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Cusum of Q chart with variable sampling intervals for monitoring the process mean

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Recently, adaptive control charts (that is, with variable sample sizes and/or sampling intervals) for the univariate or multivariate quality characteristics have received considerable attention in Phase II analysis in the literature. Due to the scarcity of enough sample to have a good knowledge of the parameters in start-up process, adding adaptive feature to self-starting control charts remains an open problem. In this paper, we propose an adaptive Cusum of Q chart with variable sampling intervals for monitoring the process mean of normally distributed variables. A Fortran program is available to assist in the design of the control chart with different parameters. The effect of the control chart parameters on the performance is studied in detail. The control chart is further enhanced by finding adaptive reference values. Due to the powerful properties of the proposed control chart, the Monte Carlo simulation results show that it provides quite satisfactory performance in various cases. The proposed control chart is applied to a real life data example to illustrate its implementation.

Keywords: Self-starting; Variable Sampling Interval; SPC; Quality Control; Reliability Engineering.

1. Introduction

Statistical process control (SPC) charts are widely used in industry for monitoring the quality of manufactured products. In Phase II analysis, the process parameters are usually assumed known. However, in practice the process parameters are usually unknown in the early stages of process improvement, and they are usually estimated by using in-control (IC) historical samples (or by the Phase I study). When the number of historical samples is small, control charts with estimated parameters generally produce a large bias in the IC average run length (ARL), and reduce the sensitivity of the chart in detecting the process changes measured by the out-of-control (OC) ARL. Moreover, after short runs, Bischak and Trietsch (2007) show that the false alarm probabilities from the charts increase drastically. This is the so-called "control charts with estimated parameters" problem.

In light of the deterioration of control chart performance that results from estimated parameters, practitioners should collect a sample of data large enough to ensure that parameter estimates are sufficiently close to the true parameters. It may not be realistic because the users usually want to monitor and adjust the process in the start-up stages. To tackle this problem, self-starting method is an efficient choice. Hawkins (1987) uses the running mean and standard deviation of all observations made on the process since start-up as substitutes for the unknown true values of the process mean and standard deviation. In particular, Hawkins

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et al. (2003) and Hawkins and Zamba (2005a,b) propose a change-point model based on the likelihood ratio for on-line monitoring which can also be seen as a self-starting method. Moreover, for start-up process and short or long runs, Quesenberry (1991) presents the important formulas so that charts for both the process mean and variance can be maintained from the start of production, whether or not prior information for estimation the parameters is available. Quesenberry (1995) studies the properties of Q charts for variables and the sensitivity of specially designed exponentially weighted moving average (EWMA) and cumulative sum (Cusum) of Q charts.

Extensive research in recent years has developed Variable Sample Rate (VSR) control charts that vary the sampling rate as a function of current and prior sample results. The advantage of using a VSR chart instead of a Fixed Sampling Rate (FSR) chart is that a VSR chart provides much faster detection of small and moderate process changes, for a given IC ARL and a given IC average sampling rate. There are several approaches that can be used to vary the sampling rate. One approach is a Variable Sampling Intervals (VSI) chart that varies the sampling intervals as a function of the sample results from the process. Another approach to varying the sample rate is a Variable Sample Size (VSS) chart that varies the sample size as a function of the sample results from the process. The VSI and VSS features can be combined to give a Variable Sample Sizes and Sampling Intervals (VSSI) control chart that allows the sample size and sampling interval to vary. There have been lots of research on conventional control chart using VSR features in literature, for $\overline{X}$ control chart, see Costa (1998), Lin and Chou (2005a) and Chen and Chiou (2005); for Cusum control chart, see Reynolds *et al* (1990), Zhang and Wu (2007) and Wu *et al* (2007); for EWMA control chart, see Saccucci *et al* (1992), Reynolds (1996) and Reynolds and Arnold (2001); for acceptance control chart, see Wu (1998); for non-central chi-square statistic chart, see Costa and De Magalhães (2007); for multivariate control chart, see Aparisi (1996) and Aparisi and Haro (2001).

Montgomery (2007) shows that one important area of SPC research continues to be the use of adaptive control chart, i.e., with variable sample sizes and/or sampling intervals. Although control chart using VSR features in Phase II analysis has been extensively investigated in the literature, there is little work on adaptive control schemes in Phase I study except Jensen *et al.* (2008), who consider the impact of parameter estimation on adaptive control chart performance. Jensen *et al.* (2008) show that adaptive control charts should only be used for mature processes, where a sufficient amount of Phase I data have been obtained to ensure that the estimated control limits are accurate. The objective of this paper is to fulfill this gap by performing a detailed investigation of Cusum of Q chart with variable sampling intervals (VSICQ) for monitoring process mean shift in which it is desirable to determine the sampling interval for the next sample before sampling is started for this sample. As Zantek (2006) points out, the expected values of the Q statistics are smaller than those of the statistics accumulated in the classic Cusum procedure following a shift, which suggests that smaller reference values being used in the Cusum of Q. That is, the magnitude of the Q statistics is masked to some extent, which greatly hampers the detection ability of the Cusum of Q chart. As there is little work on how much the magnitude of the Q statistics is masked exactly, the design of VSICQ is not at all trivial. We overcome this difficulty by finding the empirical distribution of the Cusum of Q statistics through Monte Carlo simulation. Inheriting the advantage from the Cusum of Q chart that the process parameters do not have to be known, our proposed VSICQ does not suffer the problem of Jensen *et al.* (2008). The proposed chart has the following good features: 1) it can be used in the start up of a process; 2) it does not take much effort in designing; 3) it is quite sensitive to a range of shifts.

Now we summarize some abbreviated expressions used in this paper for easy reference and recapitulation.

- CDF: cumulative distribution function; RMI: relative mean index.
- IC: in-control; OC: out-of-control.
- ARL: average run length; ATS: average time to signal; AATS: the adjusted average time to signal; SSATS: steady-state average time to signal.
- EWMA: exponentially weighted moving average; Cusum: cumulative sum.
- VSR: variable sample rate; FSR: fixed sampling rate; VSI: variable sampling intervals; VSS: Variable Sample Size; VSSI: variable sample sizes and sampling intervals.
- VSICQ: Cusum of Q chart with variable sampling intervals; VSIACQ: adaptive Cusum of Q chart with variable sampling intervals.

The remainder of this paper is organized as follows. In the next section, our proposed VSICQ and its designing strategies are presented. An enhancement of VSICQ with adaptive reference values is also provided. The numerical comparisons with the improved Cusum of Q scheme of Zantek (2006) and the enhanced VSICQ are carried out in Section 3. A real data example from two laboratories carrying out routine indirect assays for precious metals of batches of a feedstock is used to illustrate the application of VSICQ in Section 4. Several remarks conclude this paper in Section 5.

2. The method of VSICQ

In this section, the Cusum of Q scheme of Quesenberry (1995) is briefly introduced. Also our proposed VSICQ and its design are presented. An enhanced VSICQ—adaptive Cusum of Q chart with variable sampling intervals (VSIACQ) is provided, too.

2.1 *The Cusum of Q chart*

A brief review of the Cusum of Q chart is introduced in this section.

Let $X_1, X_2, \ldots, X_i, \ldots$ be a sequence of i.i.d. normal variables with mean μ and variance σ^2, where neither μ nor σ^2 is known, whether the process is in control or not. From simulation results (not shown here), we find that with the same magnitude of the mean shift, if the variance has upper-sided change, our proposed control chart would signal more quickly and if the variance has lower-sided change, our proposed control chart would signal more slowly. That means the performance of our proposed control chart is affected by the variance shifts. As our focus in this paper is on monitoring the change of process mean, we assume that the unknown σ^2 remains unchanged. However, it is not difficult to generalize the idea to detecting process variability shift if one uses the statistics of Equation (9) in Quesenberry (1991). Further we assume that the process mean shift pattern is $\mu + \delta\sigma$, where δ is neither known nor fixed.

Following Quesenberry (1991), denote

$$T_i = a_i(X_i - \bar{X}_{i-1})/\hat{\sigma}_{i-1}, \qquad i = 3, 4, \ldots$$

where

$$a_i = \sqrt{(i-1)/i}, \quad \bar{X}_i = \frac{1}{i}\sum_{k=1}^{i} X_k, \quad \hat{\sigma}_i^2 = \frac{1}{i-1}\sum_{k=1}^{i}(X_k - \bar{X}_i)^2.$$

Let $G_i(\cdot)$ denote the cumulative distribution function (CDF) of a Student t random variable with i degrees of freedom and $\Phi^{-1}(\cdot)$ denote the inverse of the standard Normal CDF. The Q statistics are defined by

$$Q_i = \Phi^{-1}(G_{i-2}(T_i)), \qquad i = 3, 4, \ldots$$

Quesenberry (1991) has shown that Q_is are i.i.d. standard Normal random variables when the process is in control. As Quesenberry (1995), the Cusum of Q statistic designed to detect an increase in the process mean is defined by

$$S_i^+ = \max\{0, S_{i-1}^+ + Q_i - k\}, \qquad i = 3, 4, \ldots$$

where $S_2^+ = 0$, and k is the reference value. It signals when $S_i^+ > h$ for the first i, where h is the control limit to specify an IC ARL. Similarly, the Cusum of Q statistic designed to detect a decrease in the process mean is defined by

$$S_2^- = 0, \quad S_i^- = \min\{0, S_{i-1}^- + Q_i + k\}, \qquad i = 3, 4, \ldots.$$

It signals when $S_i^- < -h$ for the first i. Here, we focus on illustrating the implementation of the upward control chart and the implementation of the downward control chart is similar. In Quesenberry (1995), the shifts δ are in the mean of the distribution of the $Q_i's$, but not in the process observations $X_i's$. Zantek (2006) points out the expected values of the Q statistics are smaller than those of the statistics accumulated in the classic Cusum procedure following a shift and obtains the reference value with golden section search method for specified shift size. The detecting performance is improved relative to Quesenberry (1995) but still not quite satisfactory. To enhance the detection ability of Cusum of Q chart, we will propose a Cusum of Q chart with variable sampling interval in the next section. As the assumption that the shifts δ are in the mean of the distribution of the $Q_i's$ is not proper in practice, we assume it is in the process observations $X_i's$ in the following of this paper.

2.2 *VSICQ*

The sampling scheme of the VSICQ is to use longer sampling interval as long as the sample point is close to the target so that there is no indication of process change. However, if the sampling point is far from the target, but still within the action limits so that there is some indication of process shift, then shorter sampling interval is used. Like conventional Cusum of Q chart, if a sample point falls in the action region, then the process is considered to be out of control.

Assume that in the conventional Cusum of Q chart, the fixed sampling interval is d_0. In general, in our VSICQ, the sampling interval function $d(\cdot)$ can be of any form, but previous research on VSI control charts has shown that it is sufficient to use only two possible values for the sampling intervals to achieve good statistical properties in VSI control chart, see, for example, Costa (1998), Wu *et al* (2007) and Reynolds and Arnold (2001). Let d_1 and d_2 represent these two possible sampling intervals, where $0 < d_1 < d_2$. Then the sampling interval function $d(\cdot)$ can be defined by partitioning C, the continuation or IC region, into two regions—warning region and center region, say R_w and R_c, such that

$$d(\cdot) = \begin{cases} d_1, \text{ if the monitor statistic falls into } R_w, \\ d_2, \text{ if the monitor statistic falls into } R_c. \end{cases}$$

It is advisable to start the control with the shorter sampling interval, d_1, so the first sample is taken quickly after the process is started in case of start-up problems.

Then the VSICQ can use warning limit ω and control limit h to divide the chart into the central region $R_c = (0, \omega)$, warning region $R_w = [\omega, h)$ and action region $[h, +\infty)$. To facilitate the derivation of ω, define p_0 as the conditional probability of a sample point S_i^+ falling in the central region given that this point does not fall in the action region, i.e.,

$$p_0 = P[S_i^+ < \omega | S_i^+ < h].$$

A large value of p_0 indicates that the value ω is close to h, and a large number of samples is taken using the long sampling interval d_2. Due to the intricacy of the distribution of S_i^+, we can only find the warning limit ω corresponding to different p_0 through Monte Carlo simulation. As expected, ω is related to the reference value k and the control limit h besides the conditional probability p_0. Given these, a Fortran program to find corresponding ω is available from the authors upon request.

2.3 *The design of VSICQ*

Traditionally, the ARL has been generally employed as a performance indicator to evaluate the effectiveness of various control schemes, provided that the sampling interval remains constant. However, when the sampling interval is variable, the time to signal is not a constant multiple of the ARL, and thus ARL is not appropriate for evaluating the effectiveness of VSI control charts. The widely used performance indicators for adaptive control charts are the average time to signal (ATS), which is defined as the expected value of time from the start of the process to the time when the charts indicate an OC signal, and the adjusted average time to signal (AATS), which is defined as the expected value of time from the occurrence of an assignable cause to the time when the charts indicate an OC signal. The AATS is also called the steady-state ATS (SSATS).

When the process is in control, the ATS may be used to develop the measures of the false alarm rate for a chart. A chart with a larger IC ATS indicates a lower false alarm rate than other charts. When the process is out of control, the AATS may be used to measure the performance of a chart. A chart with a smaller OC AATS indicates a better detection ability of process shifts than other charts. To make the Cusum of Q chart with and without VSI comparable, the same IC average sample rate is used, i.e.,

$$(1 - p_0)d_1 + p_0 d_2 = d_0. \tag{1}$$

The performance of VSICQ is related to determination of the following parameters: the warning limit ω, the sampling interval d_1 and d_2. Of course, the control limit h and the reference value k involved in computing the statistics S_i^+ should be specified first to give a given IC ATS. They can be determined by golden section search in conjunction with Monte Carlo simulation according to Zantek (2006).

A thorough study of the effect of different shift size δ, different shift position v, different reference value k and different IC ATS is deferred in the next section.

In this paper, it is assumed that $d_0 = 1$ without loss of generality. Otherwise, the results can be obtained by multiplying d_0. Zantek (2006) shows that Cusum of Q chart with $k = 0.125, h = 12.0842$ gives an overall one-sided IC ARL of roughly 740. Thus the ATS is 740 under the assumption that $d_0 = 1$.

To facilitate the determination of ω, the conditional probability p_0, which can be considered as the proportion of samples taken using the longer sampling interval d_2 when the process is in control, needs to be specified. Figure 1 provides the ATS and AATS for several VSICQ with various p_0 when the process mean undergoes different shifts, which are in unit of the standard deviation throughout this paper. These are on a log scale for clearer comparison. In Figure 1, $d_1 = 0.1$ and d_2 computed from (1) are used. Figure 1 shows that the VSICQ with smaller p_0 value has smaller OC AATS when the IC ATS is approximately 740, which results in quicker detection of the process shift. It seems that p_0 should be as small as possible from statistical point of view. However, too small a p_0 gives too large a d_2 for fixed d_1, which implies that once S_i^+ falls in R_c, too long a sampling interval will be used. This is not realistic in practice. For $\overline{X}$ control chart, Reynolds (1996) and Lin and Chou (2005a) suggest that the p_0 value should not be too small. Moreover, from the results (not shown here) of the Fortran program (available from the authors), too small a p_0 results in too small an ω, which is not convenient for practitioners to use. Further note that there is little gain from $p_0 = 0.4$ to $p_0 = 0.5$. Therefore, the p_0 value may be in the vicinity of 0.5 for VSICQ.

Insert Figure 1 about here.

When VSICQ is used, the sampling intervals may vary. The detection ability depends on the sampling intervals d_1 and d_2. Figure 2 provides the ATS and AATS (on a log scale) for several VSICQ with various d_1 and d_2 computed from (1). In Figure 2, $p_0 = 0.5$ is used based on the discussions above. Figure 2 shows that the VSICQ with smaller d_1 value has smaller OC AATS when the IC ATS is approximately 740, which results in quicker detection of the process shift. It seems that d_1 should be as small as possible from statistical point of view. However, d_1 depends on the shortest time required to sample each item in practice. Thus, d_1 may be the shortest time to sample each item.

Insert Figure 2 about here.

For a particular application, the steps of designing VSICQ may be proposed as follows.

(1) Set the IC ATS according to the administrative consideration and determine the value of h and k with golden section search method.
(2) Choose the conditional probability p_0. It may be in the vicinity of 0.5 if there is no special requirement based on the findings above.
(3) Choose the warning limit ω based on an application a sequence of trial and error selections. A Fortran program to find the warning limit is available from the authors upon request.
(4) Determine the shorter sampling interval d_1. It may be the shortest time to sample each item, which depends on the practical consideration of how quick it is possible to sample again after the samples have been taken. Then d_2 is computed from (1).

Although the application of our proposed control chart in practice will generally require the use of a computer, we do not feel that it is a major consideration today because many manufacturing facilities have been well stocked with computers and moreover, once these parameters are chosen, they are not changed during the monitoring process. Considering the great time reduction (from simulation results in next section) in detecting process shifts, we believe it is worth taking the effort in designing the proposed control chart and employing the variable sampling schemes. Moreover, it is better to show what would be gained and at what expense in time and complexity. There are lots of papers studying this issue from economic models, so we do not study this in detail to save space. Interested practitioners are referred to Chen and Chiou (2005), among others.

2.4 *Enhanced VSICQ—VSIACQ*

Zantek (2006) obtains the reference value with golden section search method for specified shift size. In practice, however, the assumption that the true magnitude of future mean shift is known is not realistic. So it has to be estimated with some sample based methods. As Sparks (2000), Shu and Jiang (2006) and Li and Wang (2008), we use the EWMA scheme with a reflecting boundary to obtain the estimator, $\hat{\delta}_i$, of the current mean level at time i, that is,

$$\hat{\delta}_i = \max\{\hat{\delta}_{min}, (1-\lambda)\hat{\delta}_{i-1} + \lambda Q_i\}, \qquad i = 3, 4, \ldots$$

where $0 < \lambda \leq 1$ is the smoothing parameter and $\hat{\delta}_2 = \hat{\delta}_{min}$. Thus, the plotted statistics of the VSICQ chart are revised by

$$AS_i^+ = \max\{0, AS_{i-1}^+ + (Q_i - \hat{\delta}_i/2)/f(\hat{\delta}_i/2)\}, \qquad i = 3, 4, \ldots$$

where $f(k)$ is an operating function which depends on the specified IC ARL of Cusum chart with reference value k and AS_2^+ is set to 0. The operating function $f(k)$ used in this paper is

$$f(k) \approx \frac{\ln(1 + 2k^2 \cdot ARL_0 + 2.332k)}{2k} - 1.166$$

with ARL_0 the IC ARL, which is introduced in Shu and Jiang (2006). We call this revised VSICQ chart adaptive Cusum of Q chart with variable sampling intervals (VSIACQ). Two additional control chart parameters in VSIACQ are $\hat{\delta}_{min}$ and λ. Following the guidelines for choosing parameters in Li and Wang (2008), we set $\hat{\delta}_{min} = 0.25$ and $\lambda = 0.1$ because the minimum potential mean shift in which we are interested is 0.5 in this paper. Interested practitioners are referred to Li and Wang (2008), so we do not study the effect of parameters on the performance of VSIACQ in detail in this paper to save space. With VSIACQ, a practitioner can detect a range of unknown mean shifts, which is more flexible in a real industrial process. Furthermore, because $\hat{\delta}_i$ is changed as each new sample is collected, a practitioner can have a more accurate and timely estimator of the current mean level, which makes the VSIACQ nearly optimal for each sample step. The comparison results in next section with fixed reference values imply this result.

3. Comparisons

In this section, a comparative study is conducted by Monte Carlo simulation to evaluate the performance of VSICQ and VSIACQ. The simulations are sufficiently long, 10,000 replications, such that the standard errors of the estimates are less than 2%, enabling us to draw reasonable conclusions. The IC ATS of each chart is set to be equal, and so is the IC average sample rate, such that the comparison can be conducted under the same criteria.

3.1 *Comparison of VSICQ and CUSUM of Q charts with fixed reference value*

Because there is no corresponding work on adaptive control chart in self-starting analysis, we compare AATS performance of our VSICQ with that of conventional Cusum of Q charts with fixed reference value k=0.125, 0.250 and 0.375, respectively. For IC ATS of 335.4 and 740 (one-sided), the control limits of Cusum of Q chart given by Zantek (2006) are

$$ATS = 335.4 : k = 0.125, h = 9.4302; k = 0.250, h = 6.5275; k = 0.375, h = 4.9792;$$

$$ATS = 740 : k = 0.125, h = 12.0842; k = 0.250, h = 8.0092; k = 0.375, h = 5.9965,$$

respectively. The parameters of our proposed VSICQ chart are $p_0 = 0.5, d_1 = 0.1$ based on discussions above. Five mean shift settings are considered: i.e., mean shift starting from $v = 11, 21, 31, 41$ and 101 observation. The resulting AATS for detecting the mean shift in the range [0,2] are presented in Table 1. Note that the column labelled "CQ" and "VSI" are the AATS of the Cusum of Q chart and the VSICQ, respectively; the column labelled "R" is the percentage gain of the VSICQ to Cusum of Q, which is defined as

$$R = \frac{AATS_{CQ} - AATS_{VSI}}{AATS_{CQ}} \times 100.$$

Insert Table 1 about here.

From Table 1, several general conclusions can be made.

- It is clear that adding the VSI feature to the Cusum of Q chart substantially improves the efficiency of the chart. The percentage gain is more than 50% in most of the cases studied here, the largest gain is 79.5%, and the smallest gain is 8.0%. Note that a process at start-up may encounter problems due to instability and if the assignable cause is not identified in time, practitioners will encounter cost by OC products. As our VSICQ can make the time to find the problem greatly reduced, the cost will be reduced accordingly.
- The effect of shift size δ: The percentage gain becomes larger as δ gets larger. This implies that adding VSI feature to Cusum of Q chart has more benefit for larger process shifts. The reason is that when a process has a larger process shift, the shorter sampling interval d_1 is used most of the time. This is different from other control charts, such as $\overline{X}$, Cusum and EWMA control chart, where adding VSI feature is more effective for small to moderate process shifts.
- The effect of shift position v: For small shift, $\delta = 0.5$ and $\delta = 1.0$, the percentage gain gets larger as v grows larger. The reason is that one needs more IC samples to have a more accurate estimate of the process mean. However, for large shift, $\delta = 2.0$, the percentage gain gets smaller as v grows larger. The reason is that once a large mean shift occurs at the start-up of a process, the VSICQ can detect this shift quickly, but if this large shift occurs at the steady stage of a process, this large shift will be masked a lot so that it is relatively difficult for

VSICQ to detect. For moderate shift, $\delta = 1.5$, the results show different pictures for different k and IC ATS.

- The effect of reference value k: The percentage gain becomes smaller as k gets larger. The reason is that when larger k is used, more of Q_i is shrunk to zero from the definition of S_i^+, which makes the mask of the magnitude of Q_i statistics more severe.
- The effect of IC ATS: The results show different pictures for different δ, v and k. Generally speaking, the AATS for different IC ATS depends mainly on different δ. When $\delta = 0.50$, the percentage gain is large for IC ATS=335.4; when $\delta = 1.50$ and $\delta = 2.00$, the percentage gain is large for IC ATS=740; when $\delta = 1.00$, either case has larger gain.

3.2 *Comparison of VSICQ and VSIACQ*

In order to evaluate the robustness of a control chart to various magnitudes of mean shifts, we use the relative mean index (RMI), which is given by Han and Tsung (2006). The RMI of a control chart C is denoted by

$$RMI(C) = \frac{1}{n}\sum_{i=1}^{n} \frac{AATS_{\delta_i}(C) - AATS^*_{\delta_i}}{AATS^*_{\delta_i}},$$

where $AATS_{\delta_i}(C)$ is the OC AATS of the control chart C for mean shift size δ_i and $AATS^*_{\delta_i}$ is the smallest OC AATS among all the considered chars for mean shift size δ_i. Table 2 shows the OC AATS of VSICQ and VSIACQ when IC ATS=740. The results for IC ATS=335.4 are similar, thus omitted here.

Insert Table 2 about here.

Note in Table 2 that

$$RI = (AATS(k_i) - \min_{1\le i\le 4}\{AATS(k_i)\})/\min_{1\le i\le 4}\{AATS(k_i)\}$$

with

$$k_1 = 0.125, k_2 = 0.250, k_3 = 0.375, k_4 = adaptive$$

and

$$RMI = \sum_{i=1}^{4} RI(\delta_i)/4$$

with

$$\delta_1 = 0.5, \delta_2 = 1.0, \delta_3 = 1.5, \delta_4 = 2.0.$$

It is obvious that a control chart with smaller RMI is more robust.

From Table 2, we can observe that a control chart with adaptive reference value indeed has an overall good performance. Concretely speaking, for small shift time, such as $v = 11$, the performance of VSIACQ is the most robust in terms of RMI among the four control charts considered here. Note that VSIACQ is a self-starting control chart, which is usually used when the process may have start-up problems. So the performance gain by finding adaptive reference values is in support of the implementation of VSIACQ. When the shift time gets larger, the RMI of VSIACQ is comparable with VSICQ with $k = 0.250$, no better than VSICQ with $k = 0.125$ and much better than VSICQ with $k = 0.375$. Moreover, the performance gain relative to VSICQ with $k = 0.375$ outweighs the performance loss relative to VSICQ with $k = 0.125$. So it is desirable for a practitioner to use adaptive reference values when he focuses on a range of unknown mean shifts.

4. A real data example

In this section, our proposed VSICQ chart is illustrated by an example of two laboratories carrying out routine indirect assays for precious metals of batches of a feedstock. It is difficult to carry out such routine assays to the level of accuracy and precision required. So this process is a start up process in a real life situation in which our prosed VSICQ chart is particularly appropriate. Interested practitioners in the data from the two laboratories are referred to Table 3 in Hawkins (1987), so the data sets are not presented here. This example is used to illustrate the self-starting Cusum charts proposed by Hawkins (1987), who shows that the process variance has not gone outside the control limit of the Cusum at any stage of operation, i.e. there has been no significant bias on the process variance. Thus, the data sets satisfy the assumption in this paper that the process variance is constant so that we can use these data sets to illustrate the implementation of the VSICQ chart. Because the performance of our proposed control chart is affected by the variance shift, the result of our proposed control chart may be misleading if the assumption that the variance is constant is violated. Because the objective of this paper is to construct a control chart to monitor mean shifts of a process, we do not study the variance shifts in detail. However, if monitoring variance shift is a concern for some practitioners, they can use another control chart based on the statistics of Equation (9) in Quesenberry (1991) to guard against variance shift.

Although in the previous section, the comparisons between our proposed VSICQ chart and three conventional Cusum of Q charts with fixed reference values are carried out, we incorporate some results in Hawkins (1987) to better illustrate the efficiency of our method. As discussed before, the parameters of our proposed VSICQ chart are $p_0 = 0.5, d_1 = 0.1$. For this VSICQ chart the control limit is $h = 6$ and the IC ATS is about 100, which is the same as the self-starting Cusum chart employed by Hawkins (1987), in which the control limit is $h = 6$ and the reference value is $k = 0.25$.

The monitoring statistics are presented in Table 3. Note that the column labelled "No." represents the index of the observations, the columns labelled "I", "S", and "H" represent the variable sampling interval of VSICQ chart, the monitoring statistics of VSICQ chart and the monitoring statistics of self-starting method of Hawkins (1987), respectively. Also note that the notations $^{\triangle}$, $^{\triangle\triangle}$, *, and ** over the values represents, respectively, that the corresponding value is not obtained at this observation, the corresponding value need not be shown because the process has gone out of control, the corresponding value is the first point in that excursion whose monitoring statistic is nonzero and the corresponding value is the point that gives an OC signal.

Insert Table 3 about here.

From Table 3, we can safely get some conclusions. For Laboratory 1 data, the statistic of our VSICQ chart and the self-starting method of Hawkins (1987) give an OC signal at the 29th and 30th observation, respectively. From the signal point, we read backwards the statistics until it is zero; then we can find out that both statistics show that the 16th observation is the last point whose value is nonzero. That is to say, both statistics indicate that the process has gone out-of-control from the 16th observation but our VSICQ signals one observation earlier. Moreover, note the time to signal of our VSICQ is 24.5, which is the sum of the column labelled "I". We can detect the shift of this process more quickly if we use variable sampling intervals. For Laboratory 2 data, both the statistic of our VSICQ chart and the self-starting method of Hawkins (1987) imply that the process has a mean shift from the 23th observation but our VSICQ gives a signal two observations earlier. That is to say, the monitor statistic of our VSICQ chart has better performance than that of Hawkins (1987) in terms of run length. Note that the time to signal of our VSICQ is 35.3. The fact that the time to signal is longer than the run length implies that the process is in-control most of the time, which can be seen from the monitoring statistics, so we use the long interval d_2 more than d_1.

5. Conclusion and extension

In this paper, we propose an adaptive Cusum of Q chart with variable sampling intervals for monitoring the process mean, which can greatly reduce the time to identify process variations compared with some competing methods. The proposed charts have the following good features: 1) it can be used in the start up of a process, which is desirable in practice, such as job-shop environment, in which production is low-volume and there is often a scarcity of relevant data available for estimating the process parameters and establishing control limits prior to a production run; 2) the effort to design the proposed control chart is not as great because no additional parameter is involved except for the conditional probability p_0 and shorter sampling interval d_1; 3)the tradition Cusum is only optimal for fixed mean shifts, but the VSIACQ chart is quite robust and sensitive to various types of shifts or a range of shifts from the index of RMI. Meanwhile, the proposed chart has some disadvantages: 1) the sampling intervals specified by the chart may not correspond to the natural periods in the process, such as work shifts for plant personnel. This disadvantage may not be particularly important in many applications, but for situations in which it is important, a modification of the VSI idea could be used, such as Reynolds (1996) and Lin and Chou (2005a). 2) the implementation of the proposed control chart will generally require the use of a computer. We feel that this is a minor consideration today because many manufacturing facilities have been well stocked with computers and moreover, a Fortran program to find the control chart parameters is available from the authors upon request. The effect of different conditional probability p_0, different sampling interval d_1 and d_2, different shift size δ, different shift position v, different reference value k and different IC ATS is studied thoroughly. An enhancement of our VSICQ is proposed as VSIACQ by finding adaptive reference values to detect a range of unknown mean shifts. In this paper, we only consider detecting mean shifts, it is not difficult to generalize the idea of this paper to detecting variance shifts.

The powerful properties of VSICQ have been developed in this paper under the assumption that the observations from the process are normally distributed. For some processes, this assumption may not be realistic. The VSI feature could of course be used with non-normal observations, but in this case the properties and design strategies would need to be developed under a model which allows for non-normality. Lin and Chou (2005b) study the design of VSS and VSI $\overline{X}$ charts under non-normality based on Burr distribution. This warrants further research.

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Table 1. AATS of Cusum of Q chart with and without VSI

k	v	δ											
		0.50			1.00			1.50			2.00		
		CQ	VSI	R	CQ	VSI	R	CQ	VSI	R	CQ	VSI	R
IC ATS 740													
0.125	11	439.6	365.1	16.9	146.7	78.2	46.7	34.2	9.0	73.7	15.6	3.2	79.5
	21	237.5	177.5	25.3	30.7	8.9	71.0	12.8	3.2	75.0	9.2	2.4	73.9
	31	134.6	86.5	35.7	19.0	4.8	74.7	10.6	2.8	73.6	7.7	2.1	72.7
	41	94.9	49.4	47.9	16.1	4.3	73.3	9.7	2.6	73.2	7.1	2.0	71.8
	101	36.0	11.5	68.1	13.2	3.6	72.7	8.3	2.3	72.3	6.2	1.8	71.0
0.250	11	485.7	430.5	11.4	206.7	132.5	35.9	42.4	20.2	52.4	12.0	2.9	75.8
	21	306.5	267.0	12.9	43.2	20.3	53.0	9.8	2.6	73.5	6.4	1.7	73.4
	31	206.6	159.0	23.0	18.7	6.0	67.9	8.0	2.1	73.8	5.7	1.6	71.9
	41	153.7	105.4	31.4	13.9	3.7	73.4	7.4	2.0	73.0	5.3	1.6	69.8
	101	47.7	20.2	57.7	11.0	3.1	71.8	6.6	1.9	71.2	4.8	1.5	68.8
0.375	11	536.8	493.6	8.0	278.0	221.8	20.2	91.8	45.7	50.2	17.9	6.3	64.8
	21	380.6	347.1	8.8	78.6	53.0	32.6	10.4	2.8	73.1	5.4	1.5	72.2
	31	292.7	247.1	15.6	30.8	15.3	50.3	7.2	1.9	73.6	4.8	1.5	68.8
	41	224.2	184.9	17.5	17.1	7.3	57.3	6.6	1.9	71.2	4.5	1.4	68.9
	101	77.2	49.3	36.1	10.6	3.0	71.7	5.8	1.8	69.0	4.1	1.4	65.9
IC ATS 335.4													
0.125	11	163.5	129.1	21.0	47.4	21.1	55.5	15.4	4.2	72.7	9.6	2.5	74.0
	21	87.3	56.9	34.8	16.8	5.0	70.2	8.9	2.6	70.8	6.5	2.0	69.2
	31	55.9	31.4	43.8	12.7	3.7	70.9	7.6	2.3	69.7	5.6	1.8	67.9
	41	42.9	20.8	51.5	11.7	3.5	70.1	7.2	2.2	69.4	5.3	1.7	67.9
	101	25.1	9.3	62.9	10.1	3.2	68.3	6.5	2.1	67.7	4.8	1.6	66.7
0.250	11	186.0	154.4	17.0	66.9	37.2	44.4	17.0	4.5	73.5	7.1	1.8	74.6
	21	111.7	85.2	23.7	18.5	6.8	63.2	7.1	2.0	71.8	4.9	1.5	69.4
	31	80.5	54.2	32.7	11.9	3.7	68.9	6.2	1.9	69.4	4.5	1.5	66.7
	41	58.1	36.2	37.7	10.4	3.2	69.2	5.9	1.9	67.8	4.3	1.5	65.1
	101	28.6	13.3	53.5	8.9	2.8	68.5	5.4	1.8	66.7	3.9	1.4	64.1
0.375	11	206.4	188.5	8.7	88.9	63.2	28.9	25.1	11.3	55.0	7.6	2.2	71.1
	21	141.5	117.3	17.1	27.4	13.6	50.4	6.8	2.1	69.1	4.3	1.4	67.4
	31	105.9	82.0	22.6	14.3	6.0	58.0	5.7	1.8	68.4	3.9	1.4	64.1
	41	79.2	60.2	24.0	11.0	3.8	65.5	5.3	1.7	67.9	3.7	1.4	62.2
	101	37.5	20.7	44.8	8.6	2.8	67.4	4.8	1.7	64.6	3.5	1.4	60.0

Figure caption list:

- Figure 1. log(AATS) for $p_0 = 0.4(0.1)0.9, d_1 = 0.1$.
- Figure 2. log(AATS) for $d_1 = 0.1(0.1)0.6, p_0 = 0.5$.

Table 2. AATS of VSICQ with fixed k and adaptive k when IC ATS=740

v	δ	k = 0.125 AATS	0.125 RI	0.250 AATS	0.250 RI	0.375 AATS	0.375 RI	adaptive AATS	adaptive RI
11	0.5	365.1	0.118	430.5	0.318	493.6	0.511	326.7	0
	1.0	78.2	0.310	132.5	1.219	221.8	2.715	59.7	0
	1.5	9.0	0.098	20.2	1.463	45.7	4.573	8.2	0
	2.0	3.2	0.103	2.9	0	6.3	1.172	3.9	0.345
	RMI		0.157		0.750		2.243		0.086
21	0.5	177.5	0.165	267.0	0.753	347.1	1.279	152.3	0
	1.0	8.9	0	20.3	1.281	53.0	2.800	10.1	0.135
	1.5	3.2	0.231	2.6	0	2.8	0.077	3.8	0.462
	2.0	2.4	0.600	1.7	0.133	1.5	0	2.8	0.867
	RMI		0.249		0.542		1.039		0.366
31	0.5	86.5	0.230	159.0	1.262	247.1	2.515	70.3	0
	1.0	4.8	0	6.0	0.250	15.3	2.800	5.7	0.188
	1.5	2.8	0.474	2.1	0.105	1.9	0	3.2	0.684
	2.0	2.1	0.400	1.6	0.067	1.5	0	2.4	0.600
	RMI		0.276		0.421		1.329		0.368
41	0.5	49.4	0.235	105.4	1.635	184.9	3.623	40.0	0
	1.0	4.3	0.162	3.7	0	7.3	2.800	5.0	0.351
	1.5	2.6	0.368	2.0	0.053	1.9	0	3.0	0.579
	2.0	2.0	0.429	1.6	0.143	1.4	0	2.2	0.571
	RMI		0.299		0.458		1.606		0.375
101	0.5	11.5	0	20.2	0.757	49.3	3.287	11.7	0.017
	1.0	3.6	0.200	3.1	0.033	3.0	0	4.1	0.367
	1.5	2.3	0.278	1.9	0.056	1.8	0	2.5	0.389
	2.0	1.8	0.286	1.5	0.071	1.4	0	1.9	0.357
	RMI		0.191		0.229		0.822		0.283

Table 3. The monitoring statistics for Lab. 1 & Lab. 2

No.	Lab. 1 I	Lab. 1 S	Lab. 1 H	Lab. 2 I	Lab. 2 S	Lab. 2 H
1	0.1	△	0.00	0.1	△	0.00
2	0.1	△	0.00	0.1	△	0.00
3	0.1	0.00	0.00	0.1	0.00	0.00
4	1.9	0.00	0.00	1.9	0.00	0.00
5	1.9	0.00	0.00	1.9	0.39	0.27
6	1.9	0.00	0.00	1.9	0.14	0.00
7	1.9	0.03	0.00	1.9	0.89	0.62
8	1.9	0.00	0.00	1.9	0.32	0.00
9	1.9	1.26	1.13	1.9	0.00	0.00
10	0.1	1.81	1.56	1.9	0.00	0.00
11	0.1	0.51	0.13	1.9	2.19	2.06
12	1.9	0.21	0.00	0.1	1.33	1.07
13	1.9	0.00	0.00	0.1	0.09	0.00
14	1.9	0.00	0.00	1.9	0.64	0.43
15	1.9	0.00	0.00	1.9	0.44	0.10
16	1.9	0.47*	0.34*	1.9	0.31	0.00
17	1.9	1.31	1.06	1.9	0.00	0.00
18	0.1	2.19	1.81	1.9	0.00	0.00
19	0.1	1.92	1.43	1.9	0.00	0.00
20	0.1	2.42	1.80	1.9	0.00	0.00
21	0.1	2.38	1.63	1.9	0.00	0.00
22	0.1	2.01	1.14	1.9	0.00	0.00
23	0.1	2.14	1.15	1.9	2.22*	2.09*
24	0.1	1.98	0.86	0.1	1.85	1.60
25	0.1	3.27	2.02	0.1	1.08	0.70
26	0.1	5.05	3.68	0.1	1.57	1.07
27	0.1	5.87	4.38	0.1	2.91	2.29
28	0.1	5.70	4.08	0.1	3.80	3.06
29	0.1	6.15**	4.41	0.1	6.84**	5.81
30	△△	△△	6.48**	△△	△△	5.70
31	△△	△△	△△	△△	△△	6.02**