Necessary and Sufficient Conditions for Non-interaction of a Pair of One-sided EWMA Schemes with Reflecting Boundaries

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Abstract

This paper deals with the necessary and sufficient conditions for non-interaction of the upper-sided and lower-sided EWMAs with reflecting boundaries, the average run length, the Laplace transform of the run length and some analysis under the condition of interaction.

Key words: Exponentially weighted moving average; Non-interaction; Average run length; Reflecting boundaries
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1 Introduction

Exponentially weighted moving average (EWMA) and Cumulative Sum (CUSUM) quality control schemes are widely used in industry for their ease to implement and interpret. They have good performances in detecting small and moderate changes, see Lucas and Saccucci (1990).

To be a good scheme, it should have small out-of-control average run length (ARL), which is defined as the average number of samples before the chart signals an out-of-control condition and large in-control ARL. Therefore, the computation of ARL is of great importance in comparing the performance of control charts. The main methods for computing ARL applied in literatures are

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Markov chain approach (Brook and Evans (1972)), integral equation approach (Crowder (1987)) and simulation. Some approximations to the run length distribution are made in Gold (1989) and Woodall (1983), among others. The comparison between Markov chain approach and integral equation approach is made in Champ and Rigdon (1991).

If one wants to detect both the increase and decrease shifts simultaneously, a pair of one-sided schemes is needed. However, if the Markov chain method is used to compute the ARL and one gets t, say, 30 discrete part of the state space of the statistic, the transition matrix will have a dimension of $30^2 \times 30^2$, which causes the computation tedious. Woodall (1984) shows that the number of states included in the Markov chain for one-sided chart is smaller than t^2 (it varies between $2t-1$ and $(t^2+t)/2$). Thus, even in the best cases we must deal with high dimensional matrices. Fortunately, under the condition of non-interaction, the ARL of a pair of one-sided schemes can be obtained from the ARL of the upper-sided and lower-sided schemes. Thus, the crucial question in this situation is whether the upper-sided and lower-sided schemes interact. For CUSUM charts, Lucas (1985) gives sufficient conditions for non-interaction, Lucas and Crosier (1982) obtain the ARL of the two-sided scheme from the ARL of the upper-sided and lower-sided schemes and Yashchin (1985) derives the necessary and sufficient conditions for various modes of interactions of the upper and lower CUSUM schemes with head starts. However, to our knowledge, the discussion of non-interaction for EWMA scheme is limited, which is the motivation of this paper.

In EWMA schemes, the use of a reflecting boundary is appealing because it ensures the EWMA is at most a certain distance. Gan (1993) and Gan (1998) give designs of EWMA charts for normal and exponential random variables, respectively. The sufficient condition of non-interaction for two one-sided EWMA charts is given by Gan (1998) but without proof. Among others, EWMA with boundaries is used in Reynolds and Stoumbos (2004) and Zhang and Chen (2004).

The rest of the paper is organized as follows. Some preliminaries are presented in Section 2. The main results are derived in Section 3. The simulation study is reported in Section 4. Concluding remarks are given in Section 5. All proofs are deferred to the Appendix.

2 Preliminaries

Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of i.i.d. random variables. The upper-sided EWMA chart with boundary is intended for detecting an increase in the mean and is obtained by plotting

$$Q_t = \max\{A, (1 - \lambda_Q)Q_{t-1} + \lambda_Q X_t\}$$

against t, for $t = 1, 2, \ldots$, where λ_Q is a smoothing constant$(0 < \lambda_Q < 1)$, A is a boundary and $A \leq Q_0 = u < h_Q$. A signal is issued at T_+, the first t for which $Q_t \geq h_Q$. Similarly, the lower-sided EWMA chart with boundary is intended for detecting a decrease in the mean and is obtained by plotting

$$q_t = \min\{B, (1 - \lambda_q)q_{t-1} + \lambda_q X_t\}$$

against t, for $t = 1, 2, \ldots$, where λ_q is a smoothing constant$(0 < \lambda_q < 1)$, B is a boundary and $h_q < q_0 = v \leq B$. A signal is issued at T_-, the first t for which $q_t \leq h_q$.

A two-sided EWMA charts is obtained by running a lower-sided and an upper-sided EWMA chart simultaneously. As in Gan (1993) and Gan (1998), only the case $\lambda_Q = \lambda_q = \lambda$ is considered in this paper. A signal is issued at $T = \min\{T_+, T_-\}$.

Let (A, h_Q, Q_0, λ_Q), (B, h_q, q_0, λ_q) and $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ be an upper-sided EWMA, lower-sided EWMA and two-sided EWMA scheme respectively.

For the one-sided and two-sided EWMA charts we define the follows:

(1) The lower-sided EWMA scheme does not act on the upper-sided EWMA scheme if for every realization $\{X_1, \ldots, X_n, \ldots\}$, $q_{T_-} \leq h_q$ implies that $Q_{T_-} = A$.
(2) The upper-sided EWMA scheme does not act on the lower-sided EWMA scheme if for every realization $\{X_1, \ldots, X_n, \ldots\}$, $Q_{T_+} \geq h_Q$ implies that $q_{T_+} = B$.
(3) The upper-sided EWMA and lower-sided EWMA schemes do not interact if neither the upper scheme nor the lower one acts on each other.

3 The main results

We assume $B \leq h_Q$ and $A \geq h_q$ throughout this paper without loss of generality.

Theorem 1 *Let $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ be a two-sided EWMA scheme.*

(i) The necessary and sufficient condition for the upper-sided EWMA not to

act on the lower-sided one is

$$1-\lambda \le \begin{cases} \min\{\frac{h_Q-B}{Q_0-q_0}, \frac{h_Q-B}{A-h_q}\}, & \text{if } Q_0-q_0>0, \\ \frac{h_Q-B}{A-h_q}, & \text{if } Q_0-q_0\le 0. \end{cases} \tag{1}$$

(ii) The necessary and sufficient condition for the lower-sided EWMA not to act on the upper-sided one is

$$1-\lambda \le \begin{cases} \min\{\frac{A-h_q}{Q_0-q_0}, \frac{A-h_q}{h_Q-B}\}, & \text{if } Q_0-q_0>0, \\ \frac{A-h_q}{h_Q-B}, & \text{if } Q_0-q_0\le 0. \end{cases} \tag{2}$$

(iii) The upper-sided and lower-sided schemes do not interact if and only if both (1) and (2) hold.

If we assume that $Q_0 = A, q_0 = B$, the necessary and sufficient condition for non-interaction for a pair of one-sided EWMAs can be more general. This is the following result.

Theorem 2 *Let $Q_0 = A, q_0 = B$, the upper-sided EWMA and lower-sided EWMA schemes do not interact if and only if*

$$(1-\lambda)^2 \le \min\left\{\frac{A-h_q}{h_Q-B}, \frac{h_Q-B}{A-h_q}\right\}. \tag{3}$$

Remark 1 *Theorem 2 is consistent with the result mentioned in Gan (1993). However, the necessity of the conditions is not mentioned. In this paper, the condition is shown not only sufficient but also necessary.*

Let $H(x) = E(T_+|Q_0 = x), L(y) = E(T_-|q_0 = y)$ and $ARL(x,y) = E(T|Q_0 = x, q_0 = y)$, respectively. Then we have the ARL of the two-sided EWMA scheme under non-interaction.

Theorem 3 *If the upper-sided EWMA and lower-sided EWMA schemes do not interact, then*

$$ARL(Q_0, q_0) = \frac{H(Q_0)L(B) + H(A)L(q_0) - H(A)L(B)}{H(A) + L(B)}. \tag{4}$$

Remark 2 *The formula obtained in Theorem 3 is similar to that mentioned in Lucas and Crosier (1982) and Lucas (1985) for CUSUM charts.*

We consider the problem of finding the Laplace transform of the run length (RL) for the two-sided EWMA scheme $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ when the upper-sided EWMA and lower-sided EWMA schemes do not interact.

Let us introduce the following notations:

$F_+(s|x_+) = E\{\exp(-sT_+)|Q_0 = x_+\}$,

$F_-(s|x_-) = E\{\exp(-sT_-)|q_0 = x_-\}$,

$F(s|x_+, x_-) = E\{\exp(-sT)|Q_0 = x_+, q_0 = x_-\}$,

$b_+ = P\{T_+ < T_-\}E\{\exp(-sT_+)|T_+ < T_-\}$,

$b_- = P\{T_- < T_+\}E\{\exp(-sT_-)|T_- < T_+\}$.

Theorem 4 *If the upper-sided EWMA and lower-sided EWMA schemes do not interact, then*

$$F(s|Q_0, q_0) = \begin{cases} \frac{F_+(s|Q_0)(1-F_-(s|B))+F_-(s|q_0)(1-F_+(s|A))}{1-F_+(s|A)F_-(s|B)}, s \neq 0, \\ 1, s = 0. \end{cases} \tag{5}$$

We shall give some results which are important in the analysis of interacting schemes. Let i be the smallest non-negative integer such that

$$(1-\lambda)^i (Q_0 - q_0) - \max\{h_Q - B, A - h_q\} \leq 0. \tag{6}$$

We shall show that if the signal is not triggered in a certain period of time $[0, i]$, the subsequent behavior of the interacting schemes is very similar to that of non-interacting ones. Here and in the following of the paper we shall further assume that $Q_0 > q_0$, because $Q_0 \leq q_0$ implies $i = 0$, which does not give a period of time $[0, i]$.

Theorem 5 *Let $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ be a two-sided EWMA scheme and let i, the smallest non-negative integer satisfying (6), be greater than* 0.

(i) Let $X_1, X_2 \ldots$ be any realization for which $T \geq i$. Then for every $0 \leq j < i$

$$\begin{cases} (a) 0 \leq (1-\lambda)^j (Q_0 - q_0) + h_q - A < Q_j - A < h_Q - A, \\ (b) 0 \leq (1-\lambda)^j (Q_0 - q_0) - h_Q + B < -q_j + B < -h_q + B. \end{cases} \tag{7}$$

(ii) For some realization $X_1, X_2, \ldots$, let one of the relations (7) hold for every $0 \leq j < i$. Then (for this realization) the other relation also holds for every $0 \leq j < i$ and $T \geq i$.

4 Simulation results

From Theorem 1 and Theorem 2, we know a small λ may violate the condition for non-interaction when other parameters are fixed. Lucas and Saccucci (1990) suggests a small λ considering the following: (1) EWMA is designed to detect small to moderate process shifts and smaller λ is more effective in this condition; (2) Fast initial response feature is most useful for EWMA control schemes with λ less than or equal to 0.25; (3) For smaller values of λ, EWMA scheme is more robust to contamination. So in real-world control charting, one might want to use parameter values that violates the conditions for non-interaction. In literature, the parameters $(A, h_Q, Q_0, B, h_q, q_0, \lambda) = (0.5, 2.137, 1.0, 2.0, 0.406, 1.0, 0.18)$ for exponentially distributed statistics are used in Table 2 of Gan (1998). However, Theorem 1 shows us that λ less than or equal to 0.31 violates the condition for non-interaction in this case. The ARL results of Gan (1998) and those obtained from Theorem 3 are listed in Table 1. Note that the lines labeled "T" are true results and "F" are false results obtained ignoring the interaction. From Table 1, we observe that the differences are substantial if one neglects the interaction, especially in-control situation. Note that the condition with shift equals 1 is the in-control condition because the statistics are exponentially distributed. The substantial decrease in ARL when the process is in control if one neglects the interaction implies more frequent false signals, which is not one wants. Table 2 shows the ARL with parameters $(A, h_Q, Q_0, B, h_q, q_0) = (-1.0, 1.154, -1.0, 1.0, -1.089, 1.0)$ for normally distributed statistics. λ less than or equal to 0.23 violates the condition for non-interaction in this case by Theorem 2. From Table 2, the ARLs have substantial differences for the two approaches.

It would be better able to handle the inertia problem EWMA charts suffer from. Due to the space limitation, however, we do not study the inertia problem further except showing the effect of the boundaries on the inertia problem. Table 3 shows the ARLs obtained from lower-sided EWMA with or without boundaries. Note that the lines labeled "N" and "B" are obtained from EWMA scheme without and with reflecting boundaries, respectively. In the simulation, the first ten statistics are assumed to have a δ standard deviation shift upward to generate the inertia problem. Table 4 has the same structure with Table 3 except that it is for upper-sided EWMA. From Table 3 and Table 4, when $|\delta|$ is small, for example, $|\delta| = 0.5$, the inertia problem is not serious; however, when $|\delta|$ is large, for example, $|\delta| = 3.0$, the inertia problem has great effect on the ARL performance. The EWMA scheme with boundaries has much smaller ARL when the process is out-of-control. Therefore, the EWMA scheme with boundaries can lessen the inertia problem so much that it warrants further research.

5 Concluding remarks

This paper deals with necessary and sufficient conditions for a pair of one-sided EWMA schemes not to interact. One can analyze the two-sided EWMA schemes by considering the upper-sided and lower-sided EWMA schemes separately under some general conditions, even in the case of interacting schemes. The inertia problem warrants further research.

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7 Appendix

Lemma 1 *Suppose that for a sequence of observations,* $(Q_t, q_t), (Q_{t+1}, q_{t+1}), \ldots, (Q_{t+k}, q_{t+k})$, *is a part of the realization of a two-sided EWMA scheme* $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$. *We have*

$$Q_{t+k} - q_{t+k} \geq (1-\lambda)^k (Q_t - q_t). \tag{8}$$

If, in addition, $Q_t \geq A, Q_{t+1}, \ldots, Q_{t+k} > A$ *and* $q_t \leq B, q_{t+1}, \ldots, q_{t+k} < B$, *then equality holds in (8).*

The proof of this lemma can be completed with the induction method. □

Lemma 2 *Let* $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ *be a two-sided EWMA scheme. Suppose that for some realization of the observations* $X_1, \ldots, X_T$, *the upper-sided EWMA signals at* T *and* $q_T < B$. *Suppose that* T *is the minimal time for which a realization with such a property is possible. Then:*

(i) $T = 1$ *if and only if*

$$(1-\lambda)(Q_0 - q_0) > h_Q - B. \tag{9}$$

(ii) $T \geq 2$ *if and only if*

$$(a)\textit{equation (9) does not hold; } (b)Q_{T-1} = A, \textit{ and } (c)\ 1 - \lambda > \frac{h_Q - B}{A - h_q}. \quad (10)$$

Proof. (i)If $T = 1$, $Q_1 = (1-\lambda)Q_0 + \lambda X_1 \geq h_Q$ and $q_1 = (1-\lambda)q_0 + \lambda X_1 < B$. We have $(1-\lambda)(Q_0 - q_0) > h_Q - B$. If $(1-\lambda)(Q_0 - q_0) > h_Q - B$, then an observation $X_1 = \frac{1}{\lambda}(-(1-\lambda)Q_0 + h_Q)$ will cause $Q_1 = (1-\lambda)Q_0 + \lambda X_1 = h_Q$ and $q_1 = (1-\lambda)q_0 + \lambda X_1 = (1-\lambda)(q_0 - Q_0) + h_Q < B$. Thus, $T = 1$ by definition.

(ii)The sufficiency of the conditions (10) follows from the sufficiency of (a). The necessity of (a) follows from the proof above.

To prove the necessity of (b), note that $X_T > q_{T-1}$. If not,

$$Q_T = (1-\lambda)Q_{T-1} + \lambda X_T \leq (1-\lambda)Q_{T-1} + \lambda q_{T-1} \leq \begin{cases} Q_{T-1}, \text{ if } Q_{T-1} \geq q_{T-1} \\ q_{T-1}, \text{ if } Q_{T-1} < q_{T-1} \end{cases},$$

which implies $Q_T < h_Q$, contradicting the fact that the upper-sided scheme signals at T. Then we have $B > q_T = (1-\lambda)q_{T-1} + \lambda X_T > q_{T-1} > h_q$. Now suppose that $A < Q_{T-1} < h_Q$. Then, by Lemma 1,

$$Q_T - q_T = (1-\lambda)^2(Q_{T-2} - q_{T-2}) > 0. \quad (11)$$

Now let us choose the $(T-1)$th observation to be $X^*_{T-1} = \frac{1}{\lambda}(q_T - (1-\lambda)q_{T-2})$. Then the corresponding values of the lower-sided and upper-sided scheme would be $q^*_{T-1} = (1-\lambda)q_{T-2} + \lambda X^*_{T-1} = q_T < B$ and by (11)

$$Q^*_{T-1} = (1-\lambda)Q_{T-2} + \lambda X^*_{T-1} = Q_T + \lambda(1-\lambda)(Q_{T-2} - q_{T-2}) > h_Q,$$
which contradicts the fact that T is the minimal time for which a realization with such a property is possible.

To prove the necessity of (c), now suppose that $T \geq 2$ and $1 - \lambda \leq \frac{h_Q - B}{A - h_q}$. Then, since $Q_{T-1} = A$ and $Q_T = (1-\lambda)A + \lambda X_T \geq h_Q$,

$$q_T = (1-\lambda)q_{T-1} + \lambda X_T \geq (1-\lambda)q_{T-1} + h_Q - (1-\lambda)A \geq (1-\lambda)(h_q - A) + h_Q \geq B,$$

which contradicts the assumption that for realization $X_1, \ldots, X_T$, we must have $q_T < B$. This lemma is proved. □

Lemma 3 *Let $(A, h_Q, Q_0, B, h_q, q_0, \lambda)$ be a two-sided EWMA scheme. Suppose that for some realization of the observations $X_1, \ldots, X_T$, the lower-sided EWMA signals at T and $Q_T > A$. Suppose that T is the minimal time for which a realization with such a property is possible. Then:*

(i) $T = 1$ *if and only if*

$$(1-\lambda)(Q_0 - q_0) > A - h_Q. \tag{12}$$

(ii) $T \geq 2$ *if and only if*

$$(a) \textit{equation (12) does not hold}; \ (b) q_{T-1} = B, \ \textit{and } (c)\ 1-\lambda > \frac{A-h_q}{h_Q - B}. \tag{13}$$

The proof of this lemma is similar to that of Lemma 2 and is omitted. □

Lemma 4 *If the upper-sided EWMA and lower-sided EWMA schemes do not interact, then*

$$F_+(s|Q_0) = b_- F_+(s|A) + b_+, \tag{14}$$

$$F_-(s|q_0) = b_+ F_-(s|B) + b_-. \tag{15}$$

Proof. To prove (14), note that

$$\begin{aligned} F_+(s|Q_0) &= P\{T_+ > T_-\}E\{\exp(-s(T_- + (T_+ - T_-))|T_+ > T_-\} \\ &+ P\{T_+ < T_-\}E\{\exp(-sT_+)|T_+ < T_-\}. \end{aligned} \tag{16}$$

But under the condition $T_+ > T_-$, the random variables T_- and $(T_+ - T_-)$ are independent since the upper-sided EWMA and lower-sided EWMA schemes do not interact. In this case, the distribution of $(T_+ - T_-)$ is the same as that of RL of the upper-sided EWMA with boundary A. Now we can obtain (14). The proof of (15) is similar, which is omitted. □

Proof of Theorem 1. By symmetry considerations, we only consider the proof of (i).

Firstly, we prove the sufficiency. Assume that (1) holds. Then both conditions (9) and (10)(c) fail to hold. Thus, by Lemma 2, it is impossible to find a realization $X_1, \ldots, X_T$ with the property that the upper-sided EWMA signals at T and $q_T < B$.

Secondly, we prove the necessity. Assume that (1) does not hold. If $(1-\lambda)(Q_0 - q_0) > h_Q - B$, then we can have a realization such that $Q_1 \geq h_Q$ and $q_1 < B$ by Lemma 2. If $1 - \lambda > \frac{h_Q - B}{A - h_q}$, then consider the realization $X_1 = \frac{1}{\lambda}(-(1-$

$\lambda)q_0 + h_q + \epsilon)$, $X_2 = \frac{1}{\lambda}(h_Q - (1-\lambda)Q_1)$, where $\epsilon > 0$ is a parameter. For this realization, $Q_2 = (1-\lambda)Q_1 + \lambda X_2 = h_Q$, i.e. the upper-sided EWMA signals at $T = 2$. If ε is chosen close enough to 0 such that $h_q + \epsilon < B$ and $(1-\lambda)(h_q - A + \epsilon) + h_Q < B$, then,

$$q_1 = \min\{B, (1-\lambda)q_0 + \lambda X_1\} = \min\{B, h_q + \epsilon\} = h_q + \epsilon,$$

$q_2 \leq (1-\lambda)q_1 + \lambda X_2 = (1-\lambda)(h_q - Q_1 + \epsilon) + h_Q < (1-\lambda)(h_q - A + \epsilon) + h_Q < B$. Thus, the fact that (1) does not hold implies that there exists a realization for which the lower-sided EWMA is below B at the moment the upper-sided signals. The proof of Theorem 1 is completed. □

Proof of Theorem 2. The necessity is trivial by Theorem 1.

To prove the sufficiency, let t be the time when the upper-sided EWMA begins to accumulate from the last time when $Q_t = A$, and w be the time when the lower-sided EWMA begins to accumulate from the last time when $q_w = B$. If the upper-sided EWMAs since time t and the lower-sided EWMAs since time w are drawn and all of them lie between their respective chart limits immediately after the rth sample, then we have a sequence of EWMAs Q_j^* and a sequence of EWMAs q_l^* where $A < Q_j^* < h_Q, j = 1, 2, \ldots, m, Q_0^* = A, h_q < q_l^* < B, l = 1, 2, \ldots, n$, and $q_0^* = B$. Note that $t + m = r$ and $w + n = r$, where r is the total number of samples taken. Then, the EWMAs Q_j^* and q_l^* can be expressed as

$$Q_j^* = (1-\lambda)^j A + \lambda \sum_{i=t+1}^{t+j} (1-\lambda)^{j-i+t} X_i, j = 1, 2, \ldots, m,$$

$$q_l^* = (1-\lambda)^l B + \lambda \sum_{i=w+1}^{w+l} (1-\lambda)^{l-i+w} X_i, l = 1, 2, \ldots, n.$$

Suppose that $Q_{m+1}^* \geq h_Q$, then $Q_{m+1}^* = (1-\lambda)Q_m^* + \lambda X_{m+1}^* \geq h_Q$, so that

$$\lambda X_{m+1}^* \geq h_Q - (1-\lambda)Q_m^*.$$

The lower-sided EWMA becomes

$$q_{n+1}^* = (1-\lambda)q_n^* + \lambda X_{m+1}^* \geq (1-\lambda)(q_n^* - Q_m^*) + h_Q.$$

If $w = t$, then $m = n$ and $Q_m^* - q_n^* = (1-\lambda)^m (A - B)$. Since the condition $(1-\lambda)^2 \leq \frac{h_Q - B}{A - h_q}$ implies that $h_Q \geq B + (1-\lambda)^2 (A - h_q)$, we have

$$q_{n+1}^* \geq h_Q - (1-\lambda)^{m+1}(A - B) \geq B + (1-\lambda)^2 (A - h_q) - (1-\lambda)^{m+1}(A - B).$$

Note that $A - h_q - (1-\lambda)^{m-1}(A-B) > 0$, then we have $q^*_{n+1} > B$.

If $w > t$, $Q^*_m - q^*_n < (1-\lambda)^n(h_Q - B)$, then $q^*_{n+1} > h_Q - (1-\lambda)^{n+1}(h_Q - B)$.

Note that the condition $h_Q \geq B$ implies that $h_Q - (1-\lambda)^{n+1}(h_Q - B) \geq B$, then we have $q^*_{n+1} > B$.

If $w < t$, $q^*_n - Q^*_m > (1-\lambda)^m(h_q - A)$, then $q^*_{n+1} > h_Q + (1-\lambda)^{m+1}(h_q - A)$.

Note that the condition $h_Q - B - (1-\lambda)^2(A - h_q) \geq 0$ implies that $h_Q + (1-\lambda)^{m+1}(h_q - A) \geq B$, then we have $q^*_{n+1} > B$.
Similarly, we can obtain that if $q^*_{n+1} \leq h_q$, $Q^*_{m+1} < A$. The proof is omitted here.□

Proof of Theorem 3 is similar to the approach of A.1 of Appendix in Lucas and Crosier (1982). □

Proof of Theorem 4. By taking expectations of both sides of the identity

$$\exp(-sT_+) + \exp(-sT_-) = \exp(-sT) + \exp(-s\max\{T_+, T_-\}),$$

and using the fact that for non-interacting schemes

$$\exp(-s\max\{T_+, T_-\}) = b_- F_+(s|A) + b_+ F_-(s|B),$$

we obtain

$$F(s|x_+, x_-) = F_+(s|Q_0) + F_-(s|q_0) - b_- F_+(s|A) - b_+ F_-(s|B). \tag{17}$$

From Lemma 4, $b_+ = \frac{F_+(s|Q_0) - F_+(s|A)F_-(s|q_0)}{1 - F_+(s|Q_0)F_-(s|q_0)}$, and $b_- = \frac{F_-(s|q_0) - F_-(s|B)F_+(s|Q_0)}{1 - F_+(s|Q_0)F_-(s|q_0)}$. Substituting b_+ and b_- into (17), we can obtain (5).□

Proof of Theorem 5. (i)For every $0 \leq j < i$, by Lemma 1 and the definition of i, we have $Q_j - q_j \geq (1-\lambda)^j(Q_0 - q_0) > \max\{h_Q - B, A - h_q\}$. Since $T > i$, it is clear that $A < Q_j < h_Q, h_q < q_j < B$ for every $0 \leq j < i$. Therefore, for every $0 \leq j < i$ we have $h_q < q_j \equiv Q_j - (1-\lambda)^j(Q_0 - q_0)$, and $(1-\lambda)^j(Q_0 - q_0) + q_j \equiv Q_j < h_Q$, which imply (7)(a) and (7)(b) respectively.

(ii)In the case $i = 1$, the proof is trivial. Now consider the case $i > 1$. Suppose that (7)(a) holds for every $0 \leq j < i$ and that $j^*(0 < j^* < i)$ is the smallest index for which $q_{j^*} \leq h_q$. Then, by the definition of i,

$$Q_{j^*} - q_{j^*} > (1-\lambda)^j(Q_0 - q_0) > \max\{h_Q - B, A - h_q\}, \tag{18}$$

which implies that $Q_j - q_j > \max\{h_Q - B, A - h_q\}$, also for any $j < j^*$. But then we must have $Q_j - q_j = (1-\lambda)^j(Q_0 - q_0)$ for any $j < j^*$. This contradicts

(18). Thus, $q_j > h_q$. And $(1-\lambda)^j(Q_0 - q_0) + q_j \equiv Q_j < h_Q$ implies

$$(1-\lambda)^j(Q_0 - q_0) - h_Q + B < -q_j + B.$$

Therefore, (7)(b) holds for every $0 \leq j < i$.

Similarly, we can show that if (7)(b) holds for every $0 \leq j < i$, then (7)(a) holds for every $0 \leq j < i$. □

References

Brook, D., Evans, D. A., 1972. An approach to the probability distribution of cusum run lengths. Biometrika. 59, 539-549.

Champ, C. W., Rigdon, S. E., 1991. A comparison of the Markov chain and the ingegral equation approaches for evaluating the run length distribution of quality control charts. J. Stat. Comput. Sim. 20, 191-204.

Crowder, S. V., 1987. A simple method for studying run-length distributions of exponentially weighted moving average charts. Technometrics. 29, 401-401.

Gan, F. F., 1993. Exponentially weighted moving average control charts with reflecting boundaries. J. Stat. Comput. Sim. 46, 45-67.

Gan, F. F., 1998. Designs of one- and two-sided exponential EWMA charts. J. Qual. Technol. 30, 55-69.

Gold, M. S., 1989. The geometric approximation to the cusum run length distribution. Biometrica. 76, 725-733.

Lucas, J. M., Crosier, R. B., 1982. Fast initial response for CUSUM quality control schems: give your CUSUM a head start. Technometrics. 24, 199-205.

Lucas, J. M., 1985. Counted data cusum's. Technometrics. 27, 129-144.

Lucas, J. M., Saccucci, M. S., 1990. Exponentially weighted moving average control schmes: properties and enhancements. Technometrics. 32, 1-12.

Reynolds, M. R. Jr., Stoumbos, Z. G., 2004. Control charts and the efficient allocation of sampling resources. Technometrics, 46, 200-214.

Woodall, W. H., 1983. The distribution of the run length of one-sided cusum procedures for continuous random variables. Technometrics. 25, 295-300.

Woodall, W. H., 1984. On the Markov chain approach to the two-sided cusum procedure. Technometrics. 26, 41-46.

Yashchin, E., 1985. On a unified approach to the analysis of two-sided cumulative sum control schemes with headstarts. Adv. Appl. Probab. 17, 562-593.

Zhang, L., Chen, G., 2004. EWMA charts for monitoring the mean of censored Weibull lifetimes. J. Qual. Technol. 36, 321-328.