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The CUSUM Control Chart for the Autocorrelated Data with Measurement Error *

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Abstract

As we know, the measurement error often exists in practice, and affects the performance of quality control in some cases. The autoregressive process with the measurement error is investigated in this paper. For detecting the step shift of the autoregressive process mean with measurement error, a CUSUM control chart based on the maximum log-likelihood ratio test is obtained. Simulated in-control and out-of-control ARL's are made for various measurement error and autocorrelation coefficients. The simulation results show that this new CUSUM scheme works well when the process is negatively autocorrelated.

Keywords: Measurement error, autocorrelated, CUSUM control chart, state-space model, the Kalman Filter.

AMS Subject Classification: 62P30.

§1. Introduction

Statistical process control (SPC) is widely used in industry for process monitoring and quality improvement. Several control charts has been appeared, such as Shewhart chart, cumulative sum (CUSUM, Page (1954)) and exponentially weighted moving average (EWMA, Roberts (1959)). CUSUM and EWMA charts perform better in detecting small or moderate shift than does the Shewhart control chart, but, Shewhart control chart is optimal for detecting the large shift.

There is no measurement error in the traditional control charts. However, the measurement error exists in practice. In general, the measurement error model is as follows:

$$y_t = x_t + e_t,$$

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which is called simple measurement error model, where y_t's are the independent observations, x_t denotes the true value of quality characteristic of interests and comes from $N(1,\sigma_p^2)$, and e_t represents the measurement error and is a normal random variable with zero-mean and variance σ_m^2. Thus, y_t's are independently distributed as $N(\mu,\sigma_p^2+\sigma_m^2)$.

The performance of control charts under the simple measurement error model had been studied by many researchers. Bennett (1954) suggested measurement error can be ignored for Shewhart control chart when the variance of measurement error is smaller than the process's. Abraham (1977) suggested to use the control limits without measurement error as the control limits with measurement error, and then considered the effect of measurement error on chart performance. Kanazuka (1986) considered the effect of measurement error on the power of joint $\overline{X}$ and R charts. He observed that significant measurement error can diminished the power and the larger sample size can recover the lost power. Walden (1990) also discussed the effect of measurement error on the ARL of control charts, including the $\overline{X}$ chart and the R chart, the joint $\overline{X}$ and R chart, and the $\overline{X}$ chart with runs rules, and provided various sampling methods that may be used to compensate for the presence of measurement error, including increased sample sizes, taking repeated measurements etc.

Moreover, Linna and Woodall (2001) generalized the simple measurement error model to a covariate measurement error model as follows:

$$y_t = A + Bx_t + e_t,$$

where A and B are constants. They discussed the impact of measurement error on the performance of control charts and indicated that multiple measurements can be used to increase the power.

Meanwhile, with the widespread use of high-speed data collection schemes, process measurement values are often serially correlated even when there are no special causes of variation affecting the system. This autocorrelation violates the in-control independence assumption associated with many SPC control charting procedures. Such a violation has a significant impact on the performance of the classical SPC procedures. A number of papers have developed various methods to detect the shifts in the mean of autocorrelated process (Johnson and Bagshaw (1974), Bagshaw and Johnson (1975), Harris and Ross (1991), Alwan (1992), Wardell, Moskowitz and Plante (1994), Runger, Willemain and Prabhu (1995), Krieger, Champ and Alwan (1992), Alwan and Alwan (1994), Apley and Tsung (2002), Atienza, Tang and Ang (2002), Yashchin (1993)).

This paper will discuss the effect of measurement error on the CUSUM chart for the autoregressive data. This CUSUM control chart is approximated to be the maximum likelihood ratio test, and its parameter design is close to the design of CUSUM when the data is independently from normal distribution. The simulation results show that measurement error can be ignored when the process is positively autocorrelated and variance of measurement error is smaller than that of the process; but when the process is negatively autocorrelated, it's necessary to decrease the measurement error.

The autocorrelated data model with measurement error is described in Section 2. A new CUSUM statistic is obtained in Section 3. The parameter design of the new CUSUM is given in Section 4. The comparisons with traditional or standard CUSUM chart and standard Shewhart chart are made in Section 5. Section 6 gives several conclusions and discussions.

§2. The Model Description and Some Donations

In this paper, the model of the measurement error is the same as simple measurement error model:

$$y_t = x_t + e_t, \tag{2.1}$$

where e_t is normally distributed as $N(0, \sigma_m^2)$ and independent of x_t, which is a stationary AR(1) series (for simplicity):

$$x_t = \psi x_{t-1} + \varepsilon_t, \tag{2.2}$$

where $|\psi| < 1$, ε_t is a sequence of IID $N(0, \sigma_\varepsilon^2)$.

Combining equations (2.1) and (2.2), the in-control model is given by

$$\begin{cases} y_t = x_t + e_t, \\ x_t = \psi x_{t-1} + \varepsilon_t, \end{cases} \tag{2.3}$$

where e_t and ε_t are independent, e_t is independent of $\{x_t, x_{t-1}, \cdots\}$, and ε_t is independent of $\{x_{t-1}, x_{t-2}, \cdots\}$. In fact, the method in this paper can be easily generalized to the ARMA(p, q) model with measurement error.

Model (2.3) is one of the classical state-space models. So, we can use the results about the state-space model to deal with our detecting problem. Here are some useful conclusions about the state-space model.

For observations $Y_t = (y_1, \cdots, y_t)$, define

$$\mathbf{X}_t = \mathsf{E}(x_t | Y_{t-1}), \qquad \mathbf{P}_t = \mathsf{E}(x_t - \mathbf{X}_t)^2.$$

Since all distributions are normal, $\mathbf{P}_t$ can be rewritten as

$$\mathbf{P}_t = \mathsf{E}((x_t - \mathbf{X}_t)^2 | Y_{t-1}).$$

Let the initial value $\mathbf{X}_0 = 0$, $\mathbf{P}_0 = \mathsf{Var}\,(x_0)$, we can obtain the recursive formula for evaluating the $\mathbf{X}_t$ and $\mathbf{P}_t$ by the Kalman Filter (see Durbin and Koopman (2001)), which is well known in time series analysis and given by

$$\begin{cases} \mathbf{X}_t = \psi[\mathbf{X}_{t-1} + K_{t-1}(y_{t-1} - \mathbf{X}_{t-1})], \\ \mathbf{P}_t = \psi^2(1 - K_{t-1})\mathbf{P}_{t-1} + \sigma_\varepsilon^2, \\ K_t = \mathbf{P}_t/(\mathbf{P}_t + \sigma_m^2), \end{cases} \tag{2.4}$$

where K_t is called *Kalman gain.*

Furthermore, it's easy to get the following conditional expectation and variance under in-control model (2.3):

$$\begin{aligned} \mathsf{E}(y_t|Y_{t-1}) &= \mathsf{E}(x_t|Y_{t-1}) = \mathbf{X}_t, \\ \mathsf{Var}\,(y_t|Y_{t-1}) &= \mathsf{E}[(y_t - \mathbf{X}_t)^2|Y_{t-1}] \\ &= \mathsf{E}[(x_t - \mathbf{X}_t)^2 + 2e_t(x_t - \mathbf{X}_t) + \mathrm{e}_t^2|Y_{t-1}] \\ &= \mathsf{E}[(x_t - \mathbf{X}_t)^2 + \mathrm{e}_t^2|Y_{t-1}] = \mathbf{P}_t + \sigma_m^2. \end{aligned} \tag{2.5}$$

§3. The CUSUM Based on the Maximum Likelihood Ratio

The cumulative sum (CUSUM) control charts were, initially, proposed by Page (1954). Bagshaw and Johnson (1975) proved that the optimal reference value k is equal to the half of the shift to be detected. Brook and Evans (1972) proposed a Markov chain method to evaluate the ARL of one-sided CUSUM procedure.

In this paper, the model parameters ψ, σ_m^2, σ_ε^2 are assumed to be known, and the method in Yashchin (1993), which transforms the autocorrelated data to an independent sequence, will be used.

Suppose a shift of size $\mu > 0$, occurs in $\{x_t\}$ on the $r + 1^{\text{th}}$ step. Therefore, the out-of-control model is

$$\begin{cases} y_t = x_t + e_t, & \\ x_t = \psi x_{t-1} + \varepsilon_t, & t = r, \\ x_t - \mu = \psi x_{t-1} + \varepsilon_t, & t = r + 1, \\ x_t - \mu = \psi(x_{t-1} - \mu) + \varepsilon_t, & t > r + 1. \end{cases} \tag{3.1}$$

Let $\mu_j = \mathsf{E}(y_{r+j} - \mathbf{X}_{r+j})$ $(j = 1)$. Then we have the following results:

Proposition Under the out-of-control model (3.1), the conditional random variables

$$\frac{y_{r+j} - \mathbf{X}_{r+j} - \mu_j}{\sqrt{\mathbf{P}_{r+j} + \sigma_m^2}}\Big|Y_{r+j-1}, \qquad j \geq 1$$

are mutually independently distributed as $N(0,1)$.

It's easy to prove this proposition, so, the proof is omitted here.

For given n observations $(y_1,\cdots,y_n)$, the interesting hypothesis to us is

$$H_0: \mathsf{E}x_1=\cdots=\mathsf{E}x_n=0 \leftrightarrow H_1: \mathsf{E}x_1=\cdots=\mathsf{E}x_r=0, \mathsf{E}x_{r+1}=\cdots=\mathsf{E}x_n=\mu.$$

Obviously, we can use the following maximum likelihood ratio test statistic T_n to test the above hypothesis:

$$T_n=\max_{0\le r\le n-1}\ln\frac{f(y_1,y_2,\cdots,y_r,\cdots,y_n|H_1)}{f(y_1,y_2,\cdots,y_r,\cdots,y_n|H_0)},$$

the rejected region is $T_n>h$, where h is a given constant.

For simplicity, let $\phi(\cdot)$ denote the probability density function of $N(0,1)$. Using the results of conditional distribution and the above proposition, we can rewrite T_n as

$$\begin{aligned} T_n &= \max_{0\le r\le n-1}\ln\frac{\phi\Big(\dfrac{y_{r+1}-\mathbf{X}_{r+1}-\mu_1}{\sqrt{\mathbf{P}_{r+1}+\sigma_m^2}}\Big)\cdots\phi\Big(\dfrac{y_n-\mathbf{X}_n-\mu_{n-r}}{\sqrt{\mathbf{P}_n+\sigma_m^2}}\Big)}{\phi\Big(\dfrac{y_{r+1}-\mathbf{X}_{r+1}}{\sqrt{\mathbf{P}_{r+1}+\sigma_m^2}}\Big)\cdots\phi\Big(\dfrac{y_n-\mathbf{X}_n}{\sqrt{\mathbf{P}_n+\sigma_m^2}}\Big)} \\ &= \max_{0\le r\le n-1}\sum_{j=1}^{n-r}\mu_j\frac{(y_{r+j}-\mathbf{X}_{r+j})-\mu_j/2}{\mathbf{P}_{r+j}+\sigma_m^2}, \end{aligned} \tag{3.2}$$

which is similar to the traditional CUSUM statistic, but there is no the recursive expression for it. However, we could use the method described in Yashchin (1993) to simplify the above statistic.

Appendix gives the convergent limits of $\mathbf{P}_t$, K_t, $\mathsf{E}\mathbf{X}_{r+j}$ and μ_j, respectively, which are

$$\begin{aligned} &\mathbf{P}_\infty=\frac{-\sigma_m^2(1-\psi^2)+\sigma_\varepsilon^2+\sqrt{(\sigma_m^2(1-\psi^2)-\sigma_\varepsilon^2)^2+4\sigma_m^2\sigma_\varepsilon^2}}{2}, \\ &K_\infty=\frac{\mathbf{P}_\infty}{\mathbf{P}_\infty+\sigma_m^2}, \qquad E_\infty=\frac{\psi K_\infty\mu}{1-\psi(1-K_\infty)}, \\ &\mu_\infty=\mu\Big[1-\frac{\psi K_\infty}{1-\psi(1-K_\infty)}\Big]. \end{aligned} \tag{3.3}$$

Using the similar method in Yashchin (1993) to substitute μ_∞ and $\mathbf{P}_\infty$ for μ_j and $\mathbf{P}_{i+j}$ into (3.2), we have

$$T_n\approx\max_{0\le r\le n-1}\sum_{j=1}^{n-r}\frac{\mu_\infty}{\sqrt{\mathbf{P}_\infty+\sigma_m^2}}\frac{(y_{r+j}-\mathbf{X}_{r+j})-\mu_\infty/2}{\sqrt{\mathbf{P}_\infty+\sigma_m^2}}. \tag{3.4}$$

From equations (2.3) we know the sign of μ is the same as μ_∞. Considering an upward shift, so the above statistic T_n is approximately equivalent to the following traditional

CUSUM statistic

$$S_n \approx \max_{0\le r\le n-1} \sum_{j=1}^{n-r} \Big\{ \frac{y_{r+j} - \mathbf{X}_{r+j}}{\sqrt{\mathbf{P}_\infty + \sigma_m^2}} - \frac{\mu_\infty/2}{\sqrt{\mathbf{P}_\infty + \sigma_m^2}} \Big\}. \tag{3.5}$$

To simplify the above expression, let

$$Z_n = \frac{y_n - \mathbf{X}_n}{\sqrt{\mathbf{P}_\infty + \sigma_m^2}}, \tag{3.6}$$

$$k = \frac{\mu_\infty/2}{\sqrt{\mathbf{P}_\infty + \sigma_m^2}} = \mu \frac{1 - \dfrac{\psi K_\infty}{1-\psi(1-K_\infty)}}{\sqrt{\mathbf{P}_\infty + \sigma_m^2}}. \tag{3.7}$$

The statistic S_n can be rewritten in the following recursive form

$$\begin{cases} S_0 = 0; \\ S_n = \max\{S_{n-1} + Z_n - k, 0\} \qquad n = 1, 2, \cdots. \end{cases} \tag{3.8}$$

Note that $\{Z_n\}$, the residual of model (2.3), are independent of each other and almost have the same distributions. And Yashchin (1993) has proved that under some conditions, this approximation is quite precise.

Similarly, if we want to detect an downward shift, then the downward CUSUM can be derived as

$$\begin{cases} D_0 = 0; \\ D_n = \min\{D_{n-1} + Z_n + k, 0\} \qquad n = 1, 2, \cdots, \end{cases} \tag{3.9}$$

where $k > 0$.

Table 1 The out-of-control ARL's of our CUSUM scheme with $\psi = \pm 0.5$, $\sigma_m^2/\sigma_x^2 = 1$, in-control ARL = 300

	$\psi = -0.5$						$\psi = 0.5$					
			mean shift						mean shift			
μ	k	h	0.5	1.0	2.0	3.0	k	h	0.5	1.0	2.0	3.0
0.50	0.188	7.45	**32.19**	13.76	6.34	4.17	0.108	9.64	**61.39**	28.42	13.34	8.70
1.00	0.375	4.84	35.93	**12.46**	4.99	3.15	0.217	6.89	64.59	**26.75**	11.12	6.98
2.00	0.750	2.74	52.08	15.19	**4.32**	2.43	0.433	4.35	79.81	30.28	**9.99**	5.63
3.00	1.125	1.83	71.27	21.73	4.73	**2.25**	0.650	3.13	96.32	37.73	10.71	**5.30**

§4. The Design of Parameters

Because the model parameters ψ, σ_m^2 and σ_ε^2 are supposed to be known, $\mathbf{P}_\infty$ and K_∞ are easily obtained by equation (3.3), the reference value k is calculated through equation (3.7) for the given shift μ. Therefore, there is only one parameter h, the decision interval, is necessary to design for given in-control ARL. We use the bisection method to search the decision interval. Moreover, we also calculate the out-of-control ARL by simulation for $\psi = \pm 0.5$, $\sigma_m^2/\sigma_x^2 = 1$, in-control ARL = 300, the results are included in Tables 1.

From Tables 1 we can see that the design of reference value k, which is given by equation (3.7) is optimal. This is the same as that of the traditional CUSUM. In fact, we had got the same conclusions for various values of σ_m^2/σ_x^2 and ψ.

In order to consider the effect of ψ, k, h on the in-control ARL of our CUSUM, we simulate some ARL's, which are shown in Table 2, where $\psi = \pm 0.5$, $\sigma_m^2/\sigma_x^2 = 1$ (for other cases, the results are the same as this case). According to Table 2, the ARL of our CUSUM scheme depends only on the values of (k, h), but ψ and σ_m^2, σ_x^2 (Of course, the reference value k here depends on ψ, σ_m^2 and σ_x^2). This property is similar to the standard CUSUM scheme.

Table 2　The in-control ARL for given (k, h) $(\sigma_m^2/\sigma_x^2 = 1)$

	$\psi = -0.5$				$\psi = 0.5$			
k	0.0	0.50	1.00	1.50	0.0	0.50	1.00	1.50
μ	0.0	3.71	7.42	11.14	0.0	4.16	8.33	12.49
h								
1.00	4.74	11.2	35.3	142.8	4.76	11.1	35.2	142.2
1.50	7.09	21.0	93.4	548.0	7.10	21.2	94.3	549.3
2.00	9.98	38.5	259.6		9.98	38.6	259.7	
3.00	17.33	118.1			17.37	117.5		
5.00	37.90	928.3			38.03	934.5		

Note that the bisection method, which is used in calculating h, will need a great deal of computing time when the in-control ARL is very large. As mentioned in Section 3, the $\{Z_n, n = 1, \cdots\}$ in our CUSUM statistic, which is given by (3.6) are mutually independently distributed as the standard normal distribution. So the decision interval h would be approximatively equal to that in the standard CUSUM design. Hence, for given fixed k and in-control ARL, we can use the decision interval of the standard CUSUM as that of ours.

As we know, for the standard CUSUM control charts, Hawkins and Olwell (1998) has evaluated the in-control ARL for a wide range of k and h values. Comparing the results in Hawkins and Olwell (1998) with those in Table 2, we observed that the in-control ARL's of our CUSUM and the standard CUSUM are almost the same for any pairs of (k, h). So, we can use the results in Hawkins and Olwell (1998) to evaluate the h or in-control ARL of our CUSUM.

Therefore, the steps of our CUSUM design are summarized as follows:

- Determine the model parameters: ψ, σ_m^2 and σ_ε^2;
- Specify the shift μ;
- Calculate $\mathbf{P}_\infty$, K_∞, E_∞ and μ_∞ by equations (3.3);
- Calculate the reference value k by equation (3.7);
- Evaluate the decision value h (in-control ARL) by the results in Hawkins and Olwell (1998) for given in-control ARL(h).

§5. Comparisons

Many researchers, such as Bennett (1954), Kanazuka (1986), have indicated that measurement error could reduce the performance of Shewhart chart for the independent observations. So, we'd like to know the effect of measurement error on the performance of our CUSUM control chart described in (3.8). We will discuss it in this section by simulations.

In this section, the in-control ARL's for all charts are taken to be 300, shifts in the mean are taken to be 0.5, 1.0, 2.0 and 3.0. For better comparisons, we take $\psi = \pm 0.1$, ± 0.9 and $\sigma_m^2/\sigma_x^2 = 0.1$, 10. Three designs of our CUSUM, which are the optimal for detecting mean shift 0.5, 1.0 and 3.0 are considered.

From our CUSUM of Tables 3 and 4, we observed:

1. The measurement error affects the performance of CUSUM chart, i.e., the out-of-control ARL's increase as measurement error increases.
2. When the data is negatively autocorrelated, the increasing of ARL is obvious as σ_m^2 increases, especially, for the strongly negatively autocorrelated data. So, in this situation, the reduction of measurement error is necessary for improving the performance of CUSUM.

3. When the data is positively autocorrelated, and the autocorrelation coefficients are small or moderate, there are two cases need to consider: One is σ_m^2 is larger a lot than σ_x^2, other is σ_m^2 is less than σ_x^2. For the fist case, the effect of measurement error is obvious, but it's not true for the latter. This is the same as obtained by Bennett (1954) for the i.i.d. data.

4. When the data is highly positively autocorrelated, such as $\psi = 0.9$, the effect of measurement error is very small. So, in this case, it is not necessary to reduce the measurement error.

Besides the ARL's of our CUSUM proposed in this paper, we also evaluated the out-of-control ARL's of traditional or standard CUSUM, which are also shown in Tables 3, 4 (See tiny numbers). For the standard CUSUM chart the reference value $k = \mu/(2\sigma_y)$.

Comparing our CUSUM with the standard CUSUM, we can see:

- When the data is negatively autocorrelated, our CUSUM performs uniformly better than the standard in terms of out-of-control ARL for the small measurement error. For example, when $\sigma_m^2/\sigma_x^2 = 0.1$, $\psi = -0.9$ and design shift is 3.0, the ARL's of our CUSUM are 55.76 for detecting an actual shift $\mu = 0.5$, and 1.13 for detecting the shift $\mu = 3$, but the ARL's of standard CUSUM are 170.93 and 3.26, respectively. But, for the large and moderate measurement error, their performance is almost the same.

- When the data is positively autocorrelated, our CUSUM performs worse a bit than the standard CUSUM.

In fact, our CUSUM is one of charts based on the residuals or the forecast error. When σ_m^2/σ_x^2 is small, i.e., the effect of measurement error is very small, the CUSUM is close to a residual chart. Harris and Ross (1991), Longnecker and Ryan (1992) show that the residual chart for AR(1) model may have poor capability to detect the process mean shift. Wardell, Moskowitz and Plante (1994) shows that when the processes were positively autocorrelated, the residual chart does not work very well. Zhang (1997a) showed that sometimes the detection capability of a residual chart is small comparing with that of the X chart. We also receive the same results as them, i.e., our CUSUM performs better than the standard CUSUM when the data is negatively autocorrelated, but when the data is positively autocorrelated. When σ_m^2/σ_x^2 is very large, such as 10, the measurement error plays an important role in the model, and the data is roughly independent, therefore, the performances of our CUSUM and the standard CUSUM are very close.

For comparing, we also evaluated the out-of-control ARL's of standard Shewhart chart. For given the model parameter ψ and in-control ARL, the control limits L are obtained by simulation. The out-of-control ARL's of standard Shewhart chart are given in Table 5.

Table 3 The out-of-control ARL's of our CUSUM and standard CUSUM ($\sigma_m^2/\sigma_x^2=0.1$)

ψ	μ	k	h	mean shift 0.5	1.0	2.0	3.0
-0.9	0.50	0.279(0.104)	5.91(3.16)	18.94(21.70)	7.64(9.28)	3.45(4.58)	2.29(3.27)
	1.50	0.837(0.312)	2.47(2.31)	29.23(72.08)	7.21(13.70)	2.18(4.41)	1.26(2.84)
	3.00	1.674(0.623)	1.08(1.95)	55.76(170.93)	13.44(77.89)	2.20(7.11)	1.13(3.26)
-0.1	0.50	0.260(0.237)	6.18(5.82)	20.99(21.27)	8.66(8.80)	4.03(4.13)	2.72(2.79)
	1.50	0.779(0.712)	2.65(2.62)	31.73(32.52)	8.36(8.60)	2.75(2.89)	1.70(1.80)
	3.00	1.558(1.423)	1.21(1.34)	59.37(64.27)	15.28(17.03)	2.91(3.22)	1.44(1.60)
0.1	0.50	0.216(0.237)	6.91(7.35)	27.02(26.81)	11.34(11.14)	5.24(5.12)	3.49(3.41)
	1.50	0.649(0.712)	3.13(3.19)	38.08(37.03)	10.89(10.58)	3.62(3.47)	2.24(2.13)
	3.00	1.298(1.423)	1.55(1.43)	66.91(62.19)	18.92(17.37)	3.89(3.56)	1.89(1.70)
0.9	0.50	0.025(0.104)	13.93(38.36)	179.77(174.72)	121.70(113.42)	70.00(60.42)	48.18(38.96)
	1.50	0.075(0.312)	11.03(25.41)	183.01(174.98)	122.25(110.28)	67.12(55.63)	44.29(33.66)
	3.00	0.150(0.623)	8.38(14.24)	191.40(179.01)	128.04(113.64)	67.66(53.71)	42.56(29.90)

Table 4 The out-of-control ARL's of our CUSUM and standard CUSUM ($\sigma_m^2/\sigma_x^2=10$)

ψ	μ	k	h	mean shift 0.5	1.0	2.0	3.0
-0.9	0.50	0.034(0.033)	13.31(12.77)	154.71(153.97)	95.61(95.78)	51.41(51.44)	34.86(34.88)
	1.50	0.103(0.099)	9.84(9.46)	158.47(157.87)	95.65(95.62)	47.38(47.39)	30.25(30.29)
	3.00	0.206(0.197)	7.09(6.83)	169.26(169.12)	103.39(103.75)	48.37(48.38)	28.67(28.71)
-0.1	0.50	0.076(0.075)	10.96(10.90)	87.20(86.80)	43.79(43.85)	21.17(21.21)	13.93(13.93)
	1.50	0.227(0.225)	6.72(6.66)	96.68(95.73)	43.44(43.26)	17.40(17.38)	10.54(10.54)
	3.00	0.454(0.450)	4.20(4.17)	116.31(116.93)	53.63(53.39)	17.87(17.94)	9.42(9.42)
0.1	0.50	0.074(0.075)	10.96(11.09)	87.66(88.34)	44.42(44.66)	21.52(21.62)	14.18(14.19)
	1.50	0.223(0.225)	6.79(6.83)	97.95(97.53)	44.54(44.29)	17.80(17.83)	10.82(10.84)
	3.00	0.446(0.450)	4.25(4.28)	117.72(118.42)	53.92(54.29)	18.31(18.41)	9.74(9.68)
0.9	0.50	0.020(0.033)	14.40(18.95)	197.78(193.63)	140.52(135.39)	84.08(79.93)	58.22(54.52)
	1.50	0.061(0.099)	11.74(14.55)	199.23(192.89)	139.60(134.20)	80.95(75.69)	54.14(49.91)
	3.00	0.121(0.197)	9.19(10.45)	203.58(197.79)	144.53(137.61)	81.13(74.99)	52.07(47.38)

Comparing our CUSUM with Shewhart chart in Table 5, we have:

- If the measurement error is very large, our CUSUM performs uniformly better than standard Shewhart chart.
- If the measurement error is very small, standard Shewhart chart performs a little better than our CUSUM in detecting large shift, especially when the data is highly positively autocorrelated.

Thus, we suggest to use our CUSUM scheme to detect the mean shift for the negatively autocorrelated data. Moreover, the design of our CUSUM is very simple, which is very close that of standard CUSUM for IID normal data, but, the design of decision interval h of standard CUSUM must be calculated by simulation. So, when parameter ψ is positive, although our CUSUM is not as good as the standard CUSUM, we still recommend to use our CUSUM rather than standard CUSUM and standard Shewhart chart.

Table 5 The out-of-control ARL's of Shewhart chart

σ_m^2/σ_x^2	ψ	L	mean shift			
			0.5	1.0	2.0	3.0
0.1	-0.9	2.57	175.31	107.09	41.59	17.07
	-0.1	2.71	78.61	25.33	4.68	1.76
	0.1	2.71	79.20	26.12	5.01	1.85
	0.9	2.41	190.58	126.61	58.90	29.87
10	-0.9	2.71	244.99	203.01	139.42	97.76
	-0.1	2.71	191.38	125.67	57.43	28.33
	0.1	2.71	191.81	125.56	57.65	28.54
	0.9	2.70	247.01	206.05	142.78	101.25

§6. Conclusions and Discussions

Through the maximum likelihood ratio test and Kalman filter, a new CUSUM scheme is introduced to detect the mean shift for the autoregressive data with measurement error, and its design is very close to that of the standard CUSUM for the IID data.

The simulation results show the measurement error could be ignored, when the data is positively autocorrelated and the measurement error is very small. When model is negatively autocorrelated, it is necessary to decrease measurement errors.

For simplicity, we only considered the AR(1) model. As a matter of fact, the results in this paper are easily generalized to the ARMA(p,q) model by the Kalman filter.

Moreover, it is not difficult to generalize our one-sided CUSUM to two-sided. According to the results in Yashchin (1985), in-control ARL of two-sided CUSUM could be derived by the following expression

$$\frac{1}{\mathrm{ARL}}=\frac{1}{\mathrm{ARL}_{\mathrm{up}}}+\frac{1}{\mathrm{ARL}_{\mathrm{down}}}.$$

Appendix

First of all, we consider the limit of $\mathbf{P}_t$. By the Kalman Filter, the recursive formula of $\mathbf{P}_t$ is

$$\mathbf{P}_t=\psi^2(1-K_{t-1})\mathbf{P}_{t-1}+\sigma_\varepsilon^2.$$

Because $|\psi^2(1-K_{t-1})|<1$, $\mathbf{P}_t$ is convergent. Then we obtain the limit of K_t

$$K_\infty=\frac{\mathbf{P}_\infty}{\mathbf{P}_\infty+\sigma_m^2}.$$

As $t\to\infty$, we have

$$\mathbf{P}_\infty=\psi^2\mathbf{P}_\infty\Big(1-\frac{\mathbf{P}_\infty}{\mathbf{P}_\infty+\sigma_m^2}\Big)+\sigma_\varepsilon^2,$$

which is a quadratic equation of one variable, and obviously, which has only one positive root. $\mathbf{P}_\infty$ is the positive root and given by

$$\mathbf{P}_\infty=\frac{-\sigma_m^2(1-\psi^2)+\sigma_\varepsilon^2+\sqrt{(\sigma_m^2(1-\psi^2)-\sigma_\varepsilon^2)^2+4\sigma_m^2\sigma_\varepsilon^2}}{2}.$$

Next we consider the convergence of μ_j. Because the expression of $\mathsf{E}(\mathbf{X}_{r+j})$ is given by

$$\begin{aligned}\mathsf{E}(\mathbf{X}_{r+j}) &= \psi\mathsf{E}[\mathbf{X}_{r+j}+K_{r+j-1}(y_{r+j-1}-\mathbf{X}_{r+j-1})]\\ &= \psi(1-K_{r+j-1})\mathsf{E}\mathbf{X}_{r+j-1}+\psi K_{r+j-1}\mathsf{E}y_{r+j-1},\end{aligned}$$

and $|\psi(1-K_{r+j-1})|<1$, both $\mathsf{E}\mathbf{X}_{r+j}$ and μ_j converge to

$$E_\infty=\frac{\psi K_\infty\mu}{1-\psi(1-K_\infty)},\qquad \mu_\infty=\mu\Big[1-\frac{\psi K_\infty}{1-\psi(1-K_\infty)}\Big].$$

Obviously, whether ψ is positive or negative, $1-\psi K_\infty/[1-\psi(1-K_\infty)]$ is greater than 0, so μ_∞ and μ have the same sign.

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带有测量误差的相关数据的CUSUM控制图

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实际中测量误差不仅存在而且在某些情况下还影响质量控制的表现, 本文将考虑带有测量误差的相关数据的监控问题. 为检测这类数据的飘移, 我们给出了一个基于极大似然比检验的CUSUM控制图及其多种情况下的可控与失控的ARL. 模拟结果显示, 当过程负相关时, 我们提出的CUSUM控制图具有良好的表现.

关键词: 测量误差, 相关, CUSUM控制图, 状态空间模型, Kalman滤波.

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