCUSUM control chart, we suggest the following guidelines to design our AMCUSUM chart to achieve an overall efficiency for any specified range shifts, not only at some particular magnitude of process shifts.

(a). Select the detecting range of interest $(\lambda_{min}, \lambda_{max})$ based on preliminary investigation.

(b). Choose $\lambda_0^* = (\lambda_{min} + \lambda_{max})/2$ to balance the performance of the AMCUSUM control chart at all points in the range $(\lambda_{min}, \lambda_{max})$.

(c). Choose $r \in (0.05, 0.25)$ based on the rule-of-thumb. Here, in practice, we recommend to use $r = 0.2$.

(d). Select an appropriate control limit H^* to achieve the desired ARL_0.

(e). Run the AMCUSUM control chart. If $y_t^* > H^*$, then a signal is issued.

4 Performance Comparisons

In this section, we compare the ARL performance of AMCUSUM chart with that of the conventional MCUSUM chart over serval specified range shifts. For a fair comparison, here, we use the criterion proposed by [15], called "IRARL", which is defined as

$$\text{IRARL}(C) = E\left[\frac{\text{ARL}_c(\delta)}{\text{ARL}_{op}(\delta)}\right] = \int \frac{\text{ARL}_c(\delta)}{\text{ARL}_{op}} \,\mathrm{d}F(\delta), \tag{9}$$

where C is the signature of any compared control chart, $\text{ARL}_c(\delta)$ represents the OC ARL of control chart C under the mean shift δ, $\text{ARL}_{op}(\delta)$ represents the OC ARL of MCUSUM chart with $k = \delta/2$ under the mean shift δ and $F(\delta)$ is the cumulative distribution function (CDF) of shifts δ. If we have no prior information of the mean shift, the CDF of uniform distribution $U[\,\delta_{min}, \delta_{max}\,]$ can be used as $F(\delta)$, as we employed in this paper. For easy computation, the discrete form of Equation (9) is used to approximate the value of IRARL, which is given by

$$\mathrm{IRARL} \approx \frac{1}{m+1}\sum_{i=0}^{m}\frac{\mathrm{ARL}_c(\delta_i)}{\mathrm{ARL}_{op}(\delta_i)}, \tag{10}$$

where m is a given integer, and $\delta_i = \delta_{min} + \frac{i}{m}(\delta_{max} - \delta_{min})$.

Apparently, a control chart with a smaller IRARL value for any specialized range is considered to be more effective and robust to detect the shifts over that range, and vice versa. Here, we choose three range shifts with different width: $(0.5, 4.0)$, $(1.0, 4.0)$ and $(0.75, 1.5)$, which represents large, moderate and narrow width, respectively. Also, three MCUSUMs with different reference values are used for comparison: the M1 MCUSUM, with $k = \lambda_{min}/2$, aims to perform better near the left side of the range; the M2 MCUSUM, with $k = (\lambda_{min} + \lambda_{max})/4$, to balance the performance over the range; the M3 MCUSUM, with $k = \lambda_{max}/2$, to perform better near the right side of the range. For a fair comparison, the ARL_0 of these charts are maintained to 200, and our proposed AMCUSUM chart is designed by the guidelines in the previous section. The numerical results are shown in Table 5. Note that H^* and H are the control limits of AMCUSUM and MCUSUM charts, respectively.

From Table 5, it is easy to see that a single MCUSUM chart can only signal either small or large shift quickly once the corresponding reference value k is chosen appropriately to that shift. On the other hand, IRARL of the AMCUSUM chart for any of the three range shifts are smaller than those of three MCUSUM charts, which means that the AMCUSUM chart has either the shortest or nearly the shortest ARL for every shift magnitude within the specified range, and performs more robustly than MCUSUM chart for range shifts. Similar conclusions can be made for AMCUSUM charts designed for other moderate and small ranges of process shifts.

Insert Table 5 here.

5 Concluding remarks and extensions

The MCUSUM chart ([1]) is mainly designed with the assumption that the magnitude of the future mean shift is available in a prior. Its detecting performance may be substantially different from what is expected when the actual magnitude of the process shift is different from the pre-specified one. However, based on the dynamically adjustment on the reference value according to the current sample readings, our proposed AMCUSUM control chart can be more efficient and robust than MCUSUM chart in detecting a broader range of process mean shifts.

Other interesting issues can be pursued in future works. For example, we only consider AMCUSUM chart with known in-control parameters. If the process parameters (mean vector and covariance matrix) are estimated from a number of preliminary training samples, the performance of the AMCUSUM chart will be affected. Another important issue is the interpretation of the signal. With the EWMA operator added in the AMCUSUM statistic, its one-step ahead forecasting property could help to give additional information about the current process mean shift when the chart signals.

Finally, we give two suggestions to further improve the detection ability of our chart: 1. Superimposing a Shewhart control limit; 2. Using a Markovian-type mean estimator. A Markovian-type mean estimator is given by $(\lambda_t^*)^2 = (\lambda_{t-1}^*)^2 + \phi(a_t)^2$, where $a_t = (\lambda_t)^2 - (\lambda_{t-1}^*)^2$ is the prediction error and $\phi(\cdot)$ is a monotone function. This type of EWMA mean estimator was first proposed by [16] and was suggested by [13] for process monitoring. Based on the Huber's score function ([17]), [18] suggested using a new Markovian-type EWMA operator to estimate the process mean level, and $\phi(\cdot)$ is given by

$$\phi_\eta(a) = \begin{cases} a + (1-r)\eta, & a < -\eta, \\ \eta a, & |a| \leq \eta, \\ e - (1-r)\eta, & a > \eta, \end{cases}$$

where η is a constant. Note that when $\eta \to \infty$, this EWMA operator reduces to the regular EWMA statistic without restriction of Equation (4).

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Appendix: Markov-Chain Representation

For the in-control situation, the ARL of MCUSUM charts can be approximated by using a discrete Markov chain model. Following [19], the possible values of y_t in Equation (1) can be represented by m+1 states, where one state is an absorbing state referring to the range $y_t > h$. The m transient states are numbered 1,2,...,m and represent values of y_t between 0 and h. It is reasonable to think of the Markov chain model in terms of a discrete random variable which takes on values 0, w, $2w$,..., mw, where $w = 2h/(2m-1)$. The transition probabilities among the transient states are needed to find the ARL. The transition probabilities are:

$$\Pr\left(y_t = jw \,|\, y_{t-1} = iw\right), \quad i, j \in \{1, 2, ..., m\}.$$

To find the transition probabilities, note that $y_t = \max\left(0, c_t - k\right)$, which implies the transition probabilities are conditional probabilities. Further note that y_{t-1} can be regarded as the Mahalanobis distance of the vector $\mathbf{S}_{t-1}$. With a fixed distance y_{t-1}, the vector $\mathbf{S}_{t-1}$ is not fixed, but it is some constant rather than a random variable. So the variance of $\mathbf{S}_{t-1}$ is 0. Therefore, for the in-control situation, $\mathrm{E}(\mathbf{S}_{t-1} + \mathbf{X}_t) = \mathbf{S}_{t-1}$ and $\mathrm{Var}(\mathbf{S}_{t-1} + \mathbf{X}_t) = \Sigma_0$. Under the assumption of a multivariate normal distribution for $\{\mathbf{X}_t\}$, c_t has a χ^2 distribution with non-centrality parameter $\sqrt{\mathbf{S}'_{t-1}\Sigma_0^{-1}\mathbf{S}_{t-1}} = y_{t-1}$, therefore

$$\Pr\left(y_t = 0 \,|\, y_{t-1} = iw\right) = \Pr\left(c_t \le k + w/2\right),$$

$$\Pr\left(y_t = jw \,|\, y_{t-1} = iw\right) \; = \Pr\left(k + (j - 0.5)w < c_t \le k + (j + 0.5)w\right), j > 0$$

where c_t has a chi distribution with non-centrality parameter iw.

For given ARL_0, p and k, the threshold value H^* can be searched by golden section method or bisection method in a range $[0, U_H]$ using the Markov-Chain representation of ARL, where U_H is an upper bound satisfying the condition that the IC ARL of our scheme is larger than the prespecified ARL_0 when $H^* = U_H$. In the ith iteration, H^* is searched in the range $[L_H^{(i)}, U_H^{(i)}]$ with $L_H^{(1)} = 0$ and $U_H^{(1)} = U_H$. The IC ARL value $ARL_0^{(i)}$ is computed by the Markov-Chain representation derived above with $H^* = H^{*(i)} = (L_H^{(i)} + U_H^{(i)})/2$. If $|ARL_0^i - ARL_0| < \epsilon_1$, where $\epsilon_1 > 0$ is a prespecified threshold value, then the searching procedure stops, and the searched valued of H^* is $H^{*(i)}$. Otherwise, define,

$$\begin{cases} L_H^{(i+1)} = H^{*(i)} \quad \text{and} \quad U_H^{(i+1)} = U_H^{(i)}, & \text{if} \quad ARL_0^{(i)} < ARL_0, \\ L_H^{(i+1)} = L_H^{(i)} \quad \text{and} \quad U_H^{(i+1)} = H^{*(i)}, & \text{if} \quad ARL_0^{(i)} > ARL_0, \end{cases}$$

and

$$H^{*(i+1)} = (L_H^{(i+1)} + U_H^{(i+1)})/2.$$

If $|H^{*(i+1)} - H^{*(i)}| < \epsilon_2$, where $\epsilon_2 > 0$ is another prespecified threshold value, then the searching procedure stops, and the searched valued of H^* is $H^{*(i)}$. In such a case, a message should be printed to remind the user of the actual IC ARL. If $|H^{*(i+1)} - H^{*(i)}| \ge \epsilon_2$, the searching procedure executes the next iteration.

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Table 1: The operating model $h(k, ARL_0) = e^{a+ln(ARL_0)b}$

$p = 2$	$a = 1.7888 - 2.9212k + 1.8454k^2 - 0.5062k^3$
	$b = 0.1855 + 0.0582k - 0.1245k^2 + 0.0482k^3$
$p = 3$	$a = 1.8599 - 2.0014k + 0.9288k^2 - 0.2384k^3$
	$b = 0.2033 - 0.0657k - 0.0037k^2 + 0.0131k^3$
$p = 4$	$a = 2.0109 - 1.7037k + 0.6312k^2 - 0.1482k^3$
	$b = 0.2027 - 0.1019k + 0.0321k^2 + 0.0021k^3$
$p = 5$	$a = 2.1453 - 1.5338k + 0.4724k^2 - 0.1014k^3$
	$b = 0.2011 - 0.1227k + 0.0515k^2 - 0.0037k^3$
$p = 6$	$a = 2.2636 - 1.4244k + 0.3780k^2 - 0.0744k^3$
	$b = 0.1996 - 0.1372k + 0.0642k^2 - 0.0073k^3$
$p = 7$	$a = 2.3618 - 1.3507k + 0.3242k^2 - 0.0595k^3$
	$b = 0.1999 - 0.1487k + 0.0726k^2 - 0.0096k^3$
$p = 8$	$a = 2.3665 - 1.1107k + 0.1777k^2 - 0.0296k^3$
	$b = 0.2124 - 0.1857k + 0.0957k^2 - 0.0143k^3$
$p = 9$	$a = 2.5175 - 1.1996k + 0.2273k^2 - 0.0363k^3$
	$b = 0.2024 - 0.1748k + 0.0904k^2 - 0.0136k^3$
$p = 10$	$a = 2.6380 - 1.2711k + 0.2743k^2 - 0.0436k^3$
	$b = 0.1954 - 0.1654k + 0.0847k^2 - 0.0128k^3$

Table 2: The ARL of AMCUSUM chart with $r = 0.1$, $\lambda_0^* = \lambda_{min}$

	λ_{min}					
shift	0.25	0.5	0.75	1.0	1.25	1.5
0.00	200.00	200.00	200.00	200.00	200.00	200.00
0.25	**62.94**	64.45	73.59	82.80	92.62	105.10
0.50	27.33	**25.54**	26.83	29.25	32.75	38.11
0.75	15.86	14.51	**14.33**	14.90	15.79	17.48
1.00	10.55	9.72	9.65	**5.54**	9.68	10.12
1.25	7.73	7.13	7.02	6.96	**6.86**	6.94
1.50	6.03	5.57	5.48	5.33	5.27	**5.25**

Table 3: The ARL of AMCUSUM chart with different values of r.

	r				
shift	0.05	0.10	0.20	0.30	0.50
0.00	200.00	200.00	200.00	200.00	200.00
0.50	26.07	25.54	26.83	27.23	28.42
1.00	10.18	9.72	10.03	10.36	11.07
1.50	6.03	5.57	5.54	5.71	6.23
2.00	4.25	3.83	3.74	3.78	4.08
2.50	3.29	2.92	2.81	2.80	2.95
3.00	2.69	2.37	2.26	2.23	2.30
3.50	2.29	2.02	1.91	1.87	1.88
4.00	2.02	1.76	1.65	1.60	1.59

Table 4: The ARL of AMCUSUM chart with different initial values of λ_0^*.

shift	A1	A2	A3	A4	A5
0.00	200.00	200.00	200.00	200.00	200.00
0.50	26.70	28.08	30.45	32.31	34.21
1.00	10.05	10.63	11.56	12.67	14.08
1.50	5.54	5.60	5.75	6.24	7.04
2.00	3.70	3.61	3.55	3.67	4.05
2.50	2.77	2.65	2.52	2.49	2.64
3.00	2.23	2.10	1.96	1.89	1.91
3.50	1.87	1.75	1.61	1.51	1.49
4.00	1.60	1.50	1.37	1.29	1.26

Table 5: The ARL comparisons between AMCUSUM and MCUSUM charts

	AMCUSUM	M1	M2	M3	OP
	$r = 0.2$	$k = \lambda_{min}/2$	$k = (\lambda_{min} + \lambda_{max})/4$	$k = \lambda_{max}/2$	
range shifts (0.5, 4.0)					
shift	$H^* = 1.058$	$H_c = 8.659$	$H_c = 2.672$	$H_c = 1.288$	-
0.00	200.00	200.00	200.00	200.00	200.00
0.50	30.45	26.50	57.55	99.86	26.50
1.00	11.56	11.44	13.24	30.36	9.80
1.50	5.75	7.30	5.57	10.32	5.18
2.00	3.55	5.41	3.37	4.57	3.38
2.50	2.52	4.33	2.44	2.64	2.39
3.00	1.96	3.63	1.93	1.84	1.80
3.50	1.61	3.16	1.62	1.43	1.43
4.00	1.37	2.82	1.38	1.21	1.21
IRARL	1.11	1.69	1.25	1.79	-
range shifts (1.0, 4.0)					
shift	$H^* = 0.973$	$H_c = 5.485$	$H_c = 2.388$	$H_c = 1.288$	-
0.00	200.00	200.00	200.00	200.00	200.00
1.00	10.63	9.88	15.00	30.36	9.80
1.50	5.37	5.76	5.90	10.32	5.18
2.00	3.31	4.12	3.41	4.57	3.38
2.50	2.35	3.23	2.40	2.64	2.39
3.00	1.81	2.69	1.87	1.84	1.80
3.50	1.49	2.33	1.54	1.43	1.43
4.00	1.28	2.09	1.32	1.21	1.21
IRARL	1.03	1.36	1.13	1.51	-
range shifts (0.75, 1.5)					
shift	$H^* = 0.987$	$H = 6.762$	$H = 5.019$	$H = 3.936$	-
0.00	200.00	200.00	200.00	200.00	200.00
0.75	14.80	15.18	15.44	17.45	15.09
1.00	9.61	10.36	9.80	10.17	9.80
1.25	6.90	7.85	7.09	6.98	7.04
1.50	5.24	6.33	5.58	5.18	5.18
IRARL	0.99	1.10	1.03	1.05	-