Computational Statistics and Data Analysis 54 (2010) 2244–2252

Contents lists available at ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

A multivariate control chart for simultaneously monitoring process mean and variability

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ARTICLE INFO

Article history:
Received 2 October 2009
Received in revised form 18 March 2010
Accepted 25 March 2010
Available online 8 April 2010

Keywords:
Likelihood ratio test
Multivariate process
Quality control
Reliability engineering
SPC

ABSTRACT

Recently, monitoring the process mean and variability simultaneously for multivariate processes by using a single control chart has drawn some attention. However, due to the complexity of multivariate distributions, existing methods in univariate processes cannot be readily extended to multivariate processes. In this paper, we propose a new single control chart which integrates the exponentially weighted moving average (EWMA) procedure with the generalized likelihood ratio (GLR) test for jointly monitoring both the multivariate process mean and variability. Due to the powerful properties of the GLR test and the EWMA procedure, the new chart provides quite robust and satisfactory performance in various cases, including detection of the decrease in variability and individual observation at the sampling point, which are very important cases in many practical applications but may not be well handled by existing approaches in the literature. The application of our proposed method is illustrated by a real data example in ambulatory monitoring.

1. Introduction

In recent years, there has been a resurgent interest in multivariate control charts. Given the voluminous research in various areas of univariate control charts, research in multivariate control charts is perhaps overdue. This is likely to be so because, in many industrial applications, the quality of a product is often related to several correlated quality characteristics. Several authors have also pointed out that multivariate control charts are an important area of research for the new century (Woodall and Montgomery, 1999; Stoumbos et al., 2000). The purpose of this paper is to contribute to this development.

Multivariate process measurement benefits from the use of inherent multivariate methods rather than a collection of univariate charting methods applied to the individual components. The development of multivariate control charts originates from the work by Hotelling (1947). Recent works have focused mostly on developing control charts for monitoring small changes in the process mean. See Woodall and Ncube (1985), Healy (1987), Crosier (1988), Pignatiello and Runger (1990) and Hawkins (1991, 1993) for accounts of multivariate cumulative SUM (MCUSUM) control charts and Lowry et al. (1992), Runger and Prabhu (1996) and Linderman and Love (2000) for accounts of multivariate exponentially weighted moving average (MEWMA) control charts. Qiu and Hawkins (2001, 2003) proposed a rank-based multivariate CUSUM procedure to detect a shift in the process mean. Other recent works focus on developing procedures for monitoring the process variability. See Alt and Bedewi (1986), Tang and Barnett (1996a,b), Liu (1995), Chan and Zhang (2001), Yeh et al. (2003, 2004, 2005) and Hawkins and Maboudou-Tchao (2008) for example. Generally, the process mean and variance may change simultaneously during the monitoring period. However, monitoring small changes in the multivariate process mean

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doi:10.1016/j.csda.2010.03.027

and variability simultaneously has received little attention in the literature. The few exceptions include the following. The traditional combination of the χ^2 chart and the $|\mathbf{S}|$ chart; Yeh and Lin (2002), in which a box chart was proposed; and Yeh et al. (2003), in which a combined EWMA M-chart and V-chart was developed. Chen et al. (2005) proposed a Max-EWMA (called MEW for abbreviation throughout this paper) chart for monitoring both location and dispersion; Khoo (2005) proposed a bivariate control chart based on the T^2 and $|\mathbf{S}|$ statistics, but this chart is slow to react to small process shifts. Reynolds and Cho (2006) proposed a combination of MEWMA control charts based on sample means and on the sum of the squared deviation from target. Hawkins and Maboudou-Tchao (2008) considered a combination of the MEWMA chart and the multivariate exponentially weighted moving covariance matrix (MEC) chart, which is called the MAC chart here.

Alt (1985) gave a review of multivariate quality control charts and pointed out that an important area worth further research was to develop a single control chart for the simultaneous monitoring of both the process location and dispersion. Therefore, it is desirable to construct a single control chart that can not only detect changes in the process mean, but also is sensitive to the shifts in the process variability. When a single chart is used, the design and operation of the monitoring scheme can be greatly simplified compared to the combination-type chart. Cheng and Thaga (2006) gave an overview of control charts in an effort to use only one chart to simultaneously monitor both the process location and spread in the univariate case. However, due to the complexity of multivariate distributions, these methods cannot be readily extended to multivariate cases. The purpose of this paper is to fulfill this demand.

In this paper, our motivation is to develop a new control chart which maintains the ability to simultaneously monitor, on a single chart, the process mean and process variability for multivariate processes. Our new chart is based on the generalized likelihood ratio (GLR) test and integrates the EWMA procedure. Note that Zhang et al. (2010) proposed a single control chart based on the GLR that simultaneously monitors the process mean and process variability, but it is based on univariate processes. Hawkins and Deng (2009) also look at the GLR-based control chart. Hawkins and Maboudou-Tchao (2008) also considered the GLR; the problem they faced was to monitor the covariance matrix of a multivariate normal process. Our proposed new chart has the following good features. (1) It can be easily designed and constructed because no additional parameter is involved except for the smoothing constant and an upper control limit. (2) Due to the advantages of the classical GLR test, it is quite robust and sensitive to various types of shifts. (3) It is able to handle the case when the sample size is one. The properties of the new chart with respect to the average run length (ARL), which is defined as the average number of samples before the control chart signals an out of control condition, are studied and we find that the new chart is quite sensitive in detecting small and moderate changes in a process.

The rest of this paper is organized as follows. In the next section, our proposed control chart is presented. Then the performance of the proposed chart, from the perspective of the ARL, is evaluated using Monte Carlo simulations and compared to that of some other existing procedures. In the following section, the application of our proposed method is illustrated by a real data example in ambulatory monitoring. In the last section, a conclusion and future research directions are presented.

2. The new chart for monitoring both the mean and the variability

Let $g = (g_1, \ldots, g_p)'$ be a random vector that represents p correlated quality characteristics from a process of interest. When the process is in control, it is assumed that the distribution of g is $N(\mu_0, \mathbf{\Sigma_0})$, a p-dimensional normal distribution with mean vector μ_0 and covariance matrix $\mathbf{\Sigma_0}$, and that both μ_0 and $\mathbf{\Sigma_0}$ are known or their values can be estimated at the end of Phase I process control. Therefore, one can find an appropriate transformation of g, $\mathbf{X} = \mathbf{\Sigma_0}^{-\frac{1}{2}}(g - \mu_0)$, such that in general $\mathbf{X}$ is distributed as $N(\mu, \mathbf{\Sigma})$ when the process is in control, where $\mu = \mathbf{\Sigma_0}^{-\frac{1}{2}}(\mu_0 - \mu_0) = \mathbf{0}$, $\mathbf{\Sigma} = \mathbf{\Sigma_0}^{-\frac{1}{2}}\mathbf{\Sigma_0}\mathbf{\Sigma_0}^{-\frac{1}{2}} = \mathbf{I}_p$ and $\mathbf{I}_p$ is a $p \times p$ identity matrix. In the subsequent discussion, the proposed charts will be developed based on the transformed variable $\mathbf{X}$.

For notational purposes, let $\mathbf{X}_{t1}, \mathbf{X}_{t2}, \ldots, \mathbf{X}_{tn}$, $t = 1, 2, \ldots$, be the tth sample of size n drawn from the process. Also we assume that the random vectors $\mathbf{X}_{tj}$, $j = 1, \ldots, n$, are independent of each other, both within the sample and between samples. Let $\bar{\mathbf{X}}_t = \sum_{j=1}^{n} \mathbf{X}_{tj}/n$ and $\mathbf{S}_t = \sum_{j=1}^{n} (\mathbf{X}_{tj} - \bar{\mathbf{X}}_t)'(\mathbf{X}_{tj} - \bar{\mathbf{X}}_t)/n$ be the tth sample mean vector and sample covariance matrix, respectively.

Next, consider the following hypothesis test.

$$H_0 : \mu = \mathbf{0} \quad \text{and} \quad \mathbf{\Sigma} = \mathbf{I}_p \quad \text{versus} \quad H_1 : \mu \neq \mathbf{0} \quad \text{or} \quad \mathbf{\Sigma} \neq \mathbf{I}_p.$$

It is relatively easy to obtain the generalized likelihood ratio statistic as follows:

$$\mathrm{LR}_t = np(a - \log g - 1) + n\|\bar{\mathbf{X}}_t\|^2, \tag{1}$$

where $a = \frac{1}{p}\mathrm{tr}(\mathbf{S}_t)$, $g = (|\mathbf{S}_t|)^{\frac{1}{p}}$, and $|\cdot|$, $\mathrm{tr}(\cdot)$ denote the determinant and trace of a square matrix and $\|\cdot\|$ represents the Euclidean distance of a vector.

It can be easily checked that $\mathrm{LR}_t \xrightarrow{\mathcal{L}} \chi^2_{\frac{1}{2}p(p+3)}$ as $n \to \infty$. Obviously, a large LR_t leads to rejection of the null hypothesis. The terms $\|\bar{\mathbf{X}}_t\|^2$ and $a - \log g$ contribute to the changes of the process mean and variance, respectively. Unlike other test statistics in the literature, LR_t is a likelihood ratio derived statistic under the setting in which the process mean vector and covariance matrix may change, and thus it naturally adapts to be sensitive to various types of shift combinations. We can give a brief explanation of why the new chart has the ability to detect shifts for $p = 2$.

Suppose that $\mu = (\mu_1, \mu_2)'$ and the variance–covariance matrix $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. We replace $\mathbf{S}$ and $\bar{\mathbf{X}}$ in Eq. (1) with $\mathbf{\Sigma}$ and μ, and we can derive

$$\mathrm{LR} = n[(\sigma_1^2 - \log\sigma_1^2) + (\sigma_2^2 - \log\sigma_2^2) - \log(1-\rho^2) - p] + n(\mu_1^2 + \mu_2^2).$$

The function $f(x) = x - \log x$ is monotonically increasing (decreasing) when $x > 1$ $(0 < x < 1)$ and attains its minimum at $x = 1$. In addition, the function $g(x) = -\log(1 - x^2)$ $(-1 < x < 1)$ attains its minimum at $x = 0$. So the LR statistic will be sensitive to the increase, decrease in variance, the change in correlation and the mean.

In order to detect small or moderate shifts effectively, next we incorporate the EWMA procedure into the construction of LR_t. Here the EWMA scheme is not to directly average the LR_t statistic but rather to get more precise "estimates" of the current process mean vector and covariance matrix, respectively. It is analogous to the construction of multivariate EWMA (Lowry et al., 1992; Chan and Zhang, 2001) control charts to some extent. To be specific, two EWMA statistics based on the sample mean vector $\bar{\mathbf{X}}_t$ and sample covariance matrix $\mathbf{S}_t$ are given by

$$\begin{aligned} \mathbf{u}_t &= \lambda\bar{\mathbf{X}}_t + (1-\lambda)\mathbf{u}_{t-1}, \\ \mathbf{v}_t &= \lambda\mathbf{S}_t^* + (1-\lambda)\mathbf{v}_{t-1}, \end{aligned} \tag{2}$$

where $\mathbf{S}_t^* = \sum_{j=1}^n (\mathbf{X}_{tj} - \mathbf{u}_t)'(\mathbf{X}_{tj} - \mathbf{u}_t)/n$, $\mathbf{u}_0 = \mathbf{0}$, $\mathbf{v}_0 = \mathbf{I}_p$, and λ is a smoothing parameter satisfying $0 < \lambda < 1$. In general, a smaller λ leads to a quicker detection of smaller shifts (Lucas and Saccucci, 1990). As pointed out by an anonymous referee, we can consider using different smoothing parameters for $\mathbf{u}_t$ and $\mathbf{v}_t$. Based on our computational results, a control chart with this modification is only sensitive to some particular shifts. It seems complicated in form and it is not so easy to discuss the optimal choices of different λ. So we do not suggest implementing this method in real practice.

It should be noted, as Huwang et al. (2007) pointed out, that, when $nt \geq p$, $\mathbf{v}_t$ can be used to estimate Σ. Also note that the moving average estimation of the process mean vector $\mathbf{u}_t$ is used in the covariance matrix estimation to replace $\bar{\mathbf{X}}_t$. It would be expected to be more accurate by using these sequentially updated estimations and thus it may improve the ability to detect the possible process change. In fact, Yeh et al. (2003) and Huwang et al. (2007) also advocated to use this formulation. From the simulation results we find that the out-of-control ARL (OC ARL) increases slightly when the variance increases, while it is not ARL biased when the variance decreases. That is to say, the OC ARL is not bigger than the in-control ARL (IC ARL). So from this point, in this paper, we consider this "estimation"-based formulation.

Finally, we substitute $\mathbf{u}_t$ and $\mathbf{v}_t$ for $\bar{\mathbf{X}}_t$ and $\mathbf{S}_t$ in Eq. (1) and obtain the charting statistic (denoted as ELR_t):

$$\mathrm{ELR}_t = np(a' - \log g' - 1) + n\|\mathbf{u}_t\|^2,$$

$t = 1, 2, \ldots,$ where $a' = \frac{1}{p}\mathrm{tr}(\mathbf{v}_t)$ and $g' = (|\mathbf{v}_t|)^{\frac{1}{p}}$. If $\mathrm{ELR}_t > h$, an alarm is triggered, where $h > 0$ is chosen to achieve a specified IC ARL. In this paper, we call this chart the ELR chart.

Our ELR chart is similar to the Hawkins and Maboudou-Tchao (2008) MEC chart, but it has some differences. First, our chart aims for simultaneously monitoring the process mean and variability with a single chart while the MEC chart aims for monitoring changes in the covariance matrix only (see Hawkins and Maboudou-Tchao (2008) for details), so the charting statistics are not the same. Second, the estimation of the process mean is used when estimating the covariance, and after this simple remedy, it can be seen from the next section that the ELR chart is ARL unbiased while the MEC chart is ARL biased. Apparently, unlike box and MEW charts, the ELR chart still works for the case $n = 1$ due to the definition of $\mathbf{v}_t$.

In this paper, the ARL values are found by using 20,000 simulated runs and they corresponded to standard errors of less than 0.5 in the simulated ARL. Table 1 provides the control limits of the ELR chart for various combinations of n and IC ARL for $p = 2$, $p = 3$ and $p = 5$, respectively, when $\lambda = 0.1$ and $\lambda = 0.2$. Note that the control limits are almost the same when n is large enough under the same IC ARL, which is expected because the ELR statistic follows an asymptotic χ^2 distribution. For other choices of parameters, the control limits are available from the authors upon request.

3. ARL comparisons

In this section, we compare the performance of our chart with that of some competing charts.

3.1. ARL comparisons for rational groups

The ARL performance of the ELR chart is studied with different values of λ, n, p, the shift in the process mean vector μ and the change in the process covariance matrix $\mathbf{\Sigma}$. We simulate 20,000 run lengths and use the average to estimate the corresponding ARL. The run length is sufficient long, enabling us to draw reasonable conclusions. In this paper, we only tabulate the zero-state ARLs in order to be consistent with the literatures.

Table 2 tabulates the simulation results for $p = 2$, $n = 2, 5$, IC ARL = 370 and different values of λ. Note that the ELR chart is as effective in detecting changes that only take place in ρ as it is in detecting changes that also occur in σ_1^2 or σ_2^2 or both. Also note that since the in-control values of the means and the correlation coefficient are zero, due to symmetry,

Table 1
The control limits of the ELR chart for various combinations of p, n and IC ARL when $\lambda = 0.1$ and $\lambda = 0.2$.

	n	$\lambda = 0.1$					$\lambda = 0.2$				
		IC ARL					IC ARL				
		185	200	370	500	1000	185	200	370	500	1000
$p = 2$	1	0.742	0.752	0.836	0.877	0.968	1.695	1.718	1.872	1.949	2.115
	2	0.745	0.758	0.847	0.888	0.983	1.711	1.728	1.896	1.977	2.156
	5	0.751	0.758	0.855	0.896	0.991	1.723	1.745	1.915	1.998	2.186
	8	0.751	0.765	0.855	0.898	0.995	1.725	1.746	1.918	2.005	2.191
	10	0.751	0.765	0.855	0.898	0.995	1.726	1.746	1.922	2.008	2.196
	15	0.751	0.765	0.855	0.898	0.995	1.726	1.747	1.923	2.010	2.201
$p = 3$	1	1.080	1.096	1.199	1.246	1.352	2.455	2.478	2.669	2.752	2.950
	2	1.090	1.105	1.208	1.256	1.365	2.464	2.490	2.685	2.781	2.985
	5	1.094	1.110	1.214	1.263	1.375	2.468	2.495	2.698	2.788	3.008
	8	1.095	1.110	1.214	1.264	1.377	2.470	2.496	2.701	2.797	3.014
	10	1.095	1.111	1.215	1.266	1.378	2.470	2.498	2.702	2.798	3.014
	15	1.096	1.111	1.217	1.266	1.378	2.471	2.498	2.703	2.799	3.015
$p = 5$	1	1.923	1.941	2.071	2.133	2.264	4.311	4.341	4.579	4.582	4.934
	2	1.926	1.945	2.077	2.143	2.276	4.308	4.340	4.588	4.713	4.974
	5	1.927	1.945	2.082	2.144	2.280	4.285	4.321	4.575	4.692	4.965
	8	1.929	1.945	2.084	2.144	2.281	4.282	4.316	4.571	4.687	4.959
	10	1.929	1.945	2.084	2.144	2.285	4.280	4.316	4.571	4.688	4.959
	15	1.929	1.945	2.084	2.145	2.285	4.280	4.316	4.571	4.688	4.959

Table 2
The OC ARL values for the ELR chart when $p = 2$, $n = 2, 5$, $\lambda = 0.1, 0.2, 0.3, 0.4$ and IC ARL $= 370$.

$(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$	$n = 2$				$n = 5$			
	λ				λ			
	0.1	0.2	0.3	0.4	0.1	0.2	0.3	0.4
(0.25, 0.25, 1.00, 1.00, 0.00)	48.7	68.3	88.4	109	21.2	26.6	35.3	46.3
(0.50, 0.50, 1.00, 1.00, 0.00)	14.5	15.9	18.7	23.5	7.7	7.1	7.4	8.1
(0.75, 0.75, 1.00, 1.00, 0.00)	8.1	7.6	8.0	8.8	4.5	3.9	3.8	3.7
(1.00, 1.00, 1.00, 1.00, 0.00)	5.4	4.9	4.8	5.0	3.1	2.7	2.6	2.4
(1.25, 1.25, 1.00, 1.00, 0.00)	3.9	3.5	3.4	3.4	2.3	2.1	1.9	1.8
(1.50, 1.50, 1.00, 1.00, 0.00)	3.0	2.7	2.6	2.6	1.9	1.6	1.5	1.4
(1.75, 1.75, 1.00, 1.00, 0.00)	2.4	2.2	2.1	2.1	1.5	1.3	1.2	1.0
(2.00, 2.00, 1.00, 1.00, 0.00)	2.0	1.8	1.8	1.7	1.2	1.1	1.0	1.6
(0.00, 0.00, 0.75, 0.75, 0.00)	10.1	26.8	44.2	72.1	1.0	4.8	10.2	16.2
(0.00, 0.00, 0.60, 0.60, 0.00)	1.0	3.2	8.4	13.6	1.0	1.0	1.0	1.9
(0.00, 0.00, 0.50, 0.50, 0.00)	1.0	1.0	2.2	5.2	1.0	1.0	1.0	1.0
(0.00, 0.00, 0.25, 0.25, 0.00)	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(0.00, 0.00, 1.25, 1.25, 0.00)	17.6	34.3	44.6	51.7	1.3	8.7	14.1	18.8
(0.00, 0.00, 1.50, 1.50, 0.00)	1.0	4.6	8.5	11.3	1.0	1.0	1.5	2.6
(0.00, 0.00, 1.60, 1.60, 0.00)	1.0	1.9	5.0	7.1	1.0	1.0	1.0	1.5
(0.00, 0.00, 1.75, 1.75, 0.00)	1.0	1.0	2.3	4.0	1.0	1.0	1.0	1.0
(0.00, 0.00, 1.25, 1.00, 0.00)	44.0	67.2	83.4	91.5	11.1	23.6	33.2	41.5
(0.00, 0.00, 1.25, 0.75, 0.00)	14.3	31.1	45.5	57.5	1.0	6.8	12.4	18.0
(0.00, 0.00, 1.00, 0.50, 0.00)	1.0	6.0	13.4	22.3	1.0	1.0	1.4	3.1
(0.00, 0.00, 1.50, 0.50, 0.00)	1.0	1.5	5.3	8.3	1.0	1.0	1.0	1.3
(0.00, 0.00, 1.75, 0.25, 0.00)	1.0	1.0	1.0	7.9	1.0	1.0	1.0	1.0
(0.00, 0.00, 1.00, 1.00, 0.25)	85.3	132	159	188	28.7	53.2	78.0	103
(0.00, 0.00, 1.00, 1.00, 0.50)	10.7	27.0	41	55.3	1.0	5.0	10.3	15.3
(0.00, 0.00, 1.00, 1.00, 0.75)	1.0	3.4	8.3	13.4	1.0	1.0	1.1	1.9
(0.00, 0.50, 1.00, 1.00, 0.50)	7.3	13.2	17.6	23	1.0	3.4	5.3	6.8
(0.00, 0.00, 1.50, 1.50, 0.50)	1.0	2.4	5.6	7.7	1.0	1.0	1.1	1.8
(0.00, 0.50, 1.50, 1.50, 0.50)	1.0	2.1	4.7	6.2	1.0	1.0	1.0	1.0
(0.50, 0.50, 1.50, 0.50, 0.50)	1.0	1.1	2.8	4.2	1.0	1.0	1.0	1.0

the simulation results for the case when ρ is negative produce similar comparisons among the competing charts as those seen when ρ is positive, and therefore they are not discussed in this paper. It can be seen that the performance of the ELR chart improves as n becomes larger (for a fixed λ). When the process shift is small, the performance improves as λ becomes smaller (for a fixed n).

We also compare the performance of the proposed ELR chart with that of the box chart, T^2–$|\mathbf{S}|$ and MEW charts aforementioned. Reynolds and Cho (2006) proposed several combinations of multivariate EWMA control charts based on sample means and on the sum of the squared deviation from target. The performance of these charts depends on the direction of the shift in mean or the variance. The result of this dependence on the direction of the shift is that conclusions about which

Table 3
Comparisons of OC ARL for the box chart, $T^2 - |\mathbf{S}|$, MEW and ELR charts when $p = 2$, $n = 4$, $\lambda = 0.2$ and IC ARL = 370.

$(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$	Box chart	$T^2 - \lvert\mathbf{S}\rvert$	MEW	ELR
(0.50, 0.50, 1.00, 1.00, 0.0)	63.5	41.4	7.6	8.5
(0.75, 0.75, 1.00, 1.00, 0.0)	14.8	10.6	3.6	4.6
(1.00, 1.00, 1.00, 1.00, 0.0)	4.9	3.9	2.3	3.1
(0.50, 0.00, 1.00, 1.00, 0.0)	145.	100.	16	16
(0.00, 0.00, 1.25, 1.25, 0.0)	39.	33.	14	13
(0.00, 0.00, 1.50, 1.50, 0.0)	9.2	8.3	4.2	1.0
(0.00, 0.00, 1.25, 1.25, 0.5)	43.	35.	13	2.6
(1.00, 1.00, 1.50, 1.50, 0.0)	2.6	2.3	1.9	1.0
(0.75, 0.75, 1.50, 1.50, 0.0)	3.8	3.4	2.5	1.0
(0.50, 0.50, 1.75, 1.75, 0.5)	3.9	3.4	2.2	1.0

Table 4
The OC ARL values of the ELR and MEW charts when $p = 2$, $n = 2$, $\rho = 0$, $\lambda = 0.2$ and IC ARL = 200.

(σ_1, σ_2)	Charts	c						
		0	0.5	1.0	1.5	2.0	2.5	3.0
(1.00, 1.00)	MEW	200.7	30.6	7.6	4.2	3.0	2.4	2.0
	ELR	200.7	25.3	7.6	4.2	2.8	2.0	1.6
(0.60, 1.00)	MEW	78.5	30.4	8.0	4.3	3.0	2.4	2.0
	ELR	12.6	7.7	4.2	2.7	2.0	1.5	1.2
(1.25, 1.00)	MEW	64.7	22.6	7.1	4.1	2.9	2.3	2.0
	ELR	48.1	17.1	6.7	3.8	2.6	1.9	1.5
(1.25, 2.00)	MEW	7.0	6.3	4.8	3.7	2.9	2.4	2.1
	ELR	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(0.50, 1.50)	MEW	44.9	17.9	7.3	4.4	3.1	2.5	2.1
	ELR	1.3	1.3	1.2	1.1	1.0	1.0	1.0
(0.50, 2.50)	MEW	6.2	5.6	4.5	3.6	2.9	2.5	2.1
	ELR	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(0.50, 0.50)	MEW	10.5	10.9	7.4	4.2	3.0	2.1	2.0
	ELR	1.0	1.0	1.0	1.1	1.1	1.1	1.0
(0.60, 0.60)	MEW	18.4	18.7	8.0	4.2	3.0	2.3	2.0
	ELR	2.1	2.3	2.6	2.5	2.2	1.8	1.3
(0.60, 0.80)	MEW	36.5	28.8	8.1	4.3	3.0	2.3	2.0
	ELR	8.5	6.1	4.0	2.8	2.1	1.6	1.3
(1.25, 1.25)	MEW	28.6	15.2	6.8	4.1	3.0	2.4	2.0
	ELR	26.1	12.5	3.5	3.2	2.2	1.6	1.3
(1.50, 1.50)	MEW	10.1	8.2	5.5	3.8	2.9	2.4	2.0
	ELR	3.6	3.3	2.2	1.6	1.3	1.1	1.0
(2.00, 2.00)	MEW	4.3	4.1	3.6	3.1	2.6	2.2	1.9
	ELR	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(2.50, 2.50)	MEW	2.9	2.9	2.7	2.5	2.2	2.0	1.8
	ELR	1.0	1.0	1.0	1.0	1.0	1.0	1.0
(3.00, 3.00)	MEW	2.3	2.3	2.2	2.1	2.0	1.9	1.7
	ELR	1.0	1.0	1.0	1.0	1.0	1.0	1.0

combination of charts is best for specific shifts are complicated, with the choice of the best combination depending on the type, direction, and size of the shift, and hence in this research we exclude this chart from further investigation.

In order to be consistent with the literatures, the IC ARL is taken as 370 and $n = 4$ is considered. For the two EWMA-type charts, $\lambda = 0.2$ is used for fair comparisons. When the process is out of control, without loss of generality, for $p = 2$, the process mean has been shifted to $\mu = (\mu_1, \mu_2)'$ and the variance–covariance matrix $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. From the top of Table 3, we observe that, if the process shift is only from the mean vector, the MEW chart performs slightly better. The difference between the performance of the MEW chart and our ELR chart, however, is relatively small. For other types of shift, our ELR chart performs significantly better than the other three charts. Other simulations for different values of p, n ($n > 4$), ρ and IC ARL were also done by authors (not reported here), and similar results could be obtained.

Sometimes, the sample size n is very small at one sampling point, say $n = 2$. From Table 3 we can see that the MEW chart does better than the box chart and the combined $T^2 - |\mathbf{S}|$ chart, so we exclude the box chart and the combined $T^2 - |\mathbf{S}|$ chart in the remainder of this paper, and hence we compare our ELR chart only with the MEW chart. In this case, when the process is out of control, we assume that $\mu = (0, c)'$ and $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$.

The results are summarized in Table 4. We also observe that our proposed method uniformly performs significantly better than the MEW chart over the entire range of shifts considered.

Table 5
Comparisons of OC ARL for the ELR and MEW charts when $p = 3$, $n = 2$, $\lambda = 0.2$ and IC ARL $= 200$.

$(c, \sigma_1, \sigma_2, \sigma_3, \rho_1, \rho_2, \rho_3)$	MEW	ELR
(0.00, 0.50, 0.50, 0.50, 0.00, 0.00, 0.00)	6.3	1.0
(0.00, 1.50, 1.50, 1.50, 0.00, 0.00, 0.00)	7.5	2.6
(0.50, 0.50, 0.50, 0.50, 0.00, 0.00, 0.00)	6.8	1.0
(0.50, 1.50, 1.50, 1.50, 0.00, 0.00, 0.00)	6.6	2.4
(0.50, 1.00, 1.00, 1.00, 0.00, 0.00, 0.00)	36.6	32.5
(1.00, 1.00, 1.00, 1.00, 0.00, 0.00, 0.00)	8.4	9.2
(0.00, 0.75, 1.00, 1.00, 0.00, 0.00, 0.00)	156.7	63.3
(0.00, 1.50, 1.00, 1.00, 0.00, 0.00, 0.00)	29.3	16
(0.00, 2.00, 1.00, 1.00, 0.00, 0.00, 0.00)	9.7	1.6
(0.50, 0.75, 1.00, 1.00, 0.00, 0.00, 0.00)	38.4	20.2
(0.50, 1.50, 1.00, 1.00, 0.00, 0.00, 0.00)	17.5	10.7
(0.50, 2.00, 1.00, 1.00, 0.00, 0.00, 0.00)	8.5	1.5
(0.00, 1.00, 1.00, 1.00, 0.50, 0.50, 0.50)	77.2	7.8
(0.00, 1.00, 1.00, 1.00, 0.25, 0.50, 0.75)	65.6	2.1
(0.50, 1.50, 1.50, 1.50, 0.25, 0.25, 0.25)	4.9	2.3
(0.50, 1.50, 1.50, 1.50, 0.25, 0.50, 0.75)	5.0	2.0
(1.00, 0.75, 1.00, 1.00, 0.00, 0.00, 0.00)	8.7	7.7
(1.00, 1.50, 1.00, 1.00, 0.00, 0.00, 0.00)	7.2	5.7

For $p = 3$, when the process is out of control, $\mu = (0, 0, c)'$, $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho_1 & \rho_2 \\ \rho_1 & \sigma_2^2 & \rho_3 \\ \rho_2 & \rho_3 & \sigma_3^2 \end{pmatrix}$ is considered. The results are summarized in Table 5. From Table 5 we can see that our proposed ELR chart still works significantly better than the MEW chart in most cases, especially for detecting only the correlation shifts. For example, when $\rho_1 = 0.25$, $\rho_2 = 0.50$, $\rho_3 = 0.75$, the OC ARL for the MEW chart is 65.6, but for the ELR chart the OC ARL reduces to 2.1. When the process shift is only from the mean vector, i.e., $c = 0.50$ or $c = 1.00$, and the variance and correlation do not change, the MEW chart does a little better than the ELR chart. We also compared our chart with the other two charts (not reported here), and the conclusions are the same.

3.2. Performance of the ELR chart for the individual observation case

In industrial practice, sampling may be expensive and time consuming, and the sample interval may be relatively long. In such cases, individual observation at sampling points is usually considered. However, the MEW chart and the box chart may not be appropriate. Yeh et al. (2005) proposed a maximum multivariate exponentially weighted moving variability control chart (MMV chart) for monitoring process variability with individual observations. They show that this chart is more sensitive than the multiple CUSUM and EWMA charts and is sensitive to the shift in the process mean. Huwang et al. (2007) proposed two control charts, MEWMS (MES) and MEWMV (MEV) charts, based on the traces of the estimated covariance matrices derived from the individual observations. The simulation results show that the MES chart is better than the MEV chart in many cases. They also checked the capability of their chart to detect the shift in the mean, and it was also effective. Hawkins and Maboudou-Tchao (2008) considered the MEC chart for detecting the covariance matrix and the MAC chart for detecting both the process mean and the covariance matrix. Recently, Zhang and Chang (2008) proposed a combined DEWMA-MEWMD (CDM) chart for monitoring the mean vector and variances in the variance–covariance matrix. This section compares the performance of the MMV, MES, MEC, MAC, CDM and ELR charts. In addition, we also compare our proposed chart with the MEWMA (MEA chart) charts of Lowry et al. (1992) for monitoring the mean vector. In order to be consistent with the literature, the IC ARL is taken as 370 and $\lambda = 0.2$ is considered for fair comparison. For the combined MAC chart, the IC ARL was chosen as 700 for each chart so that the combined MAC chart has an IC ARL of about 370.

In our study, we compared the performance of these charts for $p = 2$, i.e., the process has a bivariate normal distribution with $\mu = (\mu_1, \mu_2)'$ and $\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. When the process is in control, it is assumed that $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = \sigma_2^2 = 1$ and $\rho = 0$. We then simulated out-of-control scenarios by generating observations from processes having different bivariate normal distributions. When an observation was generated, it was used to test all competing charts. All the simulated OC ARL values were obtained based on 20,000 Monte Carlo simulations. Note that the focus of the simulation was on cases when either σ_1^2 or both σ_1^2 and σ_2^2 increase with an increase in ρ, or when ρ changes only or the mean vector changes only, or both the mean shifts and the covariance matrix changes occur at the same time in the process.

Table 6 tabulates the simulation results. We can see that, for detecting only the covariance, the MES chart does better than other charts. For detecting only the mean vector, the MEA chart performs better. This is not surprising, because the MEA chart and the MES chart were specially designed for detecting changes in the mean and the covariance matrix, respectively. The MAC, CDM and the ELR charts are designed for monitoring the process mean and variance simultaneously. From the last three columns of this table, we can see that when only σ_1^2 increases, the MAC chart has the best performance for detecting small shifts, i.e., $\sigma_1^2 = 1.25$, and the CDM chart does better for detecting large shifts, $\sigma_1^2 = 1.75$. When both σ_1^2 and σ_2^2 and ρ increase with a small size, i.e., $\sigma_1^2 = \sigma_2^2 = 1.25$, $\rho = 0.25$, the MAC chart performs better, while when both σ_1^2 and σ_2^2

Table 6
Comparisons of OC ARL for various charts with individual observations when $p = 2$, $\lambda = 0.2$ and IC ARL = 370.

$(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$	MEA	MMV	MES	MEC	MAC	CDM	ELR
(0.00, 0.00, 1.25, 1.00, 0.00)	205.1	154.5	144.0	166.7	177.8	207.4	249.6
(0.00, 0.00, 1.75, 1.00, 0.00)	86.1	50.8	43.2	44.8	52.2	27.0	80.7
(0.00, 0.00, 1.25, 1.25, 0.25)	122.1	66.2	69.2	70.0	81.3	139.2	118.3
(0.00, 0.00, 1.75, 1.75, 0.25)	43.7	21.6	18.2	19.2	23.6	12.6	35.3
(0.00, 0.00, 1.25, 1.25, 0.75)	70.1	19.7	46.0	10.6	20.3	109.2	14.1
(0.00, 0.00, 1.75, 1.75, 0.75)	33.1	11.7	17.0	6.1	12.4	12.8	9.0
(0.00, 0.00, 1.00, 1.00, 0.25)	298.3	184.1	298.5	174.2	213.2	363.4	198.8
(0.00, 0.00, 1.00, 1.00, 0.50)	196.1	68.2	191.9	52.8	76.0	350.4	63.4
(0.00, 0.00, 1.00, 1.00, 0.75)	132.8	32.3	126.2	12.6	26.4	272.9	15.0
(0.25, 0.25, 1.00, 1.00, 0.00)	95.6	241.0	228.2	258.2	127.5	228.3	116.2
(0.50, 0.50, 1.00, 1.00, 0.00)	23.6	84.3	80.9	94.1	29.8	84.9	28.2
(1.00, 1.00, 1.00, 1.00, 0.00)	6.7	13.1	13.0	13.1	7.3	13.6	6.4
(0.25, 0.25, 1.25, 1.25, 0.50)	44.6	29.2	45.4	26.1	30.9	105.6	30.3
(0.25, 0.25, 1.25, 1.25, 0.75)	38.8	17.7	38.3	9.6	17.7	96.4	11.4
(0.50, 0.50, 1.25, 1.25, 0.50)	18.4	19.5	27.6	16.9	16.9	63.0	15.0
(0.50, 0.50, 1.25, 1.25, 0.75)	17.6	13.5	24.8	7.7	12.9	61.1	7.5
(0.25, 0.25, 0.75, 0.75, 0.50)	112.5	135.3	838.5	55.1	78.8	123.1	31.6
(0.25, 0.25, 0.75, 0.75, 0.75)	88.9	57.2	450.8	10.2	26.4	104.5	9.5
(0.50, 0.50, 0.75, 0.75, 0.50)	25.9	48.9	192.4	30.5	28.4	56.3	13.9
(0.50, 0.50, 0.75, 0.75, 0.75)	24.4	28.3	133.4	7.9	18.1	54.1	6.2

increase with a large size, i.e., $\sigma_1^2 = \sigma_2^2 = 1.75$, and $\rho = 0.25$, the CDM chart does better. When ρ increases with a large size, i.e., $\rho = 0.75$, and the process variance also increases, the MAC chart does better than the CDM chart, but the ELR chart has the best performance. In other cases, our ELR chart always outperforms the other two charts. Also, we can see that the CDM chart is insensitive to changes in ρ.

Note that, in this paper, we only provide a general guideline on the choice of λ which produces a reasonably good performance for the ELR chart, under a variety of out-of-control scenarios. On the other hand, for a specific λ in $0.1 < \lambda < 0.3$, the ELR chart may not produce the smallest OC ARL for a predetermined IC ARL and a prespecified change in parameters. Although Markovian mean estimation (Shu et al., 2008) should perform better in detecting a range of shifts, we do not investigate it here, for simplicity. In summary, we suggest that a smaller smoothing constant λ, e.g., 0.1, be used in setting the ELR control chart since it gives smaller OC ARL values.

3.3. Diagnosis

When choosing a control chart or combination of control charts to detect and eliminate special causes, a primary consideration should be the ability to signal quickly after a special cause occurs. Another important issue, particularly in the multivariate setting, is the development of procedures that can be employed after a signal for diagnostic purposes. In particular, it is necessary to be able to pinpoint which parameter or parameters have shifted after a signal occurs.

From the traditional perspective on diagnostics, our proposed chart would be problematic because our proposed method is an omnibus chart, and it is sensitive to both mean vector and variance–covariance matrix changes, so it is not easy to diagnose which parameter or parameters have shifted. But just as Reynolds and Cho (2006) pointed out that, in today's environment, control charts are almost always plotted by computer, so after a signal by a control chart, additional control charts or other plots can easily be called up when needed to help diagnose which parameters have changed. For this type of control chart, some diagnostic aids have been proposed and developed in the literature (see, for example, Healy (1987), Hawkins (1991), Runger (1996) and Mason et al. (1995)).

4. A real data example

In this section, the application of our proposed ELR chart is illustrated by a real data example Hawkins and Maboudou-Tchao (2008) used to show the implementation of their MEWMC chart for covariance shifts. The data set is from a long-standing research project in ambulatory monitoring. In this work, subjects were equipped with instruments that measure and record physiological variables. The wearer's blood pressure and heart rate were measured and recorded every 15 min for 6 years. Before analysis using statistical process control (SPC) methods, each week's raw data are condensed into weekly summary numbers, which include mean systolic blood pressure (SBP), mean diastolic blood pressure (DBP), mean of heart rate (HR), and overall mean arterial pressure (MAP). Interested readers are referred to Hawkins and Maboudou-Tchao (2008) for more detail.

In Hawkins and Maboudou-Tchao (2008), the smoothing parameter λ is set to 0.1 and the IC ARL is set to 500. Although we have made a detailed comparative study in last section, we set the same smoothing parameter λ and IC ARL with Hawkins and Maboudou-Tchao (2008) to show the application of our ELR chart more clearly. Note that, for our chart, the control limit

Table 7
Ambulatory monitoring data.

n	U_1	U_2	U_3	U_4	ELR_n
1	0.497	−0.259	−1.249	0.398	0.038
2	1.052	−0.602	−0.878	−2.061	0.186
3	0.510	2.327	0.244	−1.167	0.282
4	1.483	0.671	0.914	0.452	0.269
5	1.664	0.099	−0.735	0.735	0.330
6	0.272	1.683	−0.085	0.519	0.407
7	0.984	1.504	−0.304	0.771	0.608
8	−0.449	1.305	0.952	1.195	0.673
9	0.717	−0.389	−0.299	−0.824	0.681
10	0.309	0.606	−0.207	−0.416	0.766
11	0.867	−1.262	−0.772	0.476	0.772
12	0.435	−1.992	0.064	1.129	0.811
13	−0.581	−1.026	0.295	1.647	0.864
14	1.184	−2.159	−1.140	1.359	1.287
15	0.121	−1.449	−0.564	0.214	1.332
16	−0.714	−0.161	0.122	−1.621	1.098
17	−0.288	−0.924	0.199	−0.625	1.108
18	−1.427	−0.782	0.565	−1.272	1.127
19	−1.327	−0.626	−0.399	−2.818	1.504
20	0.381	1.367	1.352	−2.552	1.518
21	0.296	−0.870	0.579	−0.068	1.401
22	−0.363	−1.029	0.781	0.469	1.389
23	0.412	−0.630	0.194	3.169	1.672
24	−0.208	−0.687	−0.674	−2.351	1.892

h is 1.664 to achieve IC ARL 500 with $\lambda = 0.1$. Table 7 shows the data set taken from Table 5 in Hawkins and Maboudou-Tchao (2008), with label "U_1", "U_2", "U_3" and "U_4", and the ELR statistics with label "ELR_n". Note that "U_1", "U_2", "U_3" and "U_4" are the standardized data for SBP, DBP, HR and MAP, respectively. From Table 7, we observe that the ELR chart gives an OC signal at observation 23, which is consistent with the result of Hawkins and Maboudou-Tchao (2008). This, again, shows that the ELR chart is quite a useful tool for practitioners.

After a signal, the chart gives no direct information on which variable or variables may undergo the shift. The standard approach that addresses this problem is a decomposition of T^2. Hawkins and Maboudou-Tchao (2008) gave a detailed discussion about the diagnosis, so we do not address this problem further here.

5. Conclusions

In this paper, we propose and study a new multivariate charting scheme for simultaneously monitoring the process mean vector and covariance matrix of a multivariate normal process by using a single chart. It is worth noting that the proposed chart can be applied to both the cases when the sample size is one or larger than one. As long as the current stage t satisfies $nt \geq p$, we can use these nt observations to construct the ELR chart for monitoring both the process mean and the covariance matrix.

Huwang et al. (2007) proposed using the trace in their paper to monitor the process variability. As they pointed out, since the trace reduces a complex matrix to a summary statistic, an apparent drawback is that it is insensitive in detecting changes in which the in-control and the out-of-control covariance matrices have the same trace. However, thanks to the good properties of the GLR test and the EWMA procedure, our chart is very effective for diverse cases, including the detection of the individual observation case. When compared with some existing charts, the ELR chart does significantly better in detecting almost all kinds of shifts in the process. The new chart can be easily designed and constructed. Taking consideration of its easy design, implementation and effectiveness, we think that the ELR scheme is a serious alternative in practical applications.

Acknowledgements

The authors gratefully acknowledge the constructive comments of a Co-Editor, an Associate Editor and two anonymous referees that helped to clarify the main points and greatly improved this paper. This research is supported by the NNSF of China Grant 10771107 and the youth foundation of Liaoning University.

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