International Journal of Production Research
Vol. 00, No. 00, 15 March 2008, 1–13

Adaptive CUSUM of Q Chart

Zhonghua Li and Zhaojun Wang*

LPMC and School of Mathematical Sciences, Nankai University, Tianjin 300071, China

(*v3.3 released February 2008*)

The CUSUM of Q chart, as a self-starting approach, was previously proposed for detecting process shifts when the in-control process parameters are unknown. Motivated by recent development of the adaptive chart, this paper proposes a new self-starting approach, which integrates the CUSUM of Q chart with the feature of adaptively varying the reference value, to better detect a range of shifts with unknown process parameters. The choice of the chart parameters and the masking effect are also studied. The simulation results show that our proposed chart offers a balanced protection against shifts of different magnitudes, and has comparable performance with the dynamic change-point control scheme. A real example from industrial manufacturing is used for demonstrating its implementation.

Keywords: Markov modeling; Masking effect; Quality Control; Self-starting; SPC.

1. Introduction

Statistical process control charts are widely used in industry for monitoring the quality of manufactured products. In Phase II analysis, the process parameters are usually assumed known. However, in practice the process parameters are usually unknown in the early stages of process improvement, and they are usually estimated by using in-control (IC) historical samples (or by the Phase I study). When the number of historical samples is small, control charts with estimated parameters generally produce a large bias in the IC average run length (ARL) , and reduce the sensitivity of the chart in detecting the process changes measured by the out-of-control (OC) ARL. Moreover, after short runs, the false alarm probabilities from the charts increase drastically, see Bischak and Trietsch (2007). This is the so-called "control charts with estimated parameters" problem. Many efforts have been devoted to this problem. Among others, Bagshaw and Johnson (1975) study the influence of estimated variance on the ARL of CUSUM tests. Jones *et al* (2004) discuss the run length distribution of the CUSUM charts with estimated parameters and Jones *et al* (2001) and Jones (2002) study the effect of estimation on the performance of the EWMA chart in a variety of practical scenarios. Wu *et al* (2002) consider the effect of several estimations on the performance of Shewhart $\bar{X}$ chart. An extensive discussion about the research problems on this topic can be found in Jensen *et al.* (2006).

In light of the deterioration of control chart performance that results from estimated parameters, practitioners should collect a sample of data large enough to ensure that parameter estimates are sufficiently close to the true parameters. It may not be feasible to wait for the accumulation of enough observations, because

*Corresponding author. Email: zjwang@nankai.edu.cn

ISSN: 1478-6451 print/ISSN 1478-646X online

DOI: 10.1080/1478645YYxxxxxxx
http://www.informaworld.com

the users usually want to monitor and adjust the process in the start-up stages. To tackle this problem, self-starting method is an efficient choice. Hawkins (1987) uses the running mean and standard deviation of all observations made on the process since start-up as substitutes for the unknown true values of the process mean and standard deviation. For more details about self-starting method, see Chapter 7 of Hawkins (1998). Moreover, for start-up process and short or long runs, Quesenberry (1991) presents the important formulas so that charts for both the process mean and variance can be maintained from the start of production, whether or not prior information for estimation the parameters is available. Quesenberry (1995) studies the properties of Q charts for variables and the sensitivity of specially designed EWMA and CUSUM of Q charts. In particular, Hawkins *et al.* (2003), Hawkins and Zamba (2005a,b) propose a change-point model based on the likelihood ratio for on-line monitoring that can also be seen as a self-starting method.

Assume observations X_i are independently and identically distributed (i.i.d.) as normal with mean μ and variance σ^2, and a step shift δ (in units of the standard deviation throughout the paper) is issued at some unknown point during the process. Here we mainly focus on the detection of changes in the process mean. It is well known that if the IC parameters, say μ and σ are known, the CUSUM with reference value $k = \delta/2$ is optimal for detecting this shift (Moustakides (1986)). However, due to the transformation of X_i to Q statistics (Quesenberry (1991)), Zantek (2006) shows that the CUSUM of Q chart with reference value $k = \delta/2$ is not optimal for detecting the shift δ, and the optimal reference values for the CUSUM of Q chart are obtained by using golden section search procedure. However, the reference value is designed for a given shift and is generally difficult to be determined especially in the start-up stages where even the IC parameters are not accurately identified. In the setting of known IC parameters μ and σ, an adaptive CUSUM procedure proposed by Sparks (2000) is designed to be effective for detecting a range of mean shifts, and a Markov chain model for the adaptive CUSUM is further investigated by Shu and Jiang (2006). Dual CUSUM chart with two different reference values is proposed by Zhao *et al* (2005) and used to detect a range of mean shifts.

In many applications, the magnitude of shift is always not known before running a chart. Therefore, it is desirable to find a control chart that performs well over a range of mean shifts. To improve the performance of CUSUM of Q chart for start-up processes and short runs, an adaptive CUSUM of Q (ACQ) chart is proposed in this paper. It has the same advantage of Q charts that the process parameters do not have to be known, which may make it useful for short production run situations. It also does not have the disadvantage of the CUSUM of Q chart that the reference value has to be designed for a given size of shift. The main objective of this paper is to develop a procedure that will perform well over a wide range of process shifts in the mean without collecting a sufficient large sample of IC observations. Without these IC observations, we do not have information of the process parameters, so our ACQ chart is a self-starting control chart when the IC process parameters μ and σ are unknown. This means that it is not necessary to collect a large number of IC samples before the control chart begins, although it is advisable to collect a few preliminary samples. Since the magnitude of the shift in Q_i statistic is no longer that in the original observations X_i, the implementation of ACQ chart seems to be more complex for practitioners. This paper is to give a thorough investigation and fulfill these demands. With our ACQ chart, practitioners can detect a range of shifts without collecting a large number of IC samples.

The remainder of this paper is organized as follows. The next section gives some background knowledge on the CUSUM of Q chart. Then the ACQ chart is proposed and the choice of control chart parameters and masking effect are studied in Section 3. Its numerical performance is investigated in Section 4. In Section 5, our proposed ACQ chart is used to a real example from the industrial manufacturing. Several remarks conclude this paper in Section 6.

2. The CUSUM of Q chart

A brief review of the CUSUM of Q chart is introduced in this section.

Let $X_1, X_2, \ldots, X_i, \ldots$ be a sequence of i.i.d. normal variables with mean μ and variance σ^2, where both μ and σ^2 are unknown, whether the process is in control or not. Following Quesenberry (1991), denote

$$T_i = a_i(X_i - \bar{X}_{i-1})/\hat{\sigma}_{i-1}, \qquad i = 3, 4, \ldots$$

where

$$a_i = \sqrt{(i-1)/i}, \quad \bar{X}_i = \frac{1}{i}\sum_{k=1}^{i} X_k, \quad \hat{\sigma}_i^2 = \frac{1}{i-1}\sum_{k=1}^{i}(X_k - \bar{X}_i)^2.$$

Let $G_i(\cdot)$ denote the cumulative distribution function (CDF) of a Student t random variable with i degrees of freedom and $\Phi^{-1}(\cdot)$ denote the inverse of the standard Normal CDF. The Q statistics are defined by

$$Q_i = \Phi^{-1}(G_{i-2}(T_i)), \qquad i = 3, 4, \ldots$$

Quesenberry (1991) has shown that Q_is are i.i.d. standard Normal random variables when the process is in control. The CUSUM of Q statistic designed to detect an increase in the process mean is defined by

$$S_i^+ = \max\{0, S_{i-1}^+ + Q_i - k\}, \qquad i = 3, 4, \ldots$$

where $S_2^+ = 0$, and k is the reference value. It signals when $S_i > c$ for the first i, where c is the control limit to specify an IC ARL. Similarly, the CUSUM of Q statistic designed to detect a decrease in the process mean is defined by

$$S_2^- = 0, \quad S_i^- = \min\{0, S_{i-1}^- + Q_i + k\}, \qquad i = 3, 4, \ldots.$$

It signals when $S_i^- < -c$ for the first i. Here, we focus on illustrating the implementation of the upward control chart and the implementation of the downward control chart is similar. Zantek (2006) obtains the reference value with golden section search method for specified shift size. To enhance the detection ability to a range of shift sizes, we will propose an adaptive method based on Q_i in the next section.

3. Adaptive CUSUM of Q chart

In this section, we propose the ACQ chart and study the effect of its parameters on the performance. Moreover, the masking effect of the size of the IC sample collected before shift is investigated. At last, the design procedure is given.

3.1 *ACQ chart*

In practice, the true magnitude of future mean shift is seldom known. So it has to be estimated. Following Sparks (2000) and Shu and Jiang (2006), we could use the EWMA scheme with a reflecting boundary as one-step-ahead forecast (OAF) to obtain the estimator, $\hat{\delta}_i$, of the current mean level at time i, that is,

$$\hat{\delta}_i = \max\{\hat{\delta}_{min}, (1-\lambda)\hat{\delta}_{i-1} + \lambda Q_i\}, \qquad i = 3, 4, \dots \tag{1}$$

where $0 < \lambda \leq 1$ is the smoothing parameter, $\hat{\delta}_2 = \hat{\delta}_{min}$. Although different schemes can be used to estimate δ, EWMA is one of the most popular schemes to estimate δ due to its simplicity and efficiency. Conventional EWMA statistics are denoted by

$$\hat{\delta}_i = (1-\lambda)\hat{\delta}_{i-1} + \lambda Q_i,$$

where $0 < \lambda \leq 1$ is a smoothing parameter. In practice, when detecting upward shifts, there is often a minimum magnitude of interest for early detection, $\hat{\delta}_{min} > 0$. To improve the efficiency in detecting shifts larger than $\hat{\delta}_{min}$, an upper EWMA statistic can be derived as (1).

Now, the plotted statistics of the ACQ chart are defined by

$$Z_i = \max\{0, Z_{i-1} + (Q_i - \hat{\delta}_i/2)/h(\hat{\delta}_i/2)\}, \qquad i = 3, 4, \dots \tag{2}$$

where $h(k)$ is an operating function which depends on the specified IC ARL of CUSUM chart with reference value k and Z_2 can be set to 0 or some other values to give a fast initial response (FIR) (Lucas and Crosier (1982)). In this paper, Z_2 is set to 0. A signal is triggered when $Z_i > c$, where c is the control limit to maintain the specified IC ARL. The operating function $h(k)$ used in this paper is

$$h(k) \approx \frac{\ln(1 + 2k^2 \cdot ARL_0 + 2.332k)}{2k} - 1.166$$

with ARL_0 the IC ARL, which is introduced in Shu and Jiang (2006). Note the fact that the operating function $h(k)$ introduced in Shu and Jiang (2006) is based on the assumption that the statistics are i.i.d standard normal variables when the process is in control and Quesenberry (1991) has shown that the Q statistics are i.i.d standard normal variables when the process is in control. We can use the operating function $h(k)$ directly. Practitioners who need a more precise operating function are referred to Sparks (2000).

A control chart is considered adaptive if at least one of its design parameters varies as a function of the process data. Among others, Aparisi (1996), Wu (1998) and Aparisi and Haro (2001) consider control chart with adaptive sample sizes and variable sampling intervals. However, what is adapting to the data in this paper is the estimate $\hat{\delta}_i$ so as to have a good knowledge of the current mean level. Once the current mean level changes, we can have a timely estimate, which makes it possible for us to detect a range of mean shifts. Note that the operating function $h(k)$ is a function of $\hat{\delta}_i$. And as a result the value of $h(\hat{\delta}_i/2)$ varies as well. With adaptive estimate $\hat{\delta}_i$, practitioners can detect a broader range of mean shifts. Also considering that practitioners can monitor and adjust the process even in the start-up stages of a process when the process parameters are unknown, our proposed ACQ chart can make a contribution to the field of SPC.

3.2 *The Parameter choice of ACQ chart*

For the ACQ chart, there are 3 parameters: the smoothing parameter λ, the lower bound parameter $\hat{\delta}_{min}$, and the control limit c.

The selection of the smoothing parameter, λ, is based on the following considerations. First, it is generally accepted that smaller smoothing parameters are most effective in quickly detecting smaller process shifts, while larger smoothing parameters are most effective in quickly detecting larger process shifts (Lucas and Saccucci (1990)). While the EWMA chart is most useful to detect small or moderate shift and slightly worse than other control chart, for example, Shewhart chart, a smaller smoothing parameter seems suitable to make a better use of the advantage of the EWMA chart. Second, when the process undergoes a shift, the statistics, T_i, are no longer distributed as a central Student t, so Q_i's are no longer distributed as a standard Normal, exactly. In this condition, the robustness of a control chart is needed. Lucas and Saccucci (1990), Borror *et al.* (1999) point out that small values of λ are more robust than large ones when the process produces occasional outliers or the IC distribution is not normal. Third, Zantek (2006) shows a disadvantage of the transformation to Q statistics is that the shift size is masked to some extent. That is, if a shift is not quickly detected, it may no longer be detected. Note the inertia of the EWMA, which is a disadvantage of the EWMA scheme, turns out to be an advantage in this self-starting condition, because it allows the shifts, if there exist, change slowly to give some extra time for the ACQ chart to detect. From the definition of EWMA operator, smaller smoothing parameters give more weight to the past observations, which implies a small smoothing parameter in this ACQ chart. Fourth, according to Lucas and Saccucci (1990), the FIR feature is most useful for EWMA schemes with λ less than or equal to 0.25. Considering all of the above, the smoothing parameter λ is taken to be 0.1 and 0.2 in this paper, which is also the consideration of Shu and Jiang (2006).

To specify the lower bound parameter $\hat{\delta}_{min}$, we note that Sparks (2000) shows that the value of $\hat{\delta}_{min}$ improves the detection performance for shifts $\delta \geq \hat{\delta}_{min}$ while reducing the efficiency for shifts $\delta < \hat{\delta}_{min}$. At the same time, note that too small a shift (e.g. $\delta < 0.1$) is difficult to detect for any control chart. Also the objective of this paper is to detect a range of mean shifts, a moderate $\hat{\delta}_{min}$ may be proper to protect against a wide range of mean shift sizes. Therefore, the $\hat{\delta}_{min}$ of 0.25, 0.5 and 0.75 are considered here.

Since Quesenberry (1991) has shown that the Q statistics are i.i.d Normal random variables under IC condition, that is, the IC mean and variance of the process are both known, the two-dimensional Markov chain model developed in Shu and Jiang (2006) can be used to evaluate the IC ARL of the ACQ chart. In Shu and Jiang (2006), the IC region is partitioned within a two-dimensional rectangle [0, c]$\times$[δ_{min}, L] to obtain a discretized Markov chain, where L is a large control limit for the process of Q_t. Then the ARL is obtained by the transition probability of $\{Z_t, Q_t\}$ from state (i, j) to state (k, l). So, the control limits c of ACQ chart are obtained by Markov chain and searching method, which are shown in Table 1.

To study the effect of different combinations of λ and $\hat{\delta}_{min}$ on the performance of ACQ chart, Figure 1 shows the OC ARLs of the ACQ chart with different combinations of λ and $\hat{\delta}_{min}$ when IC ARL=500 and shift position is at 100. For other shift positions, the results are similar, thus omitted here. These are on a log scale for clearer comparison.

From Figure 1 we observe that ACQ chart with small λ and small $\hat{\delta}_{min}$ gives better performance for detecting small shifts, which is consistent with the results in Sparks (2000) and Shu and Jiang (2006). The effect of the lower bound parameter,

$\hat{\delta}_{min}$, on the performance of the ACQ chart is more severe than the smoothing parameter, λ. In other words, different λ have less effect on the performance than different $\hat{\delta}_{min}$. In the rest of this paper, we use $\hat{\delta}_{min} = 0.5$ and $\lambda = 0.1$ for trade-off.

3.3 *The masking effect*

When we are discussing the selection of the smoothing parameter λ, it's necessary to note that the transformation to Q statistics could result in the shift size being masked to some extent. Zantek (2006) suggests to use smaller reference values, for example $k < \delta/2$, for improving the detection performance. This implies that the magnitude of the shift size of the Q statistics is smaller than that of the original observations. To show this masking effect, we compare the performance of the ACQ chart and the adaptive CUSUM chart (Note that when the process mean and variance are known, the ACQ chart becomes adaptive CUSUM chart).

Figure 2 shows the performance (on log scale) of the ACQ chart with shift time at observation 10, 25, 50, 100, 250 and the adaptive CUSUM chart with the process mean $\mu = 0$ and variance $\sigma^2 = 1$ when the IC ARL is designed to be 500. Similar results for other IC ARLs are omitted here.

Note that the ACQ chart is designed with unknown process parameters while the adaptive CUSUM chart is constructed with known parameters. From Figure 2, as expected, the performance of the ACQ chart, whatever the shift position is, is no better than the adaptive CUSUM chart due to the masking effect. However, as the shift position becomes larger, the performance of the ACQ chart becomes better. When the shift position is more than 250, the performances of the ACQ chart and the adaptive CUSUM chart are nearly the same. This implies that if a practitioner needs an ACQ chart good enough to compete with a control chart with true parameters, he should collect IC observations more than 250, not a large number compared with the result of Jones *et al* (2001), which show that for values of $\lambda = 0.2$ and $\lambda = 0.1$, 300 and 400 respective samples of five observations are needed to achieve the desired level of IC performance. This shows the superiority of the ACQ chart again.

It would be more interesting to show the differences of the ACQ chart and the adaptive CUSUM chart when the process parameters are estimated, as would generally have to be the case in real applications. Figure 3 gives the performance (on log scale) of the ACQ chart and adaptive CUSUM chart with estimated parameters when the IC ARL=500 with shift time at observation 10, 100 and 250, respectively. From Figure 3, although the adaptive CUSUM chart with estimated parameters has comparable performance with ACQ chart when the IC observations are more than 100, the former has worse performance than the latter when the IC observations are not so much, such as 10. Note that our ACQ chart is a self-starting control chart which may be mainly used immediately after start-up and in this condition it is more efficient than the adaptive CUSUM with estimated parameters. This is just an advantage of the ACQ chart.

Now we show how to get more precise knowledge of the masking effect of the ACQ chart with different shift positions. For example, we take the shift position to be 10. If one wants to know how much the masking effect on the performance of the ACQ chart when the shift size of the original observation is 1.5, the following procedure is suggested.

(1) From the point 1.5 in the shift-axis, draw a vertical line to the shift-axis in Figure 2 to get a crossing point with the line corresponding to "start=10".

(2) From this crossing point, draw a vertical line to the ARL-axis to get another crossing point with the line corresponding to "ACUSUM".

(3) From the second crossing point, draw a vertical line to the shift-axis. The crossing point with the shift-axis, 0.5, is just the shift size not masked, i.e., a shift size of $1.5 - 0.5 = 1$ is masked in this case.

From the procedure above, a practitioner can note that the masking effect is so severe that he should collect IC observations as many as possible to achieve good enough performance of the ACQ chart.

3.4 *The design of the ACQ chart*

For designing the ACQ chart, we should find a combination of parameters $(\hat{\delta}_{min}, \lambda, c)$ that achieves the specified IC ARL and the chart signals quickly when the process undergoes a mean shift. For $\hat{\delta}_{min} = 0.25(0.25)0.75$, $\lambda = 0.1$ and 0.2, and a wide range of IC ARLs, the desired control limit c is obtained by Markov chain method. The results are listed in Table 1. From the performance of the ACQ chart shown in Figure 1, the following design procedure is recommended.

(1) Specify the IC ARL.
(2) If there is little knowledge of the mean shift size, $\hat{\delta}_{min} = 0.5, \lambda = 0.1$ are suggested, which are also useful choice in practice.
(3) If there is some knowledge of the mean shift size and the shift size is small, small $\hat{\delta}_{min}$ and λ are suggested, and vice versa.
(4) Select control limit c in Table 1 to achieve the desired IC ARL. If this is not shown in Table 1, the interpolation method can be used. Moreover, a Fortran program to find control limit c is available from the authors upon request.

4. Comparisons

In this section, the comparisons between our proposed ACQ chart and CUSUM of Q chart (Quesenberry (1991)), and the chart based on the change point model (Hawkins *et al.* (2003)) are carried out by Monte Carlo simulation based on 10,000 replications, such that the standard errors of the estimates are less than 2%, enabling us to draw reasonable conclusions.

4.1 *Comparison with CUSUM of Q chart with fixed k*

In this section, we compare ARL performance of our ACQ chart with that of conventional CUSUM of Q charts with fixed reference value $k = 0.25$, 0.5 and 1.0, respectively. For IC ARL of 100 and 500 (two-sided), the control limits of CUSUM of Q chart given by Hawkins *et al.* (2003) are

for ARL=100: k=0.25, c=5.69; k=0.5, c=3.51; k=1.0, c=1.87;
for ARL=500: k=0.25, c=8.76; k=0.5, c=5.14; k=1.0, c=2.71.

For our proposed ACQ chart, the control limits $c = 1.147$ and $c = 1.212$ are used to give an IC ARL of 200 and 1000 so that we can make the ACQ chart and the three conventional CUSUM of Q charts comparable.

Two mean-shift settings are considered: one mean shift starting from the 10th observation, while the other one starting from the 100th observation. The resulting ARLs (on log scale) for detecting the mean shift in the range [0,3] are presented in Figure 4 (a)-(d).

From Figure 4, several conclusions can be made.

- As a result of the masking effect, the CUSUM of Q chart with fixed reference value may not perform best at its designed shift size. For example, the CUSUM of Q chart with $k = 0.5$ does not signal the most quickly when the shift size is less than 1 in Figure 4 (a) and (c), although it has the best performance among the three CUSUM of Q chart in Figure 4 (b) and (d). Note that the shift position between the left and right two panels is the difference. This suggests that the more IC sample, the less masking effect on the performance.
- When the IC ARL is 100, shown in Figure 4 (a) and (b), whatever the shift position is, the ACQ chart always has the best performance, although the difference is slight between the CUSUM of Q chart with $k = 0.25$ when the shift size is less than 1.5 and with $k = 0.5$ when the shift size is larger than 2.
- However, when the IC ARL is 500, from Figure 4 (c) we can see that the CUSUM of Q chart with $k = 0.25$ has a little better performance when the shift is about 2 and the one with $k = 0.5$ has a little better performance when the shift size is larger than 2.7. If note that in this case the shift position is only 10 and the CUSUM of Q chart with fixed reference values is designed for a particular shift size, this is not a surprise.
- As expected, among the three conventional CUSUM of Q charts, Figure 4 (d) shows that the CUSUM of Q chart with $k = 0.25$ has best performance when the shift size is less than 1, the one with $k = 0.5$ has best performance when the shift size is in the range [1,1.7] and the one with $k = 1.0$ performs best when the shift size is larger than 1.7. To our delight, the ACQ chart has nearly the same performance with the one that performs best for different shift sizes.

4.2 *Comparison with the change point approach*

The change point approach is proposed by Hawkins *et al.* (2003). The simulation results show this approach is always nearly the best for detecting the mean shifts. Thus, it makes an appropriate benchmark against which to compare the ACQ chart.

The ACQ chart and the change point approach of Hawkins *et al.* (2003) are compared when the IC ARL are made comparable. Note that the ACQ chart in this paper is one-sided while the change point approach is two-sided. Thus, IC ARL of 1000, 400, 200 and 100 for the ACQ chart are considered to compare with IC ARL of 500, 200, 100 and 50 for the change point approach, respectively. As Hawkins *et al.* (2003) and Zantek (2006) point out, the ARL performance will be affected by the size of the IC sample, so we also study the differences of ARLs for different shift positions. The ARL performances are listed in Table 2. The columns labelled "CP" are obtained from the change point approach while those labelled "ACQ" are derived from the ACQ chart. Note that the bolded values in Table 2 are smaller.

From Table 2, we can draw some conclusions as follows:

- As a result of the masking effect, the ACQ chart has worse performance than the change point approach when the shift position is less than 50 for detecting $\delta = 0.25$ and less than 25 for detecting $\delta = 0.5$.
- When $\delta = 2.0$, the ACQ chart performs worse relative to the change point method.
- When the shift position is larger than 50, the ACQ chart detects small to moderate process mean shift ($\delta < 2.0$) more quickly than the change point approach except two conditions, i.e. start=250, $\delta = 1.0$ when the IC ARL is 500 and 200.

Although our proposed ACQ chart does not always perform better than the change point approach, it has its own advantages. First, it is easy to design for ACQ chart in practice. All a practitioner need to do is specify a combination of IC ARL, smoothing parameter λ, lower bound parameter $\hat{\delta}_{min}$ and control limit c, which can be found in Table 1. Once these parameters are chosen, they are not changed during the monitoring process. This is not the case for the change point approach, in which the critical value h_n has to be changed as a new observation enters the monitoring process. Second, it is easy to compute the monitoring statistics for ACQ chart. For the change point approach, because it is needed to calculate the two sample statistic T_{jn} for every possible split point $1 \leq j < n$ to find the $T_{max,n}$, the ever-growing storage requirement for the two samples seems less convenient.

5. A real data example

In this section, our proposed ACQ chart is illustrated by an example of two laboratories carrying out routine indirect assays for precious metals of batches of a feedstock. Interested practitioners in the data from the two laboratories are referred to Table 3 in Hawkins (1987), so the data sets are not presented here. This example is used to illustrate the self-starting CUSUM charts proposed by Hawkins (1987), who shows that the process variance has not gone outside the control limit of the CUSUM at any stage of operation, i.e. there has been no significant bias on the process variance. Thus, the data sets satisfy the assumption in this paper that the process variance is constant so that we can use these data sets to illustrate the implementation of the ACQ chart.

Although in the previous section, the comparisons among our proposed ACQ chart, three conventional CUSUM of Q charts with fixed reference values, and the change point method proposed by Hawkins *et al.* (2003) are carried out, we incorporate some results in Hawkins (1987) to better illustrate the efficiency of our method. As discussed before, the parameters of our proposed ACQ chart are $\delta_{min} = 0.5, \lambda = 0.1$. For this ACQ chart the control limit is $c = 1.137$ and the IC ARL is about 100, which is the same as the self-starting CUSUM chart employed by Hawkins (1987), in which the control limit is $h = 6$ and the reference value is $k = 0.25$.

The monitoring statistics are presented in Table 3. Note that the column labelled "No." represents the index of the observations, the columns labelled "k", "S", and "H" represent the adaptive reference values of ACQ chart, the monitoring statistics of ACQ chart and the monitoring statistics of self-starting method of Hawkins (1987), respectively. Also note that $\ldots^1$ represents that the corresponding value in the entry is not obtained at this observation. $\ldots^2$ implies that the corresponding value in the entry need not be shown for the process has gone out of control. * shows that the corresponding value in the entry is the first point in that excursion whose monitoring statistic is nonzero and ** says that the corresponding value in the entry is the point that gives an OC signal.

From Table 3, we can safely get some conclusions. For Laboratory 1 data, the statistic of our ACQ chart and the self-starting method of Hawkins (1987) give an OC signal at the same time (the 30th observation). From the signal point, we read backwards the statistics until it is zero; then we can find out that both statistics show that the 16th observation is the last point whose value is nonzero. That is to say, the monitor statistic of our ACQ chart has equally the same performance with that of Hawkins (1987), at least for Laboratory 1 dada.

For Laboratory 2 data, the statistics indicate that there is significant change from the 23th observation on. However, the statistic of our ACQ chart gives an

OC signal at the 29th observation while the statistic of Hawkins (1987) signals at the 31th observation, which is 2 observations later. There is some difference on the interpretation of the 29th observation. We take it as an OC signal while Hawkins (1987) shows that it is an outlier because U_{29} exceeds 3 where

$$U_i = \frac{8\nu + 1}{8\nu + 3}[\nu \ln(1 + T_i^2/\nu)]^{\frac{1}{2}}, \nu = i - 2.$$

Then U_i is approximately distributed as normal $N(0, 1)$. The interpretation is not immediately clear. It is a function of the studentized deviation $(X_i - \bar{X}_{i-1})/\hat{\sigma}_{i-1}$, and as $i \to \infty$, $U_i \to (X_i - \mu)/\sigma$. Thus we may think of U_i as essentially just a studentized residual, restandardised to a $N(0, 1)$ distribution. Interested practitioners are referred to Wallace (1959) for more details. In Hawkins (1987), any U_i value exceeds a preset cutoff is edited down to equal the cutoff and the edited value is used in the CUSUM. As a consequence of this, the CUSUM is protected from signals generated solely by isolated outliers. The number 3 is a generally used preset cutoff and this is appropriate to applications such as this, where the primary role of the control is detecting small shifts in mean–the outliers are diagnosed using the Shewhart part of the control, while still being allowed to make a limited contribution to the CUSUM.

From our point of view, it may not be an outlier. First, from Table 3 of Hawkins (1987), the 29th observation is 2.51, while the 36th and the 38th are both 2.24. It is not probable that a process with run length no more than 10 (from the 29th to the 38th) should contain three outliers. Second, Hawkins (1987) does show that there is significant bias from the 23th observation onward. Third, the chart parameters employed in this example are relatively robust as mentioned above.

From Table 3 only, our ACQ chart does show some superiority over the self-starting method of Hawkins (1987). As shown by Hawkins (1987), the shifts for the data from Laboratory 1 and Laboratory 2 are less than one standard deviation in magnitude, they are, however, detected relatively quickly by our ACQ chart — after only 15 and 7 observations, respectively. The mean bias estimated by Hawkins (1987) for Laboratory 1 and Laboratory 2 is $0.286 - (-0.670) = 0.956$ and $0.481 - (-0.387) = 0.868$, respectively. This suggests reference values of $k = 0.956/2 = 0.478$ and $k = 0.868/2 = 0.434$ are optimal for detecting the biases. However, note that Hawkins (1987) makes use of some knowledge about the magnitude of the process bias to imply a reference value $k = 0.25$. From Table 3, we can see that the reference values used for the ACQ chart are larger than 0.25, which shows the efficiency of adaptive estimate of $\hat{\delta}_i$.

6. Conclusion and extension

Based on the Q statistics, the ACQ chart proposed in this paper can be used to detect a range of mean shifts with unknown IC process parameters μ and σ^2. This chart can be easily designed and its performance is satisfactory when the parameters are unknown. Monte Carlo simulation results indicate that our ACQ chart method has higher IC ARL values than the adaptive CUSUM with known parameters but performs well when compared with the CUSUM of Q chart with fixed reference value k and the change point method of Hawkins *et al.* (2003), particularly when the shift does not occur immediately after start-up and the shift size is small or moderate. A table is presented to assist in the design of the ACQ chart with parameters $\hat{\delta}_{min} = 0.25$, 0.5 and 0.75, $\lambda = 0.1$ and 0.2 and a range of IC ARLs. Moreover, if there are some historical IC samples, say, no less than 200, the performance of the ACQ chart is almost equally good relative to the control chart with true known process parameters. The application to the Laboratory Data shows the implementation of the ACQ chart. In this paper, we only consider detecting mean shifts, it is not difficult to generalize the idea of this paper to detecting variance shifts.

Acknowledgement

The authors are grateful to the editor and the anonymous referees for their valuable comments that have vastly improved this paper. This paper is supported by Natural Sciences Foundation of China (10771107, 10711120448) and Tianjin (07JCYBJC04300).

References

Aparisi, F., 1996. Hotelling‘s T^2 control chart with adaptive sample sizes. *International Journal of Production Research*, 34 (10), 2853–2862.

Aparisi, F. and Haro, C. L., 2001. Hotelling‘s T^2 control chart with variable sampling intervals. *International Journal of Production Research*, 39 (14), 3127–3140.

Bagshaw, M. and Johnson, R. A., 1975. The influence of reference values and estimated variance on the ARL of CUSUM tests. *Journal of the Royal Statistical Society B*, 37 (3), 413–420.

Bischak, D. P. and Trietsch, D., 2007. The rate of false signals in $\overline{X}$ control charts with estimated limits. *Journal of Quality Technology*, 39 (1), 54–65.

Borror, C. M., Montgomery, D. C. and Runger, G, C., 1999. Robustness of the EWMA control chart to nonnormality. *Journal of Quality Technology*, 31 (3), 309–316.

Hawkins, D. M., 1987. Self-starting CUSUM charts for location and scale. *The Statistician*, 36 (4), 299–316.

Hawkins, D. M. and Olwell, D. H., 1998. *Cumulative sum charts and charting for quality improvement*. New York: SpringerVerlag.

Hawkins, D. M., Qiu, P. and Kang, C. W., 2003. The change point model for statistical process control. *Journal of Quality Technology*, 35 (4), 355–366.

Hawkins, D. M. and Zamba, K. D., 2005a. A change-point model for a shift in variance. *Journal of Quality Technology*, 37 (1), 21–31.

Hawkins, D. M. and Zamba, K. D., 2005b. Statistical process control for shifts in mean or variance using a change-point formulation. *Technometrics*, 47 (2), 164–173.

Jensen, W. A., Jones-Farmer, L. A., Champ, C. W. and Woodall, W. H., 2006. Effects of parameter estimation on control chart properties: a literature review. *Journal of Quality Technology*, 38 (4), 349–364.

Jones, L. A., 2002. The statistical design of EWMA control charts with estimated parameters. *Journal of Quality Technology*, 34 (3), 277–288.

Jones, L. A., Champ, C. W. and Rigdon, S. E., 2001. The performance of exponentially weighted moving average charts with estimated parameters. *Technometrics*, 43 (2), 156–167.

Jones, L. A., Champ, C. W. and Rigdon, S. E., 2004. The run length distribution of the CUSUM with estimated parameters. *Journal of Quality Technology*, 36 (1), 95–108.

Lucas, J. M. and Crosier, R. B., 1982. Fast initial response for CUSUM quality control schemes: give your CUSUM a head start. *Technometrics*, 24 (3), 199–205.

Lucas, J. M. and Saccucci, M. S., 1990. Exponentially weighted moving average control schemes: properties and enhancements. *Technometrics*, 32 (1), 1–12.

Moustakides, G. V., 1986. Optimal stopping times for detecting changes in distributions. *The Annals of Statistics*, 14 (4), 1379–1387.

Quesenberry, C. P., 1991. SPC Q charts for start-up processes and short or long runs. *Journal of Quality Technology*, 23 (3), 213–224.

Quesenberry, C. P., 1995. On properties of Q charts for variables. *Journal of Quality Technology*, 27 (3), 184–203.

Shu, L. and Jiang, W., 2006. A Markov chain model for the adaptive CUSUM control chart. *Journal of Quality Technology*, 38 (2), 135–147.

Sparks, R. S., 2000. CUSUM charts for signaling varying location shifts. *Journal of Quality Technology*, 32 (2), 157–171.

Wallace, D. L., 1959. Bounds on normal approximations to Student's and the Chi-square distributions. *Annals of Mathematical Statistics*, 30, 1121–1130.

Wu, C., Zhao, Y., and Wang, Z., 2002. The average absolute deviations to median and their application to Shewhart $\bar{X}$ control chart. *Communication in Statistics: Simulation and Computations*, 31, 425–442.

Wu, Z., 1998. An adaptive acceptance control chart for tool wear. *International Journal of Production Research*, 36 (6), 1571–1586.

Zantek, P. F., 2006. Design of cumulative sum schemes for start-up processes and short runs. *Journal of Quality Technology*, 38 (4), 365–375.

Zhao, Y., Tsung, F., and Wang, Z., 2005. Dual CUSUM control scheme for detecting a range of mean shifts. *IIE Transactions*, 37, 1047–1058.

Table 1. The control limits for various IC ARL

IC	$\hat{\delta}_{min} = 0.25$		$\hat{\delta}_{min} = 0.5$		$\hat{\delta}_{min} = 0.75$	
ARL	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.1$	$\lambda = 0.2$
50	1.205	1.252	1.132	1.190	1.091	1.127
100	1.205	1.306	1.137	1.220	1.091	1.138
200	1.255	1.470	1.147	1.260	1.091	1.157
300	1.311	1.596	1.159	1.300	1.097	1.175
400	1.366	1.732	1.167	1.330	1.097	1.190
500	1.400	1.844	1.177	1.340	1.105	1.190
600	1.441	1.950	1.187	1.350	1.105	1.196
700	1.473	2.053	1.197	1.370	1.106	1.197
800	1.502	2.151	1.203	1.378	1.108	1.201
900	1.532	2.251	1.206	1.390	1.110	1.206
1000	1.560	2.331	1.212	1.400	1.114	1.211

Table 2. ARLs of change point approach and ACQ chart.

IC	Shift	δ									
ARL	pos.	0		0.25		0.5		1.0		2.0	
		CP	ACQ	CP	ACQ	CP	ACQ	CP	ACQ	CP	ACQ
500	10	535.8	1004.9	**531.1**	855.5	**492.7**	695.8	280.9	**279.8**	**10.8**	14.4
	25	538.7	1003.9	**504.3**	696.9	**357.5**	374.2	43.7	**33.7**	**4.3**	6.1
	50	539.5	994.0	**457.8**	538.5	195.4	**156.9**	15.7	**13.8**	**3.4**	5.0
	100	542.8	1007.9	373.1	**363.5**	77.3	**55.5**	12.3	**12.0**	**3.0**	4.7
	250	546.2	1015.2	222.9	**180.2**	43.3	**33.5**	**10.8**	11.2	**2.8**	4.5
200	10	195.9	392.2	**186.2**	307.9	**168.3**	220.9	84.9	**72.2**	**7.0**	7.3
	25	196.6	397.6	**174.7**	233.4	113.9	**105.9**	20.4	**15.0**	**3.3**	4.5
	50	196.3	388.9	**154.9**	175.2	68.7	**53.9**	11.2	**10.2**	**2.7**	4.0
	100	196.9	390.9	131.3	**124.0**	42.2	**31.5**	9.4	**9.2**	**2.5**	3.8
	250	194.8	391.2	99.0	**82.6**	29.7	**24.1**	**8.2**	8.6	**2.3**	3.7
100	10	99.3	190.6	**95.9**	137.5	**82.4**	90.6	39.1	**28.6**	5.3	**5.1**
	25	99.4	194.0	**86.9**	106.9	55.2	**48.0**	13.4	**10.2**	**2.8**	3.6
	50	99.2	193.1	**76.3**	80.5	38.2	**28.8**	8.8	**8.0**	**2.3**	3.3
	100	99.7	194.9	69.3	**63.6**	28.5	**21.2**	7.6	**7.4**	**2.2**	3.2
	250	96.9	194.1	55.4	**50.5**	23.9	**18.1**	7.0	**7.0**	**2.1**	3.1
50	10	54.5	99.2	**51.8**	66.3	43.3	**41.9**	21.3	**14.1**	4.1	**3.8**
	25	54.2	100.5	**46.7**	50.4	30.8	**24.9**	9.4	**7.3**	**2.4**	2.9
	50	53.8	99.9	**42.8**	42.9	23.3	**18.0**	7.0	**6.3**	**2.1**	2.7
	100	54.7	100.6	41.4	**35.3**	19.4	**15.2**	6.0	**6.0**	**1.9**	2.6
	250	58.7	100.0	33.4	**32.2**	16.6	**14.2**	6.3	**6.0**	**1.8**	2.6

Figure caption list:

- Figure 1. The ARLs of ACQ chart when shift occurs at the "start=100" position with units of the sifts standard deviation when IC ARL=500.
- Figure 2. The ARLs of ACQ chart when shift occurs at the "start" position and adaptive CUSUM charts (ACUSUM) with units of the sifts standard deviation when IC ARL=500.
- Figure 3. The ARLs of ACQ chart and adaptive CUSUM chart with estimated parameters (EACUSUM) with units of the sifts standard deviation when IC ARL=500.
- Figure 4. The ARLs of ACQ and CUSUM of Q charts when the IC ARL and shift position in (a)-(d) are (100, 10), (100, 100), (500, 10), and (500, 100) with units of the sifts standard deviation.

Table 3. The monitoring statistics for Lab. 1 & Lab. 2

	For Laboratory 1			For Laboratory 2		
No.	k	S	H	k	S	H
1	$\ldots^1$	$\ldots^1$	0.00	$\ldots^1$	$\ldots^1$	0.00
2	$\ldots^1$	$\ldots^1$	0.00	$\ldots^1$	$\ldots^1$	0.00
3	0.37	0.00	0.00	0.35	0.00	0.00
4	0.34	0.00	0.00	0.31	0.00	0.00
5	0.25	0.00	0.00	0.31	0.06	0.27
6	0.25	0.00	0.00	0.27	0.00	0.00
7	0.25	0.00	0.00	0.29	0.15	0.62
8	0.25	0.00	0.00	0.25	0.00	0.00
9	0.30	0.28	1.13	0.25	0.00	0.00
10	0.30	0.38	1.56	0.25	0.00	0.00
11	0.25	0.04	0.13	0.34	0.55	2.06
12	0.25	0.00	0.00	0.27	0.30	1.07
13	0.25	0.00	0.00	0.25	0.00	0.00
14	0.25	0.00	0.00	0.26	0.10	0.43
15	0.25	0.00	0.00	0.25	0.02	0.10
16	0.26	0.08*	0.34*	0.25	0.00	0.00
17	0.28	0.26	1.06	0.25	0.00	0.00
18	0.30	0.44	1.81	0.25	0.00	0.00
19	0.27	0.34	1.43	0.25	0.00	0.00
20	0.27	0.43	1.80	0.25	0.00	0.00
21	0.25	0.39	1.63	0.25	0.00	0.00
22	0.25	0.27	1.14	0.25	0.00	0.00
23	0.25	0.27	1.15	0.34	0.56*	2.09*
24	0.25	0.20	0.86	0.30	0.42	1.60
25	0.30	0.49	2.02	0.25	0.21	0.70
26	0.36	0.94	3.68	0.26	0.29	1.07
27	0.37	1.11	4.38	0.31	0.60	2.29
28	0.34	1.01	4.08	0.33	0.79	3.06
29	0.33	1.08	4.41	0.45	1.69**	5.81
30	0.41	1.68**	6.48**	$\ldots^2$	$\ldots^2$	5.70
31	$\ldots^2$	$\ldots^2$	$\ldots^2$	$\ldots^2$	$\ldots^2$	6.02**