Adaptive CUSUM Control Chart with Variable Sampling Intervals

Yunzhao Luo, Zhonghua Li, Zhaojun Wang *

LPMC and School of Mathematical Sciences, Nankai University, Tianjin 300071, P.R.China

Abstract

The standard cumulative sum chart (CUSUM) is widely used for detecting small and moderate process mean shifts, and its optimal detection ability for any pre-specified mean shift has been demonstrated by its equivalence to continuous sequential tests. In real practice, the assumption of knowing the true mean shift in prior can not be always met. So it is desirable to design a procedure that is efficient for detecting a range of future expected but unknown mean shifts. Adaptive CUSUM control chart, which can continuously adjust itself by an one-step forecasting operator, has been proposed to detect efficiently and robustly for a range of mean shifts in the early literatures. Moreover, in terms of sampling time to signal, control chart with the VSI (variable sampling intervals) feature can detect the process changes more quickly than the traditional FSI (fixed sample intervals) chart. In this paper, a new CUSUM control chart which is based on both adaptive and VSI features is discussed. Also, a two-dimensional Markov chain model is developed to evaluate its run-time performance.

Key words: Adaptive CUSUM control chart; EWMA; Markov chain model; Variable sampling intervals
1991 MSC: 62P30

1 Introduction

The cumulative sum control chart (CUSUM) has been widely used for detecting the small and moderate mean shifts since first introduced by Page (1954).

* Corresponding author.
Email addresses: 0110016@mail.nankai.edu.cn (Yunzhao Luo), 0210008@mail.nankai.edu.cn (Zhonghua Li), zjwang@nankai.edu.cn (Zhaojun Wang).

Preprint submitted to Computational Statistics & Data Analysis 2 January 2009

For simplicity, we suppose that the process readings $\{X_i\}$ can be modeled by two normal distributions, which refer to IC (in-control) and OC (out-of-control), respectively:

$$\begin{cases} X_i \sim N(\,\mu_1, \sigma_1^2\,)\,, & \text{if the process is in-control}\,, \\ X_i \sim N(\,\mu_2, \sigma_2^2\,)\,, & \text{if the process is out-of-control.} \end{cases} \tag{1}$$

In this paper, we only focus on detecting the changes in the process mean, that is $\mu_1 \neq \mu_2$, but $\sigma_1 = \sigma_2 = \sigma$ over time. Also, we assume that $\mu_2 > \mu_1$, the case that $\mu_2 < \mu_1$ can be handled analogously. A CUSUM chart with the optimal detection for a pre-known magnitude of the mean shift (denoted by $\delta = \mu_2 - \mu_1$) is defined by:

$$\begin{cases} C_0 = 0\,, \\ C_i = \max(\,0, C_{i-1} + X_i - k\,)\,, \end{cases} \tag{2}$$

where the reference value k is $\delta/2$. The CUSUM chart signals as soon as $C_i > h$, where h is called by "decision interval", which is chosen to maintain a desirable IC average run length (denoted by ARL_0). The CUSUM procedure has an attractive theoretical property: it is the optimal detection procedure for the mean shift δ when its reference value k is set at $\delta/2$, see Lorden (1971), Pollack (1985), Siegmund (1985), Moustakieds (1986), Hawkins and Olwell (1998) for more detailed discussions. Briefly speaking, the CUSUM's optimal performance requires accurate information about the magnitude of the future mean shift δ, which can not always be known in real practice. Therefore, it is reasonable to design a new procedure to be robust for the range shifts rather than only sensitive to a particular shift. There are many literatures about monitoring the range shifts. Lucas (1982) suggested a combined Shewhart-CUSUM control chart to monitor the process within a broader range of mean shift. Zhao et al. (2005) provided a dual CUSUM control chart to monitor the process range shifts, in which two CUSUM charts are used simultaneously to deal with the small and large mean shifts, respectively. Also, there are some other control charts based on change-point model and generalized likelihood ratio test for detecting the range shifts, see Nikiforov (2001), Lai (2001), and Hawkins and Qiu (2003). But these methods are usually too complicated to execute for their excessive memories to implement and time-consuming simulations to evaluate the run-length performance. Meanwhile, Sparks (2000) suggested an adaptive CUSUM (ACUSUM) chart which can continuously adjust itself according to the reference value k by an EWMA operator, to facilitate the implement of detecting the range shifts with high efficiency and robustness. Moreover, Shu and Jiang (2006) developed a two-dimensional Markov chain model to facilitate the evaluation of the run-length performance of ACUSUM chart instead of simulation.

From another aspect, the usual practice of using a control chart to monitor a process is to take samples with fixed time intervals or sample sizes. Recent literatures have shown that control chart with variable sampling schemes can detect the process shifts much faster than the traditional static one. For example, control chart with variable sampling interval (VSI) or variable sample size (VSS) feature allows the sampling time intervals or sample sizes to be changed according to the values of sample readings which indicate the current process state, see Reynolds et al. (1988), Reynolds and Arnold (1990), Runger and Pignatiello (1991), Reyonlds (1995), Tagaras (1998), Arnold and Reynolds (2001), Chen and Chiou (2005), Lin and Chou (2005), Costa and De Magalhães (2007), Wu et al. (2007), Zou et al. (2008) for more detailed discussions. Throughout this paper, a new modified CUSUM control chart based on both self-adaption and VSI features will be discussed to achieve an overall performance for the range shifts, which is called "VSI ACUSUM" control chart. For the case of VSS or VSSI, however, we will not discuss here for shorting the literature, in that the analysis is similar.

The rest of this paper is organized as follows: a brief description for ACUSUM and VSI control chart is given in Section 2. Our proposed control chart is proposed in Section 3, which is followed by some numerical comparisons in Section 4. The last section contains some concluding remarks. A two-dimensional Markov chain model for evaluating the performance of our proposed control chart will be shown in the Appendix.

2 Brief Introduction of Existing Work

Before providing our proposed control chart, some brief introductions of existing work are presented in this section.

2.1 ACUSUM Control Charts

For simplicity, the process is said to be in-control when all readings $\{X_i\}$ are sampled from standard normal distribution $N(0, 1)$, otherwise, the process is said to be out-of-control when it switches to another normal distribution differing in mean, denoted by $N(\delta, 1)$. Sparks (2000) suggested using the ACUSUM chart in the situation where efficient one-step ahead forecast of the future mean shift δ can be made. If δ can be efficiently predicted one-step-ahead at time i, denoted by $\hat{\delta}_i$, then one can optimize the CUSUM statistic by setting k at half size of this predicted value at time i, that is $k_i = \hat{\delta}_i/2$.

An upper-sided ACUSUM chart for detecting the unknown mean shift $\delta > 0$, can be defined by Equation (3), and the lower-sided one can be defined analogously:

$$\begin{cases} C_0 = 0\,, \\ C_i = \max(\,0, C_{i-1} + X_i - k_i\,)\,, \\ k_i = \hat{\delta}_i/2\,. \end{cases} \tag{3}$$

The control chart signals if any C_i exceeds the decision interval $h(k_i)$. However, this procedure is complicated to implement for its control limit $h(k_i)$ always changing over time, according to the value of k, to achieve the same desired ARL_0 at every step. Therefore, Sparks (2000) used the following modified ACUSUM statistic:

$$\begin{cases} C_0 = 0\,, \\ C_i = \max\{\,0, C_{i-1} + (\,X_i - k_i\,)/h(k_i)\,\}\,, \\ k_i = \hat{\delta}_i/2\,. \end{cases} \tag{4}$$

An alarm is triggered as soon as $C_i > h$, where h is a given constant and close to but not exactly 1 because of the errors in estimates of $\hat{\delta}_i$. Different schemes can be used to obtain $\hat{\delta}_i$. The EWMA operator is the most popular one due to its simplicity and efficiency, see Roberts (1959). The traditional EWMA can be described as a recursive form:

$$Q_i = (1-\lambda)Q_{i-1} + \lambda X_i \tag{5}$$

where λ is a smoothing parameter, and $0 \leq \lambda \leq 1$. In real practice, for detecting the upward mean shifts, there is often a minimum magnitude of interest for early detection, say $\delta_{\min} > 0$. Therefore, for the purpose of improving the efficiency in detecting the shifts larger than $\delta_{\min}$, a modified EWMA operator used for obtaining $\hat{\delta}_i$ can be defined as:

$$\hat{\delta}_i = \max\left\{\,\delta_{\min}, (1-\lambda)\hat{\delta}_{i-1} + \lambda X_i\,\right\}\,, \tag{6}$$

where the initial value $\hat{\delta}_0$ can be set to $\delta_{\min}$ or some other values. It will be shown later, this ACUSUM procedure can be viewed as a two-dimensional transition process $V_i = (\,\hat{\delta}_i, C_i\,)$, which can be modeled as a two-dimensional Markov chain model. Then the run-length performance of this control chart can be evaluated by Markov transition method.

The VSI control chart is first introduced by Reynolds et al. (1988). Resulted from its time flexibility for sampling, this control scheme performs more effective in statistical and economic aspects than its static form. From Reynolds and Arnold (1990), an upper-sided VSI CUSUM control chart can be described as the following procedure:

$$\begin{cases} C_0 = 0\,, \\ C_i = \max(\,0, C_{i-1}\,) + X_i - k\,, \\ k = \delta/2. \end{cases} \tag{7}$$

Let $\{T_i\}$ denote the sampling time interval between ith and $(i+1)$th samples. g is called the "warning line", which makes $\{T_i\}$ switch between t_1 and t_2, where $t_2 < 1 < t_1$. The VSI scheme works out as follows:

$$T_i = \begin{cases} t_1, \text{ if } C_i < g\,, \\ t_2, \text{ if } g \leq C_i < h. \end{cases} \tag{8}$$

For the VSI control chart, the time up to signal is no longer a constant multiple of run length. Therefore, Reynolds et al. (1988) introduced another two measurements to reinforce the comparison, which are:

I. Average time to signal (ATS): the expected value of the time from the start of the process to the time when chart signals (zero-state).
II. Adjusted average time to signal (AATS): the expected value of the time from process shifts to the time when chart signals (steady-state).

3 Our proposed VSI ACUSUM control scheme

The design and some properties of our proposed VSI ACUSUM chart are provided thoroughly in this section.

3.1 The design of our proposed control scheme

As mentioned above, by continuously using data information to adjust itself based an EWMA operator, the ACUSUM control chart is expected to achieve an overall performance for detecting range mean shifts. Also, by adding the variable sampling scheme, one could expect a great increase of the control chart's detection ability from time and economical aspects. It is reasonable to integrate these two features to design a modified ACUSUM control chart with variable sampling scheme to perform more efficiently and economically for range shifts, which we call "VSI ACUSUM" control chart. Here, one upper-sided CUSUM chart is used to illustrate the basic idea of our adaptive chart. Assume that a process $\{X_i\}$ is expected to experience an upward mean shift ($\delta > 0$) at some unknown time. When the mean shift magnitude δ is known, the CUSUM statistic in Equation (2) with $k = \delta/2$ has been shown optimal in detecting the occurrence of the shift δ. To maintain a pre-defined ARL_0, the threshold varies with the reference value k. According to the ARL approximation derived in Siegmund (1985), Shu and Jiang (2006) established the relationship between the threshold and reference value for the conventional upper CUSUM chart as

$$h(k) = \frac{\ln(1 + 2k^2\text{ARL}_0 + 2.332k)}{2k} - 1.166\,. \tag{9}$$

In practice, when δ is unknown, one can use its estimate $\hat{\delta}_i$ to substitute for δ in the reference value of the upper-sided CUSUM chart to obtain an adaptive CUSUM procedure. This gives rise a varying reference value derived from the real time observations. Note that for a fixed value of ARL_0, $h(k)$ is a decreasing function of k. Thus, monitoring the CUSUM statistic with a fixed threshold implies relatively tight control for small shifts and relatively loose control for large shifts. Clearly, this results in different sensitivities to different levels of mean shifts. To balance the detection sensitivity to both small and larger shifts, a standardization of the offset, X_i, $\hat{\delta}_i/2$, based on $h(\hat{\delta}_i/2)$ would be desirable. From another aspect, with the VSI feature added in the control scheme, it is necessary to modify the new adaptive CUSUM statistic based on tradition VSI CUSUM form (for more detailed information about such modification, see Reynolds and Arnold (1990)). For such two reasons above, our proposed VSI ACUSUM control chart (upper-sided) is described as follows, and the lower-sided one can be derived analogously,

$$\begin{cases} C_0 = 0\,, \\ C_i = \max(\,0, C_{i-1}\,) + (X_i - k_i)/h(k_i), \\ k_i = \hat{\delta}_i/2\,, \end{cases} \tag{10}$$

where $\hat{\delta}_i$ can be obtained by Equation (6) and the VSI scheme works out as described in Equation (8). The control chart signals if C_i exceeds h and g is the warning line for partition the in-control region $(-\infty, h\,)$, and theocratically can be computed from the following equations (also see Reynolds and Arnold (1990) for more detailed information),

$$\begin{cases} \rho_1 t_1 + \rho_2 t_2 = 1\,, \\ \qquad \rho_1 + \rho_2 = 1\,, \end{cases} \tag{11}$$

where ρ_1 and ρ_2 denote the probabilities of the ACUSUM statistic falling into $(-\infty, g\,]$ and $(\,g, h\,]$, respectively, with the condition that no false alarm happens before.

3.2 The effects of parameters

Sparks (2000) suggested choosing $\hat{\delta}_0$ as the best guess of the future mean shift or assuming $\hat{\delta}_0 = 1$ without much investigation on its effect. Because $\delta_{\min}$ is the lower bound of the EWMA estimator, it is natural that $\hat{\delta}_0 \geq \delta_{\min}$. In general, a large value of $\hat{\delta}_0$ improves the sensitivity of control chart to large shifts but reduces the sensitivity to small shifts, and vice versa for a small value of $\hat{\delta}_0$. It is interesting to note that $\hat{\delta}_0$ has small effect on the ATS_0 but does have great effect on the OC ATS. The zero-state ATS of VSI ACUSUM charts with $\hat{\delta}_0$ = 0.5, 1.0, 1.5, 2.0 and 2.25 are given in Table 1, where the VSI ACUSUM charts are designed with the same parameters: h = 1.1681, g = 0.122, $\lambda = 0.1$, $\delta_{\min} = 0.5$, $t_1 = 1.9$, and $t_2 = 0.1$.

Table 1
The zero-state ATS of VSI ACUSUM charts with different $\hat{\delta}_0$.

	$\hat{\delta}_0$				
δ	0.5	1.0	1.5	2.0	2.25
0.00	392.80	395.14	396.62	398.54	400
0.50	9.97	10.81	13.12	15.97	17.56
1.00	2.92	3.16	3.91	5.19	5.97
1.50	1.35	1.35	1.56	2.00	2.30
2.00	0.74	0.69	0.74	0.88	0.99
2.50	0.45	0.40	0.40	0.44	0.48
3.00	0.31	0.26	0.25	0.25	0.26
3.50	0.24	0.20	0.18	0.17	0.17
4.00	0.20	0.17	0.15	0.13	0.13

It can be seen that the same control limit and warning line provide nearly the same ATS_0 for all VSI ACUSUM charts with different values of $\hat{\delta}_0$. This insensitivity of the ATS_0 to $\hat{\delta}_0$ indicates a very good property because the OC run-time performance can be adjusted only by changing $\hat{\delta}_0$ without changing other parameters. This is very similar to the effect of a head-start on the traditional CUSUM and EWMA charts, see Lucas (1982).

Suppose $[\delta_1, \delta_2]$ is the range of potential mean shifts needed for detection. Based on the discussion of Sparks (2000) and Shu and Jiang (2006), we recommend the following guidelines for working out our VSI ACUSUM scheme:

1. Select $\delta_{\min} = \delta_1$, which improves the detection performance for shifts $\delta \geq \delta_1$.
2. Set $\hat{\delta}_0 = (\delta_1 + \delta_2)/2$. Although large value of $\hat{\delta}_0$ can offer a very good performance for detecting large shift δ_2, it often substantially deteriorates the performance of detecting other small shifts. Therefore, we choose $\hat{\delta}_0$ to be the midpoint of $[\delta_1, \delta_2]$ to balance the detection efficiency.
3. For the VSI scheme, we recommend to use the symmetric form, that is t_1 and t_2 are symmetric about one unit time, for example, $t_1 = 1.9$, $t_2 = 0.1$.
4. Choose λ based on the rule of thumb. Here, we recommend $\lambda \in [0.1, 0.2]$.
5. Find the h and g to achieve the desired ATS_0 by using either the Markov method given in the Appendix or simulation.

3.3 Some computational aspects

Based on Markov chain method in Appendix, all the results are implemented in Fortran 95 program with IMSL package. Just as mentioned in Appendix, the performance of VSI ACUSUM chart can be evaluated by a two-dimensional Markov chain model. Therefore, it means that the transition space L: $[h^*, h] \times [\delta_{\min}, \delta_{\max}]$ can be divided into two subregions, L_1: $[h^*, g] \times [\delta_{\min}, \delta_{\max}]$ and L_2: $(g, h] \times [\delta_{\min}, \delta_{\max}]$, where h is the control limit, h^* could be chosen small enough and $\delta_{\max}$ be large enough to approximate the values of ATS. Assume that the number of states along axis $\hat{\delta}$ over the range $[\delta_{\min}, \delta_{\max}]$ is m_δ, along axis C over the range $[h^*, g]$ is m_{c1} and along the range $[g, h]$ is m_{c2}. Then, the in-control region L is divided into a number of $N = (m_{c1} + m_{c2}) \times m_\delta$ two-dimensional rectangles. Routine "nordf" is used to evaluate the corresponding probabilities of the VSI ACUSUM statistic transiting from one state to another. Routine "blinf" is used to obtain the ARL and ATS by the expressions in Appendix. The computation time is highly correlated to the total dimension N. That implies larger value of N will lead the result with higher accuracy, but more time consuming, and vice versa. To balance the time and accuracy, we recommend that $N = 2400$, where $m_{c1} = 30$, $m_{c2} = 30$ and $m_\delta = 40$. The execution time is less than 5 minutes on a Pentium 4 with CPU processor 3.00 GHz. However, smaller N never represents lack of accuracy. For example, for zero-state situation, if the desired ARL_0 and ATS_0 is 400,

$t_1 = 1.9$, $t_2 = 0.1$ and $\delta_{min} = 0.5$, by using $N = 2400$, the results is $h = 1.1681$ and $g = 0.118$, the computational time is around 5 minutes; while, by using $N = 600$ ($m_{c1} = 15$, $m_{c2} = 15$ and $m_\delta = 20$), which is a quarter of 2400, the results is $h = 1.1663$ and $g = 0.115$, however it only takes less than 1 minute to compute, and also accurate enough. We recommend to practitioners to use high-dimensional transition space ($N = 2400$ is enough), once the parameter is obtained, then it can be used at all times.

4 Performance Comparisons

To investigate the performance of our proposed VSI ACUSUM chart for detecting the range shifts in the process mean, some comparisons are carried out in this section.

4.1 The comparisons between VSI ACUSUM and FSI charts

In this subsection, it is reasonable to compare the performance of VSI ACUSUM with other fixed sampling interval (FSI) control charts for detecting range shifts. Here, the charts used for comparison are FSI ACUSUM, Dual CUSUM suggested by Zhao et al. (2005), the Combined Shewhart-CUSUM suggested by Lucas (1982). The Combined Shewhart-CUSUM(SC) control chart can achieve an overall performance in detecting a wide range shifts by adding the Shewhart $\bar{X}$ feature to increase its sensitivity to large mean shifts. The other one, Dual CUSUM(DCUSUM), combines two traditional CUSUMs to monitor the process simultaneously, in which one CUSUM with small reference value k_1 intends to be sensitive to small shifts, and the other with a large reference value k_2 to increase its ability to large shifts. From the aspect of the sampling scheme, here we adopt the most popular symmetric form to work out the VSI scheme: t_1 =1.9, t_2 =0.1. For a fair comparison between VSI and FSI control schemes, all these charts are designed to maintain the equation $\mathrm{ARL}_0 = \mathrm{ATS}_0 = 400$. The numerical results of zero-state ATS and steady-state AATS of the four charts are listed in Tables 2 and 3.

From Tables 2 and 3, from the aspect of time to signal, it is clear to see that the VSI ACUSUM chart substantially improves the detection efficiency of ACUSUM control chart for both zero and steady states, and it performs uniformly better than DCUSUM and combined Shewhart-CUSUM charts. Especially, it greatly improves the detection ability for small and moderate shifts. For example, in the situation of zero-state, when the true magnitude of the process mean shift δ is 0.25 (small mean shift), other three charts (FSI ACUSUM, DCUSUM and SC) takes 67.93, 69.36 and 74.56 to give a signal, respectively.

Table 2
The zero-state ATS for detecting a range shift [0.5,4.0].

δ	VSI ACUSUM h=1.1681 $\delta_{\min}$=0.5 λ=0.1, g=0.118	FSI ACUSUM h=1.1681 $\delta_{\min}$=0.5 λ=0.1	DCUSUM h_1=7.46 h_2=1.21 k_1=0.25, k_2=2.0	SC h_c=8.04 $h_{\bar{X}}$=3.00
0.00	400	400	400	400
0.25	46.19	67.93	69.36	74.56
0.50	17.56	28.35	25.82	27.19
0.75	9.91	16.37	14.83	15.49
1.00	5.97	10.54	10.23	10.62
1.25	3.67	7.22	7.70	7.94
1.50	2.30	5.22	6.07	6.22
1.75	1.49	3.99	4.09	4.99
2.00	0.99	3.20	4.00	4.06
2.25	0.68	2.66	3.29	3.32
2.50	0.48	2.27	2.72	2.74
2.75	0.35	1.99	1.93	2.27
3.00	0.26	1.77	1.68	1.91
3.25	0.21	1.59	1.48	1.64
3.50	0.17	1.45	1.38	1.44
3.75	0.15	1.34	1.34	1.29
4.00	0.13	1.24	1.23	1.19

However, our VSI ACUSUM chart only takes 46.19, which is 47%, 50% and 61% faster than other three charts. Similar conclusion can be obtained for moderate and large mean shifts. Another important advantage of our chart should be noted is that, since the SC and DCUSUM charts are both combined charts, it means two thresholds should be used simultaneously to monitor the process. Such design works always encounter the difficulties, such as the optimal selection of the parameters. On the other side, our chart can utilize only one threshold to monitor the process and its optimal selection can be easily obtained, see Appendix for more detailed computational information.

Table 3
The steady-state AATS for detecting a range shift [0.5,4.0]

δ	VSI ACUSUM	FSI ACUSUM	DCUSUM	SC
	h=1.181 $\delta_{\min}$=0.5 λ=0.1, g=0.122	h=1.181 $\delta_{\min}$=0.5 λ=0.1	h_1=7.51 h_2=1.21 k_1=0.25, k_2=2.0	h_c=8.11 $h_{\bar{X}}$=3.00
0.00	400	400	400	400
0.25	38.98	60.09	64.76	71.08
0.50	10.54	21.88	23.02	24.76
0.75	5.24	12.39	12.95	13.83
1.00	3.61	8.45	8.88	9.42
1.25	2.65	6.34	6.69	7.05
1.50	2.09	5.06	5.30	5.54
1.75	1.74	4.20	4.33	4.48
2.00	1.50	3.60	3.59	3.69
2.25	1.34	3.15	3.00	3.06
2.50	1.23	2.81	2.53	2.56
2.75	1.16	2.54	2.16	2.16
3.00	1.10	2.32	1.86	1.84
3.25	1.07	2.14	1.64	1.60
3.50	1.04	1.99	1.46	1.41
3.75	1.02	1.87	1.33	1.28
4.00	1.01	1.77	1.23	1.18

4.2 The comparisons between VSI ACUSUM and VSI CUSUM charts

In this subsection, we mainly focus on the comparison of the performance of VSI ACUSUM charts with that of the traditional VSI CUSUM charts.

For fair comparison for detecting the range shifts, a new criterion is introduced, which is similar to the IRARL provided by Zhao et al. (2005). The new criterion, called the integral of the relative AATS (IRAATS), is defined by

$$\mathrm{IRAATS}(C) = E\left[\frac{\mathrm{AATS}_c(\delta)}{\mathrm{AATS}_{op}(\delta)}\right] = \int \frac{\mathrm{AATS}_c(\delta)}{\mathrm{AATS}_{op}}\,\mathrm{d}F(\delta), \tag{12}$$

where $\mathrm{AATS}_c(\delta)$, $\mathrm{AATS}_{op}(\delta)$ are the OC AATS of control chart C and CUSUM chart with $k = \delta/2$ for detecting the shift δ, respectively. $F(\delta)$ is the cumulative distribution function (CDF) of shifts δ. If we have no idea about prior information of the shift, the CDF of uniform distribution $U[\,\delta_1, \delta_2\,]$ could be used as $F(\delta)$, which we employ in this paper. To ease the computation, the discrete form of Equation (12) is used to approximate the value of IRAATS, which is given by

$$\mathrm{IRAATS} \approx \frac{1}{m+1} \sum_{i=0}^{m} \frac{\mathrm{AATS}_c(\delta_i)}{\mathrm{AATS}_{op}(\delta_i)}, \tag{13}$$

where m is a given integer, and $\delta_i = \delta_1 + \frac{i}{m}(\delta_2 - \delta_1)$. When comparing the performance of VSI ACUSUM with that of VSI CUSUM control charts, it is necessary to evaluate the values of IRAATS. Apparently, a control chart with a smaller IRAATS value of a particular region is considered to be more effective to detect the shifts in that region, and vice versa. In the following, the IRAATS comparison between VSI ACUSUM and VSI CUSUM charts with different reference value k is carried out. The conclusion for the zero-state ATS are similar, which we would not discuss here for shorting the paper. The comparison is designed as follows: for any pre-specified shifts range $[\,\delta_1, \delta_2\,]$ (here, we use $[\,0.5,\ 4.0\,]$, $[\,0.5,\ 2\,]$, $[\,1,\ 3\,]$ for investigation, respectively), two VSI ACUSUM charts with λ=0.1 and 0.2, and three VSI CUSUM charts with reference values $k = \delta_1/2$, $(\,\delta_1 + \delta_2\,)/2$, and $\delta_2/2$, respectively. The numerical results are given in Tables 4–6.

It can be seen from tables 4–6 that the VSI ACUSUM charts performance better than the traditional VSI CUSUM for detecting specified range shifts in terms of IRAATS. Also, it is shown that VSI ACUSUM chart performs nearly optimally at each point of the specified range. In the real practice, increasing the value of λ will lead the control chart to be more sensitive to moderate and large shifts. But it reduces the ability to detect the small shifts, just as mentioned in section 3. However, based on our computation, we found that λ has small effect on the performance of VSI ACUSUM chart in terms of IRAATS. To balance the performance of detecting both small and large mean shifts, we recommend $\lambda \in [\,0.1,\ 0.2\,]$ for practical use based on the rule of thumb. Though it can be seen that VSI ACUSUM performs slightly better but nearly the same as a VSI CUSUM chart with the reference value k set at $\delta_{\min}/2$, but here we recommend to use the VSI ACUSUM scheme for the following three reasons. First, the true magnitude of the process shift is always seldom known in real practice, the VSI ACUSUM can be a better monitor scheme due to its self-adaption feature. Second, by using the EWMA operator, one can also get a one-step ahead forecast about the process mean level, which can not in the traditional CUSUM. Third, it is well known that the VSI feature never effects the ARL performance of a control chart. But the

Table 4
The steady-state AATS and IRAATS (range shift [0.5, 4.0])

δ	VSI ACUSUM		VSI CUSUM			Optimal AATS
	λ		k			
	0.1	0.2	0.25	1.125	2	
0.00	400	400	400	400	400	400
0.50	10.54	10.62	10.44	34.29	59.06	10.44
0.75	5.24	5.33	5.21	11.59	23.75	5.04
1.00	3.61	3.52	3.40	4.77	10.13	2.99
1.25	2.65	2.60	2.52	2.53	4.79	2.05
1.50	2.09	2.05	2.01	1.70	2.64	1.58
1.75	1.74	1.71	1.69	1.35	1.75	1.32
2.00	1.5	1.48	1.49	1.18	1.35	1.17
2.25	1.34	1.32	1.34	1.09	1.16	1.09
2.50	1.23	1.21	1.25	1.03	1.06	1.03
2.75	1.16	1.14	1.18	1.00	1.01	0.99
3.00	1.10	1.08	1.13	0.97	0.97	0.97
3.25	1.07	1.05	1.10	0.95	0.95	0.95
3.50	1.04	1.02	1.07	0.94	0.93	0.93
3.75	1.02	1.00	1.05	0.93	0.92	0.92
4.00	1.01	0.99	1.03	0.92	0.92	0.92
IRAATS	1.17	1.15	1.18	1.31	1.89	\

VSI ACUSUM works better in aspect of ATS performance, which means the VSI ACUSUM tends to take fewer samples on average to give a signal than that of VSI CUSUM, see Sparks (2000).

5 Conclusion and Remarks

The ACUSUM control chart has been developed to achieve an overall performance for range shifts. By adding the VSI feature, the ACUSUM control chart performs better than its FSI form in terms of run-time performance. Moreover, just like its FSI form, the VSI ACUSUM chart inherits its good overall property for detecting the range shifts when compared with different VSI CUSUM charts. There is no technical difficulty in extending the result

Table 5
The steady-state AATS and IRAATS (range shift [0.5, 2.0])

δ	VSI ACUSUM		VSI CUSUM			Optimal AATS
	λ		k			
	0.1	0.2	0.25	0.625	1	
0.00	400	400	400	400	400	400
0.50	10.54	10.62	10.44	17.20	29.63	10.44
0.75	5.24	5.33	5.21	6.01	9.83	5.04
1.00	3.61	3.52	3.40	3.08	4.16	2.99
1.25	2.65	2.60	2.52	2.05	2.33	2.05
1.50	2.09	2.05	2.01	1.59	1.64	1.58
1.75	1.74	1.71	1.69	1.35	1.33	1.32
2.00	1.5	1.48	1.49	1.21	1.17	1.17
IRAATS	1.21	1.17	1.18	1.14	1.48	\

Table 6
The steady-state AATS and IRAATS (range shift [1.0, 3.0])

δ	VSI ACUSUM		VSI CUSUM			Optimal AATS
	λ		k			
	0.1	0.2	0.5	1	1.5	
0.00	400	400	400	400	400	400
1.00	3.12	3.06	2.99	4.16	7.09	2.99
1.25	2.13	2.08	2.09	2.33	3.41	2.05
1.50	1.61	1.64	1.65	1.64	2.03	1.58
1.75	1.37	1.39	1.41	1.33	1.47	1.32
2.00	1.23	1.24	1.26	1.17	1.23	1.17
2.25	1.14	1.15	1.17	1.09	1.10	1.09
2.50	1.08	1.09	1.11	1.04	1.03	1.03
2.75	1.04	1.04	1.06	1.00	0.99	0.99
3.00	1.02	1.01	1.04	0.98	0.97	0.97
IRAATS	1.04	1.04	1.06	1.09	1.28	\

to the two-sided control scheme. However, the only possible obstacle is that the ATS computation could be much more complicated by too many transient states in the Markov Chain. There are 4 parameters need to be defined instead of two in one-sided scheme. So the transient space may be totally large and out of computation ability. If the state space of each parameter has 10 values, then there will be nearly 10^4 states in the Markov chain. Currently it is not easy to invert such a large matrix. In that case, we have to resort to simulation.

Some other interesting issues about ACUSUM scheme can be pursued in the future works. For example, the EWMA operator here only acts as a smoother, but there is no use of it to give a signal. Just as we know, EWMA chart with large value of λ has similar performance as a Shewhart $\bar{X}$ chart for large shifts. It is reasonable to use this good property for evaluating the process mean level and also giving a signal. Moreover, the VSS, VSSI features added to ACUSUM are also reasonable to increase the detection performance.

Finally, something important should be noted here: like any literature about CUSUM or EWMA control charts, our proposed VSI ACUSUM charts make good detection performance in a boarder range of mean shifts, especially in small and moderate shifts. Here, we give two suggestions to further improve the detection for large shifts of our charts: 1. Superimposing a Shewhart control limit on our VSI ACUSUM chart will make our chart perform more robustly for both small and large mean shifts, just like combined Shewhart-CUSUM charts; 2. Another way, using an Markovian-type mean estimator instead of Equation (6), where the restriction $\delta \geq \delta_{min}$ is no longer considered, will help the EWMA estimator to overcome the steady-state inertia, which may not lead it to perform well in estimating both small and large mean shifts. A Markovian-type mean estimator is given by

$$\hat{\delta}_i = \hat{\delta}_{i-1} + \phi(a_i), \tag{14}$$

where $a_t = X_i - \hat{\delta}_{i-1}$ is the prediction error and $\phi(\cdot)$ is a monotone function. This type of EWMA mean estimator was first proposed by Yashchin (1995) for estimating the current process mean subject to abrupt changes and was suggested by Capizzi and Masarotto (2003) for process monitoring. Based on the Huber's score function (Huber (1981)), Shu et al. (2008) suggested using a new Markovian-type EWMA operator to estimate the process mean level, where the monotone function $\phi(\cdot)$ is given by

$$\phi_\eta(a) = \begin{cases} a + (1-r)\eta, & a < -\eta, \\ \eta\, a, & |\,a| \leq \eta, \\ e - (1-r)\eta, & a > \eta, \end{cases} \tag{15}$$

where η is a constant. It is interesting to note that when $\eta \to \infty$, this EWMA operator reduces to the regular EWMA statistic without restriction (4). However, it is not the main purpose of this paper. By using these two ways above, it could indeed improve the detection ability at large mean shifts, and at the same time it may sacrifice the efficiencies for both small and moderate mean shifts. Therefore, whether suggestion 1 or 2 is used, it could hardly change the properties of the overall performance on range shifts of our VSI ACUSUM charts.

Appendix: The Markov model for computing the ARL and ATS of VSI ACUSUM

Here, g is the warning line which decides the next sampling interval corresponding to the current sample reading, h is the upper control limit, h^* could be chosen small enough and $\delta_{\max}$ be large enough to approximate the values of ATS. Then the transition space L: $[\,h^*, h\,] \times [\,\delta_{\min}, \delta_{\max}\,]$ can be divided into two subregions, L_1: $[\,h^*, g\,] \times [\,\delta_{\min}, \delta_{\max}\,]$ and L_2: $(\,g, h\,] \times [\,\delta_{\min}, \delta_{\max}\,]$. Assume that the number of states along axis $\hat{\hat{\delta}}$ over the range $[\,\delta_{\min}, \delta_{\max}\,]$ is m_δ, along axis C over the range $[\,h^*, g\,]$ is m_{c1} and along the range $[\,g, h\,]$ is m_{c2}. Then the width of each segment of axis $\hat{\hat{\delta}}$, denoted by w_δ, is $2(\,\delta_{\max} - \delta_{\min}\,)/(\,2m_\delta - 1\,)$, except the width of the first segment is $w_\delta/2$. Similarly, the width of each segment of axis C over the range $[\,h^*, g\,]$, denoted by w_{c1}, is $2(\,g-h^*\,)/(\,2m_{c1}-1\,)$, except the width of the first segment is $w_{c1}/2$; the width over the ranger $[\,g, h\,]$, denoted by w_{c2}, is $(\,h-g)\,/m_{c2}$. The states along the axis C and $\hat{\hat{\delta}}$ are labeled respectively as $i = 1, 2, .., m_{c1}, m_{c1}+1, ..., m_{c1}+m_{c2}$ and $j = 1, 2, ..., m_\delta$. The center points of state i along the axis C and state j along the axis $\hat{\hat{\delta}}$ are denoted by $v_c(i)$ and $v_\delta(j)$. Therefore, the in-control region L is divided into a number of $N = (\,m_{c1} + m_{c2}\,)\times m_\delta$ two-dimensional rectangles.

Let $f_{(\,i,j\,),(\,m,n\,)}$ be the transition probability of $(\,C, \hat{\hat{\delta}}\,)$ from state $(\,i, j\,)$ to state $(\,m, n\,)$. Define

$$U_c(i,m,n) = \left\{\, v_c(m) - \max\,[\,0,\, v_c(i)\,] + \frac{w_c}{2} \right\}\, h\left(\frac{v_\delta(n)}{2}\right) + \frac{v_\delta(n)}{2}\,,$$

$$L_c(i,m,n) = \begin{cases} \left\{\, v_c(m) - \max\,[\,0,\, vc(i)\,] - \frac{w_c}{2} \right\}\, h\left(\frac{v_\delta(n)}{2}\right) + \frac{v_\delta(n)}{2}\,, & m \neq 1, \\ \\ -\infty\,, & m = 1, \end{cases}$$

$$U_\delta(j,n) = \frac{1}{r}\,\{\, v_\delta(n) - (1-r)\, v_\delta(j) + w_\delta(n)/2 \,\}\ ,$$

$$L_\delta(j,n) = \begin{cases} \frac{1}{r}\{\, v_\delta(n) - (1-r)\, v_\delta(j) - w_\delta(n)/2 \,\} \ , & n \neq 1, \\ -\infty \, , & n = 1. \end{cases}$$

Then, when $m \neq 1$ and $n \neq 1$, the transition probability $f_{(i,j)(m,n)}$ can be evaluated as follows

$$\begin{aligned} f_{(i,j)(m,n)} &= Pr(\, C_t \in state\ m,\, \hat{\delta}_t \in state\ n \mid C_{t-1} \in state\ i,\, \hat{\delta}_{t-1} \in state\ j\,) \\ &\approx Pr\{\, v_c(m) - \frac{w_c(m)}{2} \ < \ \max\,[\,0, v_c(i)\,] + \frac{X_t - \frac{v_\delta(n)}{2}}{h(\frac{v_\delta(n)}{2})} \ \leq \ vc(m) \\ &\quad + \frac{w_c(m)}{2} \ , \ v_\delta(n) - \frac{w_\delta(n)}{2} \ < \ (1-r)\, v_\delta(j) + rX_t \ \leq \ v_\delta(n) + \frac{w_\delta(n)}{2} \} \\ &= Pr\{\, L_c(i,m,n) < X_t \leq U_c(i,m,n) \ , \ L_\delta(j,n) < X_t \leq U_\delta(j,n)\} \\ &= Pr\,\{\, \max\,[\, L_c(i,m,n) \ , \ L_\delta(j,n)\,] \ < \ X_t \ \leq \ \min\,[\, U_c(i,m,n) \ , \ U_\delta(j,n)\,]\,\}\,. \end{aligned}$$

Similarly, we obtain

$$f_{(i,j),(m,n)} = \begin{cases} Pr\{\, L_\delta(j,n) \ < \ X_t \ \leq \ \min\,[\, U_c(i,m,n), U_\delta(j,n)\,]\,\}, & m = 1, n \neq 1, \\ Pr\{\, L_c(i,m,n) < X_t \ \leq \ \min\,[\, U_c(i,m,n), U_\delta(j,n)\,]\,\}, & m \neq 1, n = 1, \\ Pr\{\, X_t \ \leq \ \min\,[\, U_c(i,m,n), U_\delta(j,n)\,]\,\}, & m = 1, n = 1. \end{cases}$$

Therefore, the VSI ACUSUM can be viewed as an two-dimensional Markovian process with transition matrix $\mathbf{R}$, which is an $N \times N$ matrix:

$$\mathbf{R}_{[(i-1)m_\delta + j,\, (m-1)m_\delta + n]} = f_{(i,j),(m,n)},$$

And then, the zero-state ARL and ATS of the VSI ACUSUM procedure can be evaluated by the following equations:

$$\text{ARL} = \pi_0^{'} \cdot (\mathbf{I} - \mathbf{R})^{-1} \cdot \mathbf{1},$$

$$\text{ATS} = \pi_0^{'} \cdot (\mathbf{I} - \mathbf{R})^{-1} \cdot \mathbf{t},$$

where $\mathbf{1}$ is a column vector with all its elements are 1, and $\mathbf{t}$ is also a column vector whose element is either t_1 or t_2, π_0 is any initial probability vector of states and $\mathbf{I}$ is an identical matrix. Moreover, the steady state ATS and AATS can be obtained by:

$$\text{ATS} = \pi_s^{'} \cdot (\mathbf{I} - \mathbf{R})^{-1} \cdot \mathbf{t},$$

$$\text{AATS} = \pi_s^{'} \cdot \{(\mathbf{I}-\mathbf{R})^{-1} - \frac{\mathbf{I}}{2}\} \cdot \mathbf{t},$$

where π_s is the steady state probability, which can be obtained by $\pi_s^{'} \cdot \mathbf{R} = \pi_s^{'}$. For more detailed information about CUSUM procedure's ARL and ATS computation, see Brood and Evans (1972).

Acknowledgements

The authors gratefully acknowledge the constructive comments of a co-editor and two anonymous referees that are very helpful for the improvement of this paper. This paper is supported by Natural Sciences Foundations of China (Grand No. 10771107, 10711120448) and Tianjin (Grand No. 07JCYBJC04300).

References

Arnold, J. C. and Reynolds, M. R. Jr., 2001. CUSUM Control Charts with Variable Sample Sizes and Sampling Intervals. J. Qual. T. 33, 66-81.

Brook, D. and Evans, D. A., 1972. An Approach to the Probability Distribution of CUSUM Run Length. Biometrika. 59, 539-549.

Capizzi, G. and Masarotto, G., 2003. An Adaptive Exponentially Weighted Moving Average Control Chart. Technometrics. 45, 199-207.

Chen, Y. K. and Chiou, K. C., 2005. Optimal design of VSI $\bar{X}$ control charts for monitoring correlated samples. Qual. Rel. Eng. Int. 21, 757-768.

Costa, A. F. B. and De Magalhães, M. S., 2007. An adaptive chart for monitoring the process mean and variance. Qual. Rel. Eng. Int. 23, 821-831.

Hawkins, D. M. and Olwell, D. H., 1998. Cumulative Sum Charts and Charting for Quality Improvement. Springer, Berlin.

Hawkins, D. M. and Qiu, P., 2003. The Change Point Model for Statistical Process Control. J. Qual. T. 35, 355-366.

Huber, P. J., 1981. Robust Statistics. John Wiley & Sons, New York.

Lai, T. L., 2001. Sequential Analysis: Some Classical Problems and New Challenges. Statist. Sinica. 11, 303-350.

Lin, Y. C., and Chou, C. Y., 2005. Adaptive $\bar{X}$ control charts with sampling at fixed times. Qual. Rel. Eng. Int. 21, 163-175.

Lorden, G., 1971. Procedures for Reacting to A Change in Distribution. Ann. Math. Statist. 42, 1897-1908.

Lucas, J. M., 1982. Combined Shewhart-CUSUM Quality Control Schemes. J. Qual. T. 14, 51-59.

Moustakieds, G. V., 1986. Optimal Stopping Time for Detecting Changes in Distributions. Ann. Statist. 14, 1379-1387.

Nikiforov, I. V., 2001. A Simple Change Detection Scheme. Signal Processing. 81, 149-172.

Page, E. S., 1954. Continuous Inspection Schemes. Biometrika. 42, 243-254.

Pollack, M., 1985. Optimal Detection of A Change in Distribution. Ann. Statist. 13, 206-227.

Reynolds, M. R. Jr., 1995. Evaluating Properties of Variable Sampling Interval Control Charts. Sequential Anal. 14, 59-97.

Reynolds, M. R. Jr, Amin, R. W., Arnold, J. C. and Nachlas, J. A., 1988. $\bar{X}$ Charts with Variable sampling intervals. Technometrics. 30 (2), 181-192.

Reynolds, M. R. Jr. and Arnold, J. C., 1990. CUSUM Charts with Variable Sampling Intervals. Technometrics 32. 371-384.

Roberts, G. C., 1959. Control Chart Tests Based on Geometric Moving Average. Technometrics. 1, 239-250.

Runger, G. C. and Pignatiello, J. J., 1991. Adaptive Sampling for Process Control. J. Qual. T. 23 (2), 135-155.

Shu, L. J. and Jiang, W., 2006. A Markov Chain Model for the Adaptive CUSUM Control Chart. J. Qual. T. 38, 135-147.

Shu, L. J., Jiang, W. and Wu, Z., 2008. Adaptive CUSUM procedures with Markovian mean estimation, Comput. Statist. Data Anal. 52, 4395-4409.

Siegmund, D., 1985. Sequential Analysis: Test and Confidence Intervals. Spring-Verlag, New York.

Sparks, R. S., 2000. CUSUM Charts for Signalling Varying Locations shifts. J. Qual. T. 32, 157-171.

Tagaras, G., 1998. A Survey of Recent Developments in The Design of Adaptive Control Charts. J. Qual. T. 30, 212-231.

Wu, Z., Zhang, S. and Wang, P. H., 2007. A CUSUM scheme with variable sample sizes and sampling intervals for monitoring the process mean and variance. Qual. Rel. Eng. Int. 23 (2), 157-170.

Yashchin, E., 1995. Estimating the current mean of a process subject to abrupt changes. Technometrics. 37. 311-323.

Zhao, Y., Tsung, F., and Wang, Z., 2005. Dual CUSUM control scheme for detecting a range of mean shift. IIE Trans. 37, 1047-1058.

Zou, C. L., Wang, Z. J. and Tsung, F., 2008. Monitoring autocorrelated processes using variable sampling schemes at fixed-times. Qual. Rel. Eng. Int. 24 (1), 55-69.