# Self-Starting Control Chart For Linear Profiles

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#### Abstract

Self-starting control chart based on recursive residuals is proposed for monitoring the linear profiles when the nominal values of the process parameters are not known. This chart can detect a shift in either the intercept or the slope or the standard deviation. Due to the good properties of the plot statistics, the proposed chart can be easily designed to match any desired In-control average run length. The simulated results show that our approach has good performance across the range of possible shifts and it can be used during the start-up stages of the process.

**Keywords:** Self-Starting; Average Run Length; Markov Chain; Linear Profiles; EWMA Charts; Recursive Residuals.

### 1 Introduction

Statistical Process Control (SPC) has been widely used to monitor various industrial processes. Most of research in SPC focuses on the charting techniques. In most SPC applications, it is assumed that the quality of a process can be adequately represented by the distribution of a quality characteristic. However, in some situations, the quality of a process is better characterized and summarized by a relationship between a response variable and one or more explanatory variables. In particular, most of studies are focused on the simple linear regression profiles. In recent years,

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some monitoring methods for the linear profiles have been proposed in the literature. Kang and Albin (2000) proposed two control charts for Phase II monitoring of linear profiles. One is multivariate  $T^2$  chart and the other is the combination of exponential weighted moving average (EWMA) chart and R chart. In Kim, Mahmoud and Woodall (2003), the method based on combination of three EWMA charts was proposed for detecting a shift in the intercept, slope and standard deviation simultaneously. Simulation study showed the for detecting the sustained shifts in the parameters the three EWMA charts perform better than the methods in Kang and Albin (2000) in terms of average run length (ARL) and their methods also seem much more interpretable. Mahmoud and Woodall (2004) studied the Phase I method for the linear profiles. A discussion about the problems of linear profiles was given in Woodall et al. (2004).

The parameters of the process are assumed to be known for all of the control charts for linear profile mentioned above. However, in the early stages of process improvement, the process parameters, the intercept, slope and standard deviation are usually unknown, and they are usually estimated by m in-control (IC) historical samples of size n. Some authors have recommended using 20 to 30 samples of size 4 or 5 to estimate the process parameters for the traditional control charts (see Montgomery (1997), Ryan (1989)). Several authors have investigated the effect of the estimated parameters on the performance of traditional control charts, such as Quesenberry (1993), Jones, Champ and Rigdon (2001, 2004), etc. They indicated that when the number of reference samples is small, the control charts with estimated parameters will produce rather large bias in the IC ARL from the nominal value and reduce the sensitivity of the chart to detect the process changes in term of out-of -control (OC) ARL. To attain the similar performance of the chart with known parameters, 20 or 30 historical samples are too small. For example, for the traditional EWMA chart with  $\lambda = 0.2$ , 300 samples of five observations are needed to achieve the desired level of IC performance. Whereas, in most cases, it may not be feasible to wait for the accumulation of sufficient large subgroups, because the users usually want to monitor the process at start-up stages. Hence, many authors had studied the design procedures of the tranditional control charts with estimated parameters, such as Hiller (1967, 1969), Yang and Hiller (1970), Nedumaran and

Pignatiello (2001) and Jones (2002).

Specially, Self-starting methods which update the parameter estimates with new observation and simultaneously check for the OC conditions are developed for the situations when samples sufficiently large to approximate control chart performance with the true parameters are unavailable, such as Hawkins (1987), Hawkins and Olwell (1997), Quesenberry (1991,1995), and Sullivan and Jones (2002). Hawkins, Qiu and Kang (2003) and Hawkins and Zamba (2005) proposed the change-point model based on the likelihood ratio for on-line monitoring which can also be seen as a self-starting method.

In this paper, a self-starting control chart based on recursive residuals is proposed for monitoring the linear profiles when the process parameters are not known. This means that it is not necessary to assemble a large number of reference samples before the control scheme begins (although it is generally advisable to collect a few preliminary observations). The combination of two EWMA charts are used which monitoring the regression coefficients and standard deviation respectively. Given the desired overall IC ARL, the control limits of each charts can be obtained through the Markov chain method. We demonstrate the effectiveness of our proposed approach by the Monte Carlo method.

This paper is organized as follows: In the next section the description and design of our proposed control chart is given; An example of our proposed control chart is illustrated in section 3; The performance assessment is considered in section 4; The discussions and conclusions are given in section 5. The involved deductions are given in the Appendix of this paper.

## 2 The Self-Starting Chart for Linear Profiles

In this section, the model of linear profiles considered in this paper and a brief description of the recursive residuals are firstly given in this section. And then, our proposed Self-starting control chart, its design, and some diagnostic aids are also considered in this section.

Assume the *j*th random sample collected over time is  $(x_i, y_{ij})$ . When the process is IC, the relationship between the response and explanatory variables is assumed to be

$$y_{ij} = A_0 + A_1 x_i + \varepsilon_{ij}, \ i = 1, 2, \cdots, n,$$
 (1)

where the  $\varepsilon_{ij}/\sigma$  are independent identically distributed (i.i.d) as a standard Normal random variable, the explanatory variable X is assumed to be fixed at n values. This is usually the case in the practical applications and is consistent with Kang and Ablin (2000), Kim et al. (2003) and Mahmoud and Woodall (2004).

When the parameters  $A_0$ ,  $A_1$  and  $\sigma^2$  are unknown, a wide used method is to estimate them by the historical data. Suppose there are total  $m \ (m \ge 1)$  IC historical samples of size  $n \ \{(x_i, y_{ij}), i = 1, 2, \dots, n, j = 1, 2, \dots, m\}$ , the most often used unbiased estimator for  $A_0$ ,  $A_1$  and  $\sigma^2$  are the average of the m least square estimators  $a_{0j}$ ,  $a_{1j}$  and  $MSE_j$  given by

$$a_{0j} = \bar{y}_j - a_{1j}\bar{x}, \qquad a_{1j} = \frac{S_{xy(j)}}{S_{xx}},$$
$$MSE_j = \frac{1}{n-2}\sum_{i=1}^n (y_{ij} - a_{1j}x_i - a_{0j})^2$$

where  $\bar{y}_j = \frac{1}{n} \sum_{i=1}^n y_{ij}, \ \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i, \ S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2 \text{ and } S_{xy(j)} = \sum_{i=1}^n (x_i - \bar{x})y_{ij}.$ 

After we got the estimations, the parameters are assumed to be known and the monitoring could be started. As more IC samples are obtained, one may wish to update the estimations and start monitoring again. However, the statistical properties of this procedure, such as the IC ARL, cannot be got easily, so that the designs of this procedure seems very difficult.

Another way to deal with the unknown parameters is to use the self-starting control charts, which update the parameter estimates with new observations and simultaneously check for the OC conditions.

The application of the recursive residuals to the regression model were introduced by Brown, Durbin and Evans (1975). First, we pool all of the historical and future samples of size n into one sample, i.e.  $\{(x_i, y_{ij}), i = 1, 2, \dots, n, j = 1, 2, \dots, m, m + 1, m + 2, \dots\}$ , and then, for convenience, let  $y_{(j-1)n+i} = y_{ij}$ ,  $i = 1, 2, \dots, n, j = 1, 2, \dots, n, j = 1, 2, \dots$ . In this paper, the standardized recursive residuals for the future samples are defined by

$$e_{ij} = \frac{y_{(j-1)n+i} - z'_i \beta_{(j-1)n+i-1}}{\sqrt{S_{(j-1)n+i-1}(1 + z'_i(\mathbf{X}'_{(j-1)n+i-1}\mathbf{X}_{(j-1)n+i-1})^{-1}z_i)}}{i = 1, 2, \cdots, n, \quad j = m+1, m+2, \cdots}$$
(2)

where

$$z'_{i} = (1, x_{i}), \quad \mathbf{Y}'_{(j-1)n+i-1} = (y_{1}, y_{2}, \cdots, y_{(j-1)n+i-1}),$$

$$\mathbf{X}'_{(j-1)n+i-1} = (\overline{z_{1}, z_{2}, \cdots, z_{n}, z_{1}, z_{2}, \cdots, z_{n}, \cdots}, z_{1}, z_{2}, \cdots, z_{i-1}),$$

$$\beta_{t} = (\mathbf{X}'_{t}\mathbf{X}_{t})^{-1}\mathbf{X}'_{t}\mathbf{Y}_{t}$$

$$S_{t} = \frac{1}{t-2}(\mathbf{Y}_{t} - \mathbf{X}_{t}\beta_{t})'(\mathbf{Y}_{t} - \mathbf{X}_{t}\beta_{t})$$

Under the IC model (1), it well know that  $e_{ij}$  has a Student-*t* distribution  $t_{(j-1)n+i-3}$  (see Brown, Durbin and Evans (1975)). Using a lemma due to Basu (Lehmann 1991), we can show that the  $e_{ij}$ 's are statistically independent. So, through a transformation, we obtain the following statistics

$$w_{ij} = \Phi^{-1} \Big[ T_{(j-1)n+i-3} \Big( e_{ij} \Big) \Big]$$
(3)

which is called *Q*-statistics in Quesenberry (1991), where  $\Phi^{-1}$  denotes the inverse of the cumulative distribution function (CDF) of standard normal random variable,  $T_{\nu}$  is the CDF of Student-*t* distribution with  $\nu$  degrees of freedom. Therefore,  $\{w_{ij}, i = 1, 2, \dots, n, j = m + 1, m + 2, \dots\}$  is a sequence of independent standard normal random variables.

When an assignable cause occurs after some observation, say kth observation, the distribution of Q-statistics  $\{w_{ij}, i = 1, 2, \dots, n, j = k, k + 1, \dots\}$  is different from that of  $\{w_{ij}, i = 1, 2, \dots, n, j = m + 1, m + 2, \dots, m + k - 1\}$ . So, the difference between them could be used to detect the assignable cause. This is the motivation of our proposed method. At the first glance, the method based on  $\{w_{ij}, i = 1, 2, \dots, n, j = m + 1, m + 2, \dots\}$  may be a omnibus one, i.e. it is not easy to diagnose which parameters have been shifted. But, a convenient method will be given to aid the method based on the transformed recursive residuals in this paper. For the transformed residuals  $\{w_{ij}, i = 1, 2, \cdots, n, j = m + 1, m + 2, \cdots\}$ , let  $\bar{w}_j = \frac{1}{n} \sum_{i=1}^n w_{ij}$  and  $S_{w_j} = \frac{1}{n-1} \sum_{i=1}^n (w_{ij} - \bar{w}_j)^2$  denote, respectively, the sample mean and variance of *j*th subgroup of them. Define two EWMA statistics, EWMA<sub>IS</sub> and EWMA<sub> $\sigma$ </sub>, as follows

$$EWMA_{IS}(j) = \lambda \sqrt{n} \bar{w}_{m+j} + (1-\lambda) EWMA_{IS}(j-1), \qquad (4)$$

$$\operatorname{EWMA}_{\sigma}(j) = \max\left(0, \lambda \sqrt{\frac{n-1}{2}} (S_{w_{m+j}} - 1) + (1-\lambda) \operatorname{EWMA}_{\sigma}(j-1)\right), (5)$$

where  $j = 1, 2, \dots$ , EWMA<sub>IS</sub>(0) = EWMA<sub> $\sigma$ </sub>(0) = 0, and  $\lambda$  (0 <  $\lambda \leq 1$ ) is a smoothing constant.

Our proposed self-starting scheme is defined to be the combination of above two EWMA charts, i. e., an out of control signal is triggered as soon as EWMA<sub>IS</sub> $(j) < LCL_{IS}$  or EWMA<sub>IS</sub> $(j) > UCL_{IS}$  or/and EWMA<sub> $\sigma$ </sub> $(j) > UCL_{\sigma}$ , where  $UCL_{IS}$ ,  $LCL_{IS}$  and  $UCL_{\sigma}$  are chosen such that to obtain the given specified IC ARL.

From the definition, we know the EWMA<sub>IS</sub> chart is used to monitor the change in slope or intercept while EWMA<sub> $\sigma$ </sub> is effective in monitoring the shift in standard deviation of the process. When the process is IC, { $w_{ij}$ ,  $i = 1, 2, \dots, n$  j = $m + 1, \dots$ } can be regarded as i.i.d samples from N(0, 1) whatever the number of historical samples  $m(m \ge 1)$  is. So, the properties of IC run-length of the EWMA<sub>IS</sub> and EWMA<sub> $\sigma$ </sub> charts are same as that of the classical EWMA chart studied by many authors, such as Lucas and Saccucci (1990).

Note that the EWMA<sub> $\sigma$ </sub> chart in equation (5) is a one-sided scheme, which is used to detect the increase in process variance only. If one want to detect decrease in variance, then some other appropriate methods can be used, such as the method discussed by Acosta-Mejia et al. (1999). In this paper, we consider the detection of the increase in the variance only.

The smoothing constants  $\lambda$  in Equation (4) and (5) are set equal to 0.2 as in the EWMA chart used by Kang and ALbin (2000) and Kim et al. (2003). Certainly, we may use different smoothing constants for each chart. In general, smaller smoothing constants lead to quicker detection of smaller shifts as shown by Lucas and Saccucci (1990). As we known, the self-starting charts have the "masking" effect in the OC situation, that is to say, relatively magnitude of the shift will get smaller as the more OC samples are observed for the reason that these observations data "contaminated"

the estimates (see Hawkins and Olwell (1997) for more detail). So, some adaptive charts such as the AEWMA chart proposed by Capizzi and Masarotto (2003) which offer a more balanced protection against shifts of different sizes can be used to substitute for our EWMA charts but we don't investigate them here.

It seems that our proposed charts require a lot of computations such as the inverse matrix in Equation (2). Actually, the calculations of the  $e_{ij}$ 's are considerably simplified by the following recursive formulas

$$(\mathbf{X}'_{t}\mathbf{X}_{t})^{-1} = (\mathbf{X}'_{t-1}\mathbf{X}_{t-1})^{-1} - \frac{(\mathbf{X}'_{t-1}\mathbf{X}_{t-1})^{-1}z_{i}z'_{i}(\mathbf{X}'_{t-1}\mathbf{X}_{t-1})^{-1}}{1 + z'_{i}(\mathbf{X}'_{t-1}\mathbf{X}_{t-1})^{-1}z_{i}},$$
(6)

$$\beta_t = \beta_{t-1} + (\mathbf{X}'_t \mathbf{X}_t)^{-1} z_i (y_t - z'_i \beta_{t-1}), \qquad (7)$$

$$(t-2)S_t = (t-3)S_{t-1} + (e_{ij})^2 S_{t-1}, (8)$$

where t = (j - 1)n + i.

Under IC condition, because  $w_{ij} \sim N(0, 1)$ , the statistics  $\sqrt{n}\bar{w}_j$  and  $\sqrt{\frac{n-1}{2}}(S_{w_j} - 1)$  are independently distributed as a standard normal and a scaled  $\chi^2$  distributions, respectively. Hence, for each chart, the IC ARL properties can be easily obtained through a classical Markov chain procedure (Brook and Evans (1972)). However, as we well known, these two charts are not independent, i.e. a shift in one parameter may be signaled by both of the charts. A Markov chain method is used to evaluate the IC ARL of our proposed chart (see Appendix of this paper). Given a desired overall IC ARL, a Fortran program is available from the authors which can find the control limits for each chart.

Because EWMA<sub>IS</sub> chart is used to monitor the shift in both of intercept and slope, EWMA<sub> $\sigma$ </sub> chart is used for monitoring the standard deviation, the ratio of EWMA<sub>IS</sub> chart's IC ARL to that of EWMA<sub> $\sigma$ </sub> is designed to be 2. For some given IC ARL and sample size n = 4 and 5, the control limits for each chart are tabulated in Table 1 (Note that  $LCL_{IS} = -UCL_{IS}$ ).

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			n = 4					n = 5		
IC ARL	200	300	370	400	500	200	300	370	400	500
$UCL_{IS}$	0.9276	0.9746	0.9978	1.0062	1.0302	0.9271	0.9748	0.9985	1.0071	1.0313
$UCL_{\sigma}$	1.2959	1.3794	1.4214	1.4369	1.4812	1.2530	1.3318	1.3717	1.3864	1.4280

Table 1 The control limits of the self-starting chart

In the practical applications of quality control, there are two issues need to detect. One is to detect if the process is in control, other is to point out the position of the shift if the process had shifted. Knowing the process change point would help engineer to identify the special cause quicker. A method based on the maximum likelihood estimator of the change point is proposed to assist our self-starting chart. We assume that the chart signals at subgroup k, i.e. there are m historical IC samples and k - m future samples, and a shift in parameters had been occurred after the  $\tau$ th samples ( $m < \tau < k$ ). The classical likelihood ratio statistic is given by

$$lr(k_1n, kn) = kn \log[\widehat{\sigma}_{kn}^2(\widehat{\sigma}_{k_1n}^2)^{-\frac{k_1}{k}}(\widehat{\sigma}_{k_2n}^2)^{-\frac{k_2}{k}}].$$
(9)

The meaning of  $\hat{\sigma}_{kn}^2$ ,  $\hat{\sigma}_{k1n}^2$   $\hat{\sigma}_{k2n}^2$  and the involved deductions are given in Appendix of this paper. Our proposed estimator of the change point  $\tau$  of a step shift in parameter(s) of linear profile is given by

$$\widehat{\tau} = \underset{m < k_1 < k}{\operatorname{arg}} \max\{ lr(k_1 n, kn) \}.$$
(10)

Note that this is consistent with that in Pignatiello and Samuel (2001).

As Kim et al. (2003) pointed out, it is very necessary to justify which parameter or parameters have been shifted after a signal occurs. Therefore, using the coded explanatory values, they obtained the following alternative form of the underlying model

$$y_{ij} = B_0 + B_1 x_i^* + \varepsilon_{ij}, \ i = 1, 2, \cdots, n,$$
 (11)

where  $B_0 = A_0 + A_1 \bar{x}$ ,  $B_1 = A_1$  and  $x_i^* = (x_i - \bar{x})$ . For model (11), the least square estimators of  $B_0$ ,  $B_1$  and  $\sigma^2$  are independent, so, they proposed to use three independent EWMA charts to detect if the intercept, slope, and standard deviation has changed, respectively.

Similar to Sullivan and Woodall (1996), a useful convenient method is introduced to enhance the ability of our proposed self-starting chart in detecting where the shift comes from. We decompose the test statistic  $lr(\hat{\tau}n, kn)$  in Equation (9) into three parts:  $I_{lr}(\hat{\tau})$ ,  $S_{lr}(\hat{\tau})$  and  $\sigma_{lr}(\hat{\tau})$  (see Appendix) which are, respectively, used as the index of the relative contribution from Y-intercept, slope and standard deviation. The simulated results show that this method is more effective than the combination of three charts used by Kim et al.(2003), when there is a shift in two or three parameters simultaneously. But a case must be pointed out is that the proposed method is effective for the model (11) but not appropriate for the model (1) because the change of  $A_1$  will result in the augment of values of both  $I_{lr}$  and  $S_{lr}$  and we think Kim at al. (2003) has the same question.

### 3 Illustrative Example

In this section, an illustrative example is given to introduce the implementation of our proposed self-starting control chart. In this example, the underlying in-control linear profile model is  $y_{ij} = 3 + 2x_i + \varepsilon_{ij}$ , which is used by Kang and Albin(2000), where the  $\varepsilon_{ij}$ 's are i.i.d normal random variable with zero and variance one. The explanatory variable takes 2(2)8. Obviously,  $\bar{x} = 5$ ,  $S_{xx} = 20$ ,  $B_0 = 13$  and  $B_1 = 2$ . There are m = 5 of size n = 4 IC historical observations, which are the first 5 rows in Table 2. We want to detect the shift in  $B_0$  and  $\sigma$  simultaneously in model (11). Suppose the Y-intercept  $B_0$  and standard deviation  $\sigma$  have shifted from 13 to 13.8 and 1.0 to 1.5, respectively, after the 15th future sample. For given overall IC ARL=200, the control limits of EWMA<sub>IS</sub> and EWMA<sub> $\sigma$ </sub> charts are, respectively, 0.9276 and 1.2959. The statistics  $\bar{w}_j$ ,  $S_{w_j}$ , EWMA<sub>IS</sub> (EW<sub>IS</sub>) and EWMA<sub> $\sigma$ </sub> (EW<sub> $\sigma$ </sub>) for  $j = 6, 7, \dots, 28$  are also tabulated in Table 2.

It is clear that the EWMA<sub> $\sigma$ </sub> chart has been signaled after only four OC samples are monitored. But in order to illustrate our chart, more OC samples were got. At j = 28, the EWMA<sub>IS</sub> chart also gives a alarm. The maximum of  $lr(k_1n, 28n)$  for  $k_1 = 6, 7, \dots, 28$  is the lr(20n, 28n) = 31.55 which indicates accurately the location  $\tau$  of the shift. The values of  $I_{lr}(20)$ ,  $S_{lr}(20)$  and  $\sigma_{lr}(20)$  are 10.24, 2.31 and 19.0 respectively. Thus, we would diagnose the shifts in the Y-intercept and variance after observation 20.

### 4 Performance Comparisons

In this section, we assess the performance of our proposed self-staring control chart through the comparisons with other methods in term of the OC ARL. For the

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j		$y_{ij}$			$\bar{w}_j$	$S_{w_j}$	$\mathrm{EW}_{\mathrm{IS}}$	$EW_{\sigma}$	lr	$I_{lr}$	$S_{lr}$	$\sigma_{lr}$
1	7.29	10.46	13.14	20.88					0.90	0.77	0.14	0.00
2	7.65	10.39	15.89	20.19					1.30	0.36	0.24	0.70
3	7.37	11.88	14.54	20.33					2.57	0.22	0.18	2.16
4	7.39	13.34	13.92	18.92					1.69	0.26	0.04	1.40
5	5.64	11.06	15.47	19.33					3.58	0.88	0.03	2.67
6	6.63	11.51	15.66	18.42	-0.18	0.43	-0.07	0.00	6.00	1.42	0.00	4.58
7	8.16	12.12	16.27	19.83	0.79	0.10	0.26	0.00	6.62	0.54	0.02	6.06
8	5.90	11.21	16.45	18.71	-0.26	1.04	0.10	0.01	8.15	0.96	0.00	7.19
9	6.49	12.56	15.78	20.42	0.49	0.69	0.28	0.00	9.45	0.57	0.08	8.80
10	8.69	11.87	17.46	18.92	0.83	1.45	0.56	0.11	7.98	0.09	0.00	7.89
11	6.05	12.61	14.09	18.44	-0.61	1.51	0.20	0.21	7.99	0.50	0.06	7.44
12	7.88	10.02	16.02	18.39	-0.30	1.05	0.04	0.18	9.55	0.89	0.24	8.42
13	9.27	12.31	14.34	17.91	0.08	2.39	0.06	0.49	9.75	0.95	1.83	6.97
14	7.91	10.68	16.15	18.46	-0.07	0.62	0.02	0.30	12.51	1.19	2.64	8.68
15	7.43	11.62	14.82	20.34	0.18	0.40	0.09	0.09	15.03	1.13	2.53	11.37
16	6.53	10.08	15.92	19.04	-0.47	0.62	-0.11	0.00	17.39	2.05	2.15	13.19
17	6.77	11.76	13.86	18.48	-0.61	0.61	-0.34	0.00	20.84	3.75	3.10	13.99
18	5.66	10.98	15.00	20.18	-0.34	1.05	-0.40	0.01	22.73	5.31	1.87	15.55
19	7.39	10.13	15.94	19.00	-0.18	0.59	-0.39	0.00	26.86	6.86	2.11	17.89
20	5.96	11.75	14.81	18.64	-0.49	0.55	-0.51	0.00	31.55	10.24	2.31	19.00
21	4.41	12.28	17.02	20.02	0.15	3.98	-0.35	0.73	27.28	11.37	0.67	15.24
22	8.85	10.99	19.13	21.25	1.63	2.41	0.37	0.93	16.55	5.78	0.30	10.47
23	7.47	10.62	17.28	21.33	0.72	1.22	0.59	0.80	15.97	4.21	0.00	11.75
24	8.46	8.92	13.08	21.56	-0.33	4.07	0.34	1.39	13.20	7.17	0.04	5.99
25	4.24	12.00	16.58	19.53	-0.23	2.27	0.18	1.42	15.74	12.12	1.36	2.27
26	6.54	12.39	17.33	20.32	0.64	0.71	0.40	1.07	18.06	12.15	4.14	1.77
27	10.23	13.70	15.63	18.29	0.86	2.77	0.66	1.29	11.84	10.27	0.02	1.54
28	10.33	13.67	14.58	23.51	1.60	2.65	1.17	1.44				

Table 2 Data for Example with a shift in the slope and standard deviation after 20th sample.

parameters known case, Kim et al. (2003) showed that the combination of three EWMA charts (EWMA<sub>3</sub>) performances better than the methods in Kang and Albin (2000) in terms of detecting the sustained shifts in the parameters. So, we only compare our proposed self-starting method with EWMA<sub>3</sub>.

For simplicity, we only consider the case of overall IC ARL=200, the smoothing constant  $\lambda$  is set to be 0.2, the parameters in the in-control model are  $A_0 = 3$ ,  $A_1 = 2$  and  $\sigma^2 = 1$ ,  $x_i=2(2)8$ . The control limits for our two EWMA charts are 0.9276 and 1.2959 which yield an IC ARL 298.4 and 596.5 for the two charts respectively whatever the number of IC historical samples. In Kim et al. (2003), the  $L_I$ ,  $L_S$  and  $L_E$  are set to be 3.0156, 3.0109 and 1.3723 for the three EWMA charts respectively. In the case of known parameters, this design will has an overall IC ARL of roughly 200 and each chart has an about 584 IC ARL. Though the IC ARL of our proposed self-starting chart can be evaluated by Markov chain procedure, the OC ARL is rather difficult to calculate by Markov chain method, the reason is the intricacy of OC distribution. Therefore, the results in this section is evaluated by 10,000 simulations. Moreover, the types of shifts considered in this paper are same as the that in Kim et al. (2003), although some other scales, instead of the scale  $\sigma$ , can be used to measure the size of shifts in all parameters.

First of all, the performance of our self-starting charts with m = 10, 30, 50, 100and 500 IC historical samples, and the performance of EWMA<sub>3</sub> charts with the true values of parameters are shown in Figure 1 (a-c).

From Figure 1 we observed

- Our proposed self-starting chart performs well in detecting large shifts in whichever parameters, regardless of how many IC historical samples are available.
- Our proposed self-starting chart has a disadvantage in detecting the small shifts compared with EWMA<sub>3</sub> charts for the small m, however, the differences get smaller as the value of m get larger. For the detection of shift in slope, the self-starting chart with m = 500 have the almost same performance as the EWMA<sub>3</sub> charts with known parameters while for shifts in intercept or standard deviation, our approach with m = 500 samples perform even nearly

uniformly better than the  $EWMA_3$  charts.



Figure 1 The ARL's for step shift in intercept (a), slope (b) and standard deviation (c).

There is shift in parameter  $B_1$  only in model (11) in Kang and Albin (2000) and Kim et al. (2003). Table 3 gives the comparison between our self-starting chart (EW<sub>2</sub>) with m = 500 IC historical samples and the EWMA<sub>3</sub> charts(EW<sub>3</sub>) with known parameters in this case. We observed EWMA<sub>3</sub> chart has much better ARL performance than our self-starting chart for the small  $\delta$ . The two methods perform similarly for the moderate-to- large shift sizes. Totally, the EWMA<sub>3</sub> are more effective than our approach in this case.

Table 3 ARL comparisons for  $B_1$  to  $B_1 + \delta \sigma$  in model (11)(IC ARL=200)

					$\delta$				
Chart	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$EW_3$	13.1	6.6	4.4	3.3	2.7	2.3	2.1	1.9	1.7
$\mathrm{EW}_2$	111.5	33.3	10.8	5.6	3.6	2.7	2.0	1.6	1.4

For a more meaningful and fair comparison, the  $EWMA_3$  and self-starting charts with unknown parameters are compared using the following steps:

- [1] To generate m subgroups of size n from the underlying model and to estimate the unknown parameters.
- [2] To construct the EWMA<sub>3</sub> charts with the estimated parameters and our selfstarting chart with the control limits given in Table 1.
- [3] The shift(s) in parameter(s) occurs after  $\tau$ th sample. To monitor the process from the (m + 1)th sample, and set the run length to be k, when the  $\tau + k$ th future sample falls outside the control limits.
- [4] Repeat Steps [S1]-[S3] 10,000 times.
- [5] Compute OC ARL, i. e. the average of the 10,000 run lengths.

The simulation results of m = 10, 80, 300 are given in Table 3-5. The values of  $\tau - m$  are chosen to be 20, 90 and 240 for a representative illustration. From these tables, we can see that

- The number of the IC historical samples m strongly affects the ability of EWMA<sub>3</sub> charts to detect the shift. As the m increases, the OC ARL get smaller quickly.
- Our proposed self-starting chart performs well in detecting moderate and large shifts in whichever parameters.
- Our proposed self-starting chart has a disadvantage in detecting the small shifts compared with EWMA<sub>3</sub> charts for the small  $\tau m$ , however, as the more IC future samples are collected, the self-starting chart will be more sensitive to the small shifts which is owe to the updating parameter estimates with new observation.

Another noteworthy feature from these tables is that the IC ARL's of self-starting and EWMA<sub>3</sub> chart. The self-starting chart's ARL's differ slightly from the nominal 200 because the chart didn't start from the zero state in this case. It's surprising

	Table 3 ARL comparisons for $m = 10$										
	$\tau - m = 20$ $\tau - m = 90$ $\tau - m = 240$										
	δ	$EW_3$	$\mathrm{EW}_2$	$EW_3$	$EW_2$	$EW_3$	$\mathrm{EW}_2$				
	0.2	119.3	139.3	120.2	88.5	119.3	61.7				
	0.4	38.5	46.7	38.1	17.4	37.6	14.5				
	0.6	10.1	11.1	10.3	7.4	10.4	7.0				
	0.8	5.4	5.5	5.3	4.7	5.5	4.6				
$A_0 + \delta \sigma$	1.0	3.8	3.8	3.8	3.5	3.8	3.5				
	1.2	3.0	3.0	3.0	2.9	3.0	2.8				
	1.4	2.5	2.6	2.6	2.4	2.6	2.4				
	1.6	2.2	2.2	2.2	2.1	2.2	2.1				
	1.8	2.0	2.0	2.0	1.9	2.0	1.9				
	2.0	1.8	1.8	1.8	1.7	1.8	1.7				
	0.025	157.7	171.7	157.8	145.2	157.6	116.5				
	0.050	85.0	115.2	84.8	57.5	84.7	39.1				
	0.075	34.4	56.7	34.3	20.4	34.3	16.3				
	0.100	15.3	21.6	15.3	10.3	15.3	9.5				
$A_1 + \delta \sigma$	0.125	8.0	10.1	8.0	6.9	8.0	6.5				
	0.150	5.5	6.2	5.5	5.2	5.5	5.0				
	0.175	4.3	4.7	4.3	4.2	4.3	4.1				
	0.200	3.6	3.8	3.6	3.5	3.6	3.4				
	0.225	3.1	3.3	3.1	3.0	3.1	3.0				
	0.250	2.7	2.9	2.7	2.7	2.7	2.7				
	1.2	40.5	90.0	41.1	46.4	39.6	32.9				
	1.4	13.3	28.3	13.3	11.1	13.5	9.6				
	1.6	6.8	8.8	6.8	5.6	6.8	5.2				
	1.8	4.5	4.5	4.5	3.7	4.5	3.5				
$\delta\sigma$	1.0	3.4	3.1	3.4	2.8	3.4	2.7				
	2.2	2.8	2.5	2.8	2.3	2.8	2.2				
	2.4	2.4	2.1	2.4	2.0	2.4	1.9				
	2.6	2.1	1.8	2.1	1.7	2.1	1.7				
	2.8	1.9	1.7	1.9	1.6	1.9	1.6				
	3.0	1.8	1.5	1.8	1.5	1.8	1.5				
IC	0.0	212.0	196.5	212.2	196.4	212.1	196.3				
	1	1		1							

Table 3 ARL comparisons for m = 10

		$\tau - m$	n = 20	$\tau - \eta$	n = 90	$\tau - m =$	= 240
	δ	$EW_3$	$\mathrm{EW}_2$	$EW_3$	$\mathrm{EW}_2$	$EW_3$	$\mathrm{EW}_2$
	0.2	65.3	88.5	65.6	71.8	64.6	58.3
	0.4	16.8	17.4	16.9	15.2	17.0	14.2
	0.6	7.8	7.4	7.8	7.1	7.9	7.0
	0.8	5.0	4.7	5.1	4.7	5.1	4.6
$A_0 + \delta \sigma$	1.0	3.7	3.5	3.8	3.5	3.8	3.5
	1.2	3.0	2.9	3.0	2.8	3.0	2.8
	1.4	2.6	2.4	2.6	2.4	2.6	2.4
	1.6	2.2	2.1	2.2	2.1	2.2	2.1
	1.8	2.0	1.9	2.0	1.9	2.0	1.9
	2.0	1.8	1.7	1.8	1.7	1.8	1.7
	0.025	105.7	147.1	105.7	130.3	105.5	112.6
	0.050	39.2	59.5	39.1	45.5	39.3	37.1
	0.075	17.1	20.7	17.5	17.5	17.3	16.0
	0.100	10.0	10.4	10.0	9.6	10.2	9.4
$A_1 + \delta \sigma$	0.125	6.9	6.9	6.9	6.6	7.0	6.5
	0.150	5.2	5.2	5.3	5.1	5.3	5.0
	0.175	4.3	4.2	4.3	4.1	4.3	4.0
	0.200	3.6	3.5	3.6	3.5	3.6	3.4
	0.225	3.2	3.0	3.2	3.0	3.2	3.0
	0.250	2.8	2.7	2.8	2.7	2.8	2.7
	0.2	33.4	46.4	32.2	35.8	32.3	30.5
	0.4	11.9	11.1	11.7	10.0	11.6	9.2
	0.6	6.6	5.6	6.5	5.2	6.5	5.0
	0.8	4.5	3.7	4.5	3.5	4.5	3.5
$\delta\sigma$	1.0	3.5	2.8	3.5	2.7	3.5	2.7
	1.2	2.9	2.3	2.9	2.2	2.9	2.2
	1.4	2.5	2.0	2.5	1.9	2.5	1.9
	1.6	2.2	1.7	2.2	1.7	2.2	1.7
	1.8	2.0	1.6	2.0	1.6	2.0	1.6
	2.0	1.8	1.5	1.8	1.5	1.9	1.5
IC	0.0	181.7	196.4	182.3	194.1	182.5.0	195.5

Table 4 ARL comparisons for m = 80

	$\tau - m = 20$ $\tau - m = 90$ $\tau - m = 240$								
	δ	$EW_3$	$EW_2$	$EW_3$	$EW_2$	$EW_3$	$EW_2$		
	0.2	58.2	58.3	58.0	55.1	58.1	52.3		
	0.4	16.2	14.2	16.2	13.9	16.2	13.8		
	0.6	7.8	7.0	7.8	6.9	7.8	6.9		
	0.8	5.0	4.6	5.0	4.5	5.0	4.5		
$A_0 + \delta \sigma$	1.0	3.7	3.5	3.7	3.4	3.7	3.4		
	1.2	3.0	2.8	3.0	2.8	3.0	2.8		
	1.4	2.5	2.4	2.5	2.4	2.5	2.4		
	1.6	2.2	2.1	2.2	2.1	2.2	2.1		
	1.8	2.0	1.9	2.0	1.9	2.0	1.9		
	2.0	1.8	1.7	1.8	1.7	1.8	1.7		
	0.025	100.5	112.6	100.3	109.2	100.2	105.0		
	0.050	36.1	37.1	36.0	35.0	36.0	34.0		
	0.075	16.8	16.0	16.8	15.7	16.8	15.4		
	0.100	9.9	9.4	9.9	9.3	9.9	9.1		
$A_1 + \delta \sigma$	0.125	6.9	6.5	6.9	6.4	6.9	6.4		
	0.150	5.3	5.0	5.3	4.9	5.3	4.9		
	0.175	4.3	4.0	4.3	4.0	4.3	4.0		
	0.200	3.6	3.4	3.6	3.4	3.6	3.4		
	0.225	3.2	3.0	3.2	3.0	3.2	3.0		
	0.250	2.8	2.7	2.8	2.7	2.8	2.7		
	0.2	32.2	30.5	31.8	30.2	32.0	30.1		
	0.4	11.5	9.2	11.6	9.1	11.7	9.0		
	0.6	6.5	5.0	6.6	5.0	6.6	5.0		
	0.8	4.5	3.5	4.6	3.5	4.6	3.5		
$\delta\sigma$	1.0	3.5	2.7	3.6	2.7	3.6	2.7		
	1.2	2.9	2.2	2.9	2.2	2.9	2.2		
	1.4	2.5	1.9	2.5	1.9	2.5	1.9		
	1.6	2.2	1.7	2.2	1.7	2.2	1.7		
	1.8	2.0	1.6	2.0	1.6	2.0	1.5		
	2.0	1.9	1.5	1.9	1.5	1.9	1.4		
IC	0.0	189.0	195.5	188.5	196.1	188.6	198.2		

Table 5 ARL comparisons for m = 300

to us the IC ARL's of the EWMA<sub>3</sub> chart are all close to the 200 for different m. In fact, based on other simulations, we can show that though the overall IC ARL is not affected severely, the difference of ARL between the case of estimated parameters and that of known parameters for each chart are rather large. For example, for m = 80, the ARL's for the EWMA chart for monitoring the intercept, slope and standard deviation are 499.4, 495.4 and 952.0 respectively and it may lead to more serious results for smaller m. Moreover, the false alarm probabilities of the EWMA<sub>3</sub> charts increases drastically after short runs when the parameters are estimated. To attain the similar performance of the known parameters, much more samples are required, however, in this paper, we don't discuss them in detail any more but just give a suggestion that the use of the EWMA<sub>3</sub> chart with estimated parameter should be careful.

## 5 Conclusions and Considerations

Basing on the recursive residuals, a self-starting control chart was introduced to detect shifts in intercept, slope and standard deviation for the linear profile. This chart can be easily designed and perform well in the case of process parameters are unknown but some historical samples are available. We also gave a useful tool based on the maximum likelihood ratio to diagnose the position of shift. For the practical applications, if one want to get the information about which the parameter(s) have been changed, a useful convenient method is given to aid the proposed chart.

The proposed approach can be easily generalized to the multiple linear regression profiles. However, it is more difficult to analysis the relative contribution of each parameter to the shift and we will discuss this setting in the future paper. Based on the likelihood ratio for self-starting monitoring, another future paper will discuss a method, which is similar to Hawkins, Qiu and Kang (2003) and Hawkins and Zamba (2005).

#### Acknowledgement

This paper was supported by The Natural Sciences Foundation of Tianjin(033603111).

## Appendix

# The Markov chain approach for evaluating the IC ARL of our proposed self-starting chart

Similar to the procedure proposed by Brook and Evans (1972), the formula to evaluate the IC ARL of the self-starting scheme can be obtained by approximating the combination of two EWMA charts with a Markov chain.

The transition probability matrix  $\mathbf{P} = (p_{ij \rightarrow kl})$  is partitioned into the following form

$$\left(\begin{array}{cc} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{array}\right)$$

where the submatrix **R** is the transition probability matrix for in-control states; **I** is the identity matrix, and **1** is a column vector of ones. Let  $t_1$  and  $t_2$  are given integers,  $w_1 = \frac{2UCL_{IS}}{2t_1+1}$  and  $w_2 = \frac{2UCL_{\sigma}}{2t_2-1}$ . A pair of integers (i, j) is denoted as a state of the two EWMA charts, where  $i = -t_1, \dots, 0, \dots, t_1, j = 0, 1, \dots, t_2 - 1$ , and (0, 0) denote the initial state of the self-starting chart. The transition probability that the plot statistics EWMA<sub>IS</sub>(q) and EWMA<sub> $\sigma$ </sub>(q) goes from state (i, j) to state (k, l) is denoted by  $R_{ij \rightarrow kl}$ , which is calculated by

$$\begin{aligned} R_{ij\to k0} \\ &= Pr\{(\text{EWMA}_{\text{IS}}(\mathbf{q}), \text{EWMA}_{\sigma}(\mathbf{q})) = (k,0) \left| (\text{EWMA}_{\text{IS}}(\mathbf{q}-1), \text{EWMA}_{\sigma}(\mathbf{q}-1)) = (i,j)) \} \\ &= Pr\{(k-i+\lambda i-0.5)\frac{w_1}{\lambda} \le \sqrt{n}\bar{w}_q < (k-i+\lambda i+0.5)\frac{w_1}{\lambda}, \\ (n-1)S_{w_q} < (n-1)[\sqrt{\frac{2}{n-1}}(-j+\lambda j+0.5)\frac{w_2}{\lambda}+1] \} \\ &= \{\Phi[(k-i+\lambda i+0.5)\frac{w_1}{\lambda}] - \Phi[(k-i+\lambda i-0.5)\frac{w_1}{\lambda}] \} \cdot \\ \chi^2_{n-1}\Big(n-1)[\sqrt{\frac{2}{n-1}}(-j+\lambda j+0.5)\frac{w_2}{\lambda}+1]\Big) \end{aligned}$$

$$\begin{aligned} R_{ij \to kl} \\ &= Pr\{(\text{EWMA}_{\text{IS}}(\mathbf{q}), \text{EWMA}_{\sigma}(\mathbf{q})) = (k, l) \left| (\text{EWMA}_{\text{IS}}(\mathbf{q}-1), \text{EWMA}_{\sigma}(\mathbf{q}-1)) = (i, j)) \} \\ &= Pr\{(k-i+\lambda i-0.5)\frac{w_1}{\lambda} \le \sqrt{n}\bar{w}_q < (k-i+\lambda i+0.5)\frac{w_1}{\lambda}, \\ (n-1)[\sqrt{\frac{2}{n-1}}(l-j+\lambda j-0.5)\frac{w_2}{\lambda}+1] \le (n-1)S_{w_q} \\ &< (n-1)[\sqrt{\frac{2}{n-1}}(l-j+\lambda j+0.5)\frac{w_2}{\lambda}+1] \} \\ &= \{\Phi[(k-i+\lambda i+0.5)\frac{w_1}{\lambda}] - \Phi[(k-i+\lambda i-0.5)\frac{w_1}{\lambda}]\} \cdot \\ &\{\chi_{n-1}^2\Big((n-1)[\sqrt{\frac{2}{n-1}}(l-j+\lambda j+0.5)\frac{w_2}{\lambda}+1]\Big) - \\ &\chi_{n-1}^2\Big((n-1)[\sqrt{\frac{2}{n-1}}(l-j+\lambda j-0.5)\frac{w_2}{\lambda}+1]\Big)\} \end{aligned}$$

where  $\Phi(\cdot)$  and  $\chi^2_{n-1}(\cdot)$  are the CDFs of the standard normal distribution and chisquare distribution with degree of freedom n-1. Following the procedure of Brook and Evans (1972), the ARL of the self-starting chart with initial state (0,0) hence is given by

$$l_0(I - R)^{-1}1$$

where  $\mathbf{l}_0 = (0, \dots, 1, \dots, 0)$  is a row vector with 1 in the  $(t_1 \times t_2 + 1)$ th element. For increasing the accuracy of our method, the following extrapolation is used

$$ARL(t) = ARL + B/t + C/t^2,$$
(12)

where ARL(t) denotes the value of ARL calculated by  $t_1 = t_2 = t$  states and t = 10, 15, 20 are used for Table 1.

#### The approximate calculation of the control limits

Let  $ARL_{IS\sigma}(UCL_{IS}, UCL_{\sigma})$ ,  $ARL_{IS}(UCL_{IS})$  and  $ARL_{\sigma}(UCL_{\sigma})$  denote the ARL functions evaluated by the Markov chain procedure for the combination of the two EWMA charts, the EWMA<sub>IS</sub> and EWMA<sub> $\sigma$ </sub> charts, respectively. Given a overall IC ARL<sub>0</sub>, the values of  $UCL_{IS}$  and  $UCL_{\sigma}$  can be found through solving the following equations

$$\begin{cases} ARL_{IS\sigma}(UCL_{IS}, UCL_{\sigma}) = ARL_{0} \\ ARL_{IS}(UCL_{IS}) = 0.5ARL_{\sigma}(UCL_{\sigma}) \end{cases}$$

The dichotomy method is used to search for the values.

#### The derivation of the Equation (9)

Let  $\{(x_i, y_{ij}), i = 1, 2, \cdots, n, j = 1, 2, \cdots, k\}$  denote all the historical and collected future samples,  $\bar{y}_{kn} = \frac{1}{kn} \sum_{j=1}^{k} \sum_{i=1}^{n} y_{ij}, \ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \ S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})^2$  and  $S_{xy(kn)} = \sum_{j=1}^{k} \sum_{i=1}^{n} (x_i - \bar{x})y_{ij}$ . Then, the logarithm of likelihood function is given by  $-\frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{n} \left[ \log(2\pi\sigma_j^2) + \frac{(y_{ij} - A_{0j} - A_{1j}x_i)^2}{2\sigma_i^2} \right].$ 

If the data is collected under in-control conditions, the maximum value of the logarithm of likelihood function is

$$l_0 = -\frac{kn}{2}\log(2\pi) - \frac{kn}{2}\log(\widehat{\sigma}_{kn}^2) - \frac{kn}{2},$$

where  $\widehat{\sigma}_{kn}^2 = \frac{1}{kn} \sum_{j=1}^k \sum_{i=1}^n (y_{ij} - \widehat{A}_{0(kn)} - \widehat{A}_{1(kn)}x_i)^2$ ,  $\widehat{A}_{1(kn)} = \frac{S_{xy(kn)}}{kS_{xx}}$ ,  $\widehat{A}_{0(kn)} = \overline{y}_{kn} - \widehat{A}_{1(kn)}\overline{x}$ . Similarly, let  $\overline{y}_{k1n} = \frac{1}{k_{1n}} \sum_{j=1}^{k1} \sum_{i=1}^n y_{ij}$ ,  $S_{xy(k_{1n})} = \sum_{j=1}^{k1} \sum_{i=1}^n (x_i - \overline{x})y_{ij}$ ,  $\overline{y}_{k_{2n}} = \frac{1}{k_{2n}} \sum_{j=k_{1}+1}^n \sum_{i=1}^n y_{ij}$ ,  $S_{xy(k_{2n})} = \sum_{j=k_{1}+1}^k \sum_{i=1}^n (x_i - \overline{x})y_{ij}$ . When there is a step shift after the  $k_1$ th sample, the corresponding maximum value is

$$l_1 = -\frac{kn}{2}\log(2\pi) - \frac{k_1n}{2}\log(\widehat{\sigma}_{k_1n}^2) - \frac{k_2n}{2}\log(\widehat{\sigma}_{k_2n}^2) - \frac{kn}{2},$$

where

$$\hat{\sigma}_{k_{1}n}^{2} = \frac{1}{k_{1}n} \sum_{j=1}^{k_{1}} \sum_{i=1}^{n} (y_{ij} - \hat{A}_{0(k_{1}n)} - \hat{A}_{1(k_{1}n)}x_{i})^{2}, \quad \hat{A}_{1(k_{1}n)} = \frac{S_{xy(k_{1}n)}}{k_{1}S_{xx}},$$
$$\hat{\sigma}_{k_{2}n}^{2} = \frac{1}{k_{2}n} \sum_{j=k_{1}+1}^{k} \sum_{i=1}^{n} (y_{ij} - \hat{A}_{0(k_{2}n)} - \hat{A}_{1(k_{2}n)}x_{i})^{2}, \quad \hat{A}_{1(k_{2}n)} = \frac{S_{xy(k_{2}n)}}{k_{2}S_{xx}},$$
$$\hat{A}_{0(k_{1}n)} = \bar{y}_{k_{1}n} - \hat{A}_{1(k_{1}n)}\bar{x}, \quad \hat{A}_{0(k_{2}n)} = \bar{y}_{k_{2}n} - \hat{A}_{1(k_{2}n)}\bar{x}.$$

So, the classical likelihood ratio statistic is defined by

$$lr(k_1n, kn) = -2(l_0 - l_1) = kn \log[\widehat{\sigma}_{kn}^2(\widehat{\sigma}_{k_1n}^2)^{-\frac{k_1}{k}}(\widehat{\sigma}_{k_2n}^2)^{-\frac{k_2}{k}}].$$

The Expression of  $I_{lr}$ ,  $S_{lr}$  and  $\sigma_{lr}$ 

Note that 
$$\widehat{\sigma}_{kn}^2 = \frac{k_1 \widehat{\sigma}_{k_1n}^2 + k_2 \widehat{\sigma}_{k_2n}^2}{k} + \frac{k_1 k_2}{k^2} (\bar{y}_{k_1n} - \bar{y}_{k_2n})^2 + \frac{k_1 k_2 (\frac{1}{k_1} S_{xy(k_1n)} - \frac{1}{k_2} S_{xy(k_2n)})^2}{k^2 n S_{xx}}$$
, then  
 $I_{lr}(k_1) = kn \log \left[ 1 + \frac{k_1 k_2 (\bar{y}_{k_1n} - \bar{y}_{k_2n})^2}{k(k_1 \widehat{\sigma}_{k_1n}^2 + k_2 \widehat{\sigma}_{k_2n}^2)} \right],$   
 $\sigma_{lr}(k_1) = kn \log \left[ \frac{k_1 \widehat{\sigma}_{k_1n}^2 + k_2 \widehat{\sigma}_{k_2n}^2}{k} (\widehat{\sigma}_{k_1n}^2)^{-\frac{k_1}{k}} (\widehat{\sigma}_{k_2n}^2)^{-\frac{k_2}{k}} \right],$   
 $S_{lr}(k_1) = kn \log \left[ 1 + \left( \frac{k_1 k_2 (\frac{1}{k_1} S_{xy(k_1n)} - \frac{1}{k_2} S_{xy(k_2n)})^2}{n S_{xx} [k(k_1 \widehat{\sigma}_{k_1n}^2 + k_2 \widehat{\sigma}_{k_2n}^2) + k_1 k_2 (\bar{y}_{k_1n} - \bar{y}_{k_2n})^2]} \right) \right].$ 

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