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# A control chart based on a change-point model for monitoring linear profiles

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A control chart based on the change-point model is proposed that is able to monitor linear profiles whose parameters are unknown but can be estimated from historical data. This chart can detect a shift in either the intercept, slope or standard deviation. Simulation results show that the proposed approach performs well across a range of possible shifts, and that it can be used during the start-up stages of a process. Simple diagnostic aids are also given to estimate the location of the change and to determine which of the parameters has changed.

## 1. Introduction

Statistical Process Control (SPC) is widely used to monitor industrial processes. Most of the academic research on SPC is focused on charting techniques. In most SPC applications, it is assumed that the quality of a process can be adequately represented by the distribution of a quality characteristic. However, in some situations, the quality of a process is better characterized and summarized by a relationship between the response variable and one or more explanatory variables. Studies focusing on simple linear-regression profiles have received particular attention. In recent years, various methods to monitor linear profiles have been proposed in the literature. Kang and Albin (2000) proposed two control charts for the phase II monitoring of linear profiles. One of them is a multivariate $T^2$ chart and the other is a combination of an Exponentially Weighted Moving Average (EWMA) chart and a range ($R$) chart. In Kim *et al*. (2003), a method based on the combination of three EWMA charts was proposed to detect shifts in either the intercept, the slope, or standard deviation.

Simulation studies showed that the three EWMA charts performed better in detecting sustained shifts in the parameters than the methods in Kang and Albin (2000) in terms of the Average Run Length (ARL). Moreover, they appear to be much simpler to interpret. Mahmoud and Woodall (2004) studied a phase I method for monitoring linear profiles. Mahmoud *et al*. (2005) proposed a change-point method, based on the likelihood ratio statistics, to detect sustained changes in a linear profile data set in phase I. They concluded that to detect both kinds of changes, sustained and randomly occurring unsustained shifts, one could use the change-point method in conjunction with the methods proposed by Mahmoud and Woodall (2004). A discussion of the problems associated with the monitoring of linear profiles is given in Woodall *et al*. (2004).

In a phase II analysis, the process parameters are usually assumed to be known. This is true for all the control charts that monitor linear profiles as mentioned above. However, the process parameters, the intercept, the slope and the standard deviation are usually unknown in the early stages of process improvement, and they are usually estimated using $m$ In-Control (IC) historical samples of size $n$ (or by the phase I study). Some authors have recommended using 20 to 30 samples of size four or five to estimate the process parameters for traditional control charts (see Montgomery (2004) or Ryan (1989)). Quesenberry (1993) and Jones *et al*. (2001, 2004), among others, have investigated the effect of the estimated parameters on the performance of traditional control charts. A recent literature review paper by Jensen *et al*. (2005) provides a thorough discussion on the effects that parameter estimation has on control chart performance. They concluded that when the number of reference samples is small, control charts with estimated parameters produce a large bias in the IC ARL, and reduce the sensitivity of the chart in detecting the process changes as measured by the Out-of-Control (OC) ARL. Moreover, after short runs, the false alarm probabilities from the charts increase dramatically. In fact, to attain a performance similar to a chart with known parameters, 20 or 30 samples are insufficient. For example, for the traditional EWMA chart

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with $\lambda = 0.2$, 300 samples of five observations are needed to achieve the desired level of IC performance (Jones *et al.*, 2001). However, in most cases, it may not be feasible to wait for the accumulation of enough subgroups, because the users usually want to monitor and adjust the process in the start-up stages. Hence, many authors have studied design procedures for traditional control charts with estimated parameters (e.g., Hillier (1967, 1969), Yang and Hillier (1970), Nedumaran and Pignatiello (2001) and Jones (2002)). In response, self-starting methods have been developed that update the parameter estimates with new observations and simultaneously check for the OC conditions (Hawkins, 1987; Quesenberry, 1991, 1995; Hawkins and Olwell, 1998; Sullivan and Jones, 2002). These methods have in fact been developed for situations where the number of available samples is too small to obtain a control chart performance comparable to that obtained with the true parameters.

As we know, the problem of detecting a step shift in a parameter of the process is similar to sequential change-point detection. A commonly used change-point model is:

$$X_i = \begin{cases} N(\mu_0, \sigma_0^2) & \text{for } i = 1, 2, \ldots, \tau, \\ N(\mu_1, \sigma_1^2) & \text{for } i = \tau + 1, \ldots, \end{cases} \tag{1}$$

where $\tau$ is known as the change point. Sequential change-point detection is addressed in Pollak and Siegmund (1991), Siegmund and Venkatraman (1995), Gombay (2000) and in an excellent review paper by Lai (2001), which presents a summary of these methods as well as a class of sequential detection rules. Pignatiello and Simpson (2002) proposed a control chart based on the likelihood ratio approach for on-line detection and show that it has a robust performance when the magnitude of the shifts varies.

Recently, based on the change-point model, Hawkins *et al.* (2003) and Hawkins and Zamba (2005a) have proposed two control charts to detect shifts in the mean and variance, respectively, when the true parameters of the process are unknown. Hawkins and Zamba (2005b) proposed an attractive alternative to the traditional charting method. They suggest a single chart to detect a change in the mean and/or variance based on the likelihood ratio test for unknown parameters. They showed that this change-point formulation not only had the desired run length behavior but was also comparable to the best of the traditional formulations for detecting step changes in parameters. The success of their method inspires us to employ a change-point detection approach in control charts. Since their method can monitor multiple process parameters at the same time, we expect it also to be effective for monitoring linear profiles with three parameters that need to be controlled simultaneously.

Since linear profiles are often modeled by a regression model, the change-point problem associated with regression models is also relevant and has been studied before (see Quandt (1958), Holbert (1982), Hawkins (1989), Kim and Siegmund (1989), Kim (1994) and Chen (1998)). It should be noted that the sample sizes are fixed in these papers which may not be appropriate for a phase II analysis. The change-point model of Mahmoud *et al.* (2005) was designed to monitor linear profiles but primarily focused on the phase I analysis, although the authors claimed that their method might be generalized to phase II settings.

In this paper, a control chart based on a likelihood ratio statistic is proposed for monitoring linear profiles when the true process parameters are unknown. The benefit of such a control chart is that it may eliminate the need to collect a large number of reference samples to estimate the process parameters before the control scheme begins (although it is still advisable to collect a few preliminary or historical samples). We demonstrate the effectiveness of our proposed approach through Monte Carlo simulations.

## 2. Control chart for linear profiles

In this section, we first present the model for linear profiles under consideration and a brief description of the change-point model. We then discuss our proposed control chart, its design, and some diagnostic aids.

### 2.1. *The change-point model for linear profiles*

Denote by $\{(x_i, y_{ij}),\ i = 1, 2, \ldots, n\}$ the $j$th random sample collected over time. For an IC process, the relationship between the response and explanatory variables is assumed to be:

$$y_{ij} = A_0 + A_1 x_i + \varepsilon_{ij} \quad i = 1, 2, \ldots, n, \tag{2}$$

where the $\varepsilon_{ij}/\sigma$ are independent, identically distributed (i.i.d) standard Normal random variables, and the explanatory variable $X$ takes on $n$ values that are fixed. This is usually the case in practical applications and is consistent with the assumptions in Kang and Albin (2000), Kim *et al.* (2003) and Mahmoud and Woodall (2004).

When the parameters $A_0$, $A_1$ and $\sigma^2$ are unknown, a widely used method is to estimate them using historical data. Suppose there is a total of $m$ ($m \geq 1$) IC historical samples $\{(x_i, y_{ij}), i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$. The unbiased estimators for $A_0$, $A_1$ and $\sigma^2$ are, respectively, the average of the $m$ least-square estimators $a_{0j}$, $a_{1j}$ and $MSE_j$ given by:

$$a_{0j} = \bar{y}_j - a_{1j}\bar{x}, \quad a_{1j} = \frac{S_{xy(j)}}{S_{xx}},$$
$$MSE_j = \frac{1}{n-2}\sum_{i=1}^{n}(y_{ij} - a_{1j}x_i - a_{0j})^2,$$

where $\bar{y}_j = (1/n)\sum_{i=1}^{n} y_{ij},\ \bar{x} = (1/n)\sum_{i=1}^{n} x_i,\ S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$ and $S_{xy(j)} = \sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$.

After we get these estimates, the parameters are treated as if they are known, and the monitoring task can then be started. As more IC samples are obtained, one may wish to update the estimates and restart monitoring. However, the statistical properties of this procedure, such as the IC ARL, cannot be obtained easily, so this procedure seems difficult to design. To deal with the unknown parameters, our method uses the sequential change-point formulation to construct the control chart. With this approach we can update the parameter estimates with new observations and also check for the OC conditions simultaneously.

Consider the following model:

$$y_{ij} = A_{0j} + A_{1j}x_i + \varepsilon_{ij}$$
$$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m, \quad m + 1, \ldots. \quad (3)$$

After observing $t$ future subgroups, the null hypothesis of interest is that the entire process is IC, namely, $A_{0j} = A_0$, $A_{1j} = A_1$ and $\sigma_j = \sigma$ for all $j$. The alternative hypothesis is that the process is initially IC, but after the change point $\tau(\tau \geq m)$, a step shift in the intercept and/or slope and/or standard deviation occurs. The sample intercept $A_{0j}$, slope $A_{1j}$ and standard deviation $\sigma_j$ are equal to $A_0$, $A_1$ and $\sigma$, respectively, for $j = 1, \ldots, m, m + 1, \ldots, \tau$. The parameters $A_{0j}$, $A_{1j}$ and $\sigma_j$ are equal to $A'_0$, $A'_1$ and $\sigma'$, respectively, for the last $m + t - \tau$ samples, starting with $j = \tau + 1$.

Assume that $t$ future samples have been collected, and let $k = m + t$, $\bar{y}_{kn} = (1/kn)\sum_{j=1}^{k}\sum_{i=1}^{n} y_{ij}$ and $S_{xy(kn)} = \sum_{j=1}^{k}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij}$. Then, the logarithm of the likelihood function for the $t$ samples is given by:

$$-\frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{n}\left[\ln\left(2\pi\sigma_j^2\right) + \frac{(y_{ij} - A_{0j} - A_{1j}x_i)^2}{\sigma_j^2}\right].$$

If the data were collected under IC conditions, the maximum value of the logarithm likelihood function would be:

$$l_0 = -\frac{kn}{2}\ln(2\pi) - \frac{kn}{2}\ln\left(\hat{\sigma}_{kn}^2\right) - \frac{kn}{2},$$

where $\hat{\sigma}_{kn}^2 = (1/kn)\sum_{j=1}^{k}\sum_{i=1}^{n}(y_{ij} - \hat{A}_{0(kn)} - \hat{A}_{1(kn)}x_i)^2$, $\hat{A}_{1(kn)} = S_{xy(kn)}/kS_{xx}$, $\hat{A}_{0(kn)} = \bar{y}_{kn} - \hat{A}_{1(kn)}\bar{x}$.

Define

$$\bar{y}_{k_1n} = \frac{1}{k_1n}\sum_{j=1}^{k_1}\sum_{i=1}^{n} y_{ij},$$

$$S_{xy(k_1n)} = \sum_{j=1}^{k_1}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij},$$

$$\bar{y}_{k_2n} = \frac{1}{k_2n}\sum_{j=k_1+1}^{k}\sum_{i=1}^{n} y_{ij},$$

$$S_{xy(k_2n)} = \sum_{j=k_1+1}^{k}\sum_{i=1}^{n}(x_i - \bar{x})y_{ij},$$

where $k_2 = k - k_1, m \leq k_1 < k$. When there is a step shift after the $k_1$th sample, the corresponding maximum value of the log-likelihood is:

$$l_1 = -\frac{kn}{2}\ln(2\pi) - \frac{k_1n}{2}\ln\left(\hat{\sigma}_{k_1n}^2\right) - \frac{k_2n}{2}\ln\left(\hat{\sigma}_{k_2n}^2\right) - \frac{kn}{2},$$

where

$$\hat{\sigma}_{k_1n}^2 = \frac{1}{k_1n}\sum_{j=1}^{k_1}\sum_{i=1}^{n}\left(y_{ij} - \hat{A}_{0(k_1n)} - \hat{A}_{1(k_1n)}x_i\right)^2,$$

$$\hat{\sigma}_{k_2n}^2 = \frac{1}{k_2n}\sum_{j=k_1+1}^{k}\sum_{i=1}^{n}\left(y_{ij} - \hat{A}_{0(k_2n)} - \hat{A}_{1(k_2n)}x_i\right)^2,$$

$$\hat{A}_{1(k_1n)} = \frac{S_{xy(k_1n)}}{k_1S_{xx}}, \quad \hat{A}_{1(k_2n)} = \frac{S_{xy(k_2n)}}{k_2S_{xx}},$$

$$\hat{A}_{0(k_1n)} = \bar{y}_{k_1n} - \hat{A}_{1(k_1n)}\bar{x},$$

$$\hat{A}_{0(k_2n)} = \bar{y}_{k_2n} - \hat{A}_{1(k_2n)}\bar{x}.$$

Combining $l_0$ and $l_1$, a classical Likelihood Ratio Test (LRT) statistic is defined by:

$$\begin{aligned} lr(k_1n, kn) &= -2(l_0 - l_1) \\ &= kn\ln\left[\hat{\sigma}_{kn}^2\left(\hat{\sigma}_{k_1n}^2\right)^{-k_1/k}\left(\hat{\sigma}_{k_2n}^2\right)^{-k_2/k}\right]. \quad (4) \end{aligned}$$

The $lr(k_1n, kn)$ is the same as the LRT statistic in Mahmoud *et al.* (2005). In fact, this LRT statistic was first given in Quandt (1958).

Define

$$lr_{\max,m,k} = \max_{m \leq k_1 < k} lr(k_1n, kn)$$

Using the methodology presented in Hawkins and Zamba (2005b), it is natural to construct the control chart for linear profiles based on the statistic $lr_{\max,m,k}$ when there are $m$ IC historical samples. Note that our statistic $lr_{\max,m,k}$ is a little different from its counterpart used in Hawkins and Zamba (2005b). Here, $lr_{\max,m,k}$ is not the maximum of $lr(k_1n, kn)$ across all values of $k_1$ but only on a constrained interval of $m \leq k_1 < k$. From the view of change points, since the first $m$ samples are IC, the shift should not occur in these samples, that is to say, the maximum of $lr(k_1n, kn)$ is expected to be one of the values in $\{lr(k_1n, kn), k_1 = m, m + 1, \ldots, k - 1\}$. This minor modification simplifies the design of the control chart because the design is almost the same for all values of $m$. This will be illustrated later in the paper.

Given $n$, unlike the two-sample statistic $T_{jn}$ used in Hawkins *et al.* (2003), the expectation of $lr(k_1n, kn)$ depends on the value of $k_1$. Sullivan and Woodall (1996) stated that the control chart would be less efficient if the expectations of the likelihood ratio statistics were not equal. In order to offset the side effect of unequal expectations, Hawkins and Zamba (2005b) used the Bartlett-corrected test statistic in place of the standard Generalized Likelihood Ratio (GLR) statistic, and Mahmoud *et al.* (2005) used the standard expansion of the expectation (see Kendall and Stuart (1977) for more details) to obtain a normalizing factor $C$ and used the statistic $LRT/C$ in their paper. In fact, the variance of $lr(k_1n, kn)$ also varies with the value of $k_1$. In this paper, in order to compensate for the varying expectation and variance of $lr(k_1n, kn)$), we naturally consider using the standardized $lr(k_1n, kn)$ which is defined as:

$$slr(k_1n, kn) = \frac{lr(k_1n, kn) - E[lr(k_1n, kn)]}{\sqrt{\mathrm{Var}[lr(k_1n, kn)]}}. \quad (5)$$

Our simulations show that the change-point approach based on the statistic $slr(k_1n, kn)$ performs almost uniformly better than the method using the Bartlett-corrected statistic for either the phase I or phase II setting, although the difference between them is not very significant.

Although $lr(k_1n, kn) \xrightarrow{\mathcal{L}} \chi^2(3)$ as $k_1, k, k - k_1 \to \infty$ (see Serfling (1980)), this is not true for fixed $k_1$. The limiting distribution of $lr(k_1n, kn)$ under the null hypothesis, for fixed $k_1$ and fixed $n$, is given in the Appendix. The expectation and the variance of this limiting distribution are given as:

$$E\,[lr(k_1n, \infty)] = k_1n\left[\ln\left(\frac{k_1n}{2}\right) - \psi_0\left(\frac{k_1n-2}{2}\right)\right], \quad (6)$$

$$\mathrm{Var}\,[lr(k_1n, \infty)] = (k_1n)^2\psi_1\left(\frac{k_1n-2}{2}\right) - 2k_1n, \quad (7)$$

where $\psi_0(\cdot)$ and $\psi_1(\cdot)$ are, respectively, the digamma and trigamma function. The recursive formulas for calculating $\psi_0$ and $\psi_1$ are also given in the Appendix. Note that for fixed $k_1$, if the process is IC, the distribution of $lr(k_1n, kn)$ is symmetric about $k_1n$, that is to say, $lr(k_1n, kn)$ and $lr((k - k_1)n, kn)$ are identically distributed.

Our simulations results (not reported here) show that there is very little difference between the simulated values and asymptotic values based on Equations (6) and (7). For simplicity, the asymptotic expectation and variance are used to redefine the standardized likelihood ratio as:

$$slr(k_1n, kn) = \frac{lr(k_1n, kn) - E[lr(k_1n, \infty)]}{\sqrt{\mathrm{Var}\,[lr(k_1n, \infty)]}}. \quad (8)$$

### 2.2. *Our proposed control chart and its design*

The maximal standardized likelihood ratio statistic for the $k = m + t$ observations is defined by:

$$slr_{\max,m,k} = \max_{m \le k_1 < k} slr(k_1n, kn). \quad (9)$$

If $slr_{\max,m,m+t} > h_{m,t}$, an OC signal is given. However, if $slr_{\max,m,m+t} \le h_{m,t}$, the monitoring continues and the $(t+1)$st future sample is obtained and the procedure is repeated. In this paper, we call this chart the LRT chart.

As one may have noted, due to the appearance of a change point, the expectation of the GLR statistic is large, not only at the change point, but also at points on both sides. Considering only the maximum value of $slr$ given by Equation (9) may be an inefficient utilization of the information available regarding the change point. When there is a small shift in the process, it seems intuitive that for small $k$, all the values of $slr(k_1n, kn)$ will be small, and hence, $slr_{\max,m,k}$ will also be small. Consequently, the chart will not signal a shift. Thus, new samples will continue to be monitored until the value of $slr_{\max,m,k}$ is so large that the control chart signals. In order to sensitize the response of our control chart to small shifts, we can use the EWMA or CUSUM method to accumulate the small increments and make the control chart respond faster. Using the above idea, based on the standardized likelihood ratio statistic $slr(k_1n, kn)$ given by Equation (8), the statistic of our proposed EWMA chart is given by:

$$\begin{aligned} Y_j(m, t) = \max(0, \lambda \times slr(jn, (m+t)n) \\ + (1-\lambda)Y_{j-1}(m, t)), \end{aligned} \quad (10)$$

where $j = m, m+1, \ldots, m+t-1$, $Y_{m-1}(m, t) = 0$ and $\lambda$ $(0 < \lambda \le 1)$ is a smoothing constant. We define $Y_{\max}(m, t) = \max_{m \le j < m+t} Y_j(m, t)$. Our charting procedure is given as follows

1. After the $t$th future sample is monitored, compute $Y_{\max}(m, t)$.
2. If $Y_{\max}(m, t) \le h_{m,t}$, where $h_{m,t}$ is chosen such that the chart attains the specified IC ARL, then conclude that there is no evidence of a shift, and monitor the $(t+1)$st future sample.
3. If $Y_{\max}(m, t) > h_{m,t}$, then an alarm is triggered to indicate that the process may be out of control.

The difference between the LRT chart based on Equation (9) and this EWMA chart is that after the $(m+t)$th sample is monitored, the former chart calculates the maximum value of $slr(k_1n, (m+t)n)$, $m \le k_1 < k$, whereas the latter chart calculates the maximum of the exponentially weighted average of $slr(k_1n, (m+t)n)$.

In this paper, the smoothing constant $\lambda$ in Equation (10) is taken to be 0.2. In general, smaller-smoothing constants lead to quicker detection of smaller shifts (Lucas and Saccucci, 1990). In fact, when $\lambda$ is equal to one, the performance of the EWMA chart is the same as that of the LRT chart defined by Equation (9). The adaptive EWMA chart proposed by Capizzi and Masarotto (2003) can also be used here in place of our EWMA chart. Although we do not investigate the adaptive EWMA chart here, we expect that it will have better detection performance over a range of shifts.

For a given false alarm probability $\alpha$, the control limit of our proposed EWMA chart, $h_{m,t}(\alpha)$ can be obtained by solving the following equations:

$$\begin{aligned} \Pr(Y_{\max}(m, t) > h_{m,t}(\alpha) \mid Y_{\max}(m, i) \le h_{m,i}(\alpha), 1 \le i < t) \\ = \alpha \quad \text{for } t > 1, \Pr(Y_{\max}(m, 1) > h_{m,1}(\alpha)) = \alpha. \end{aligned}$$

Due to the intricacy of this conditional probability, it seems impossible to solve analytically for $h_{m,t}$. Therefore, similar to the methods in Hawkins *et al.* (2003) and Hawkins and Zamba (2005a, 2005b), we use 1000 000 sequences of length 500 to estimate them. As in Hawkins *et al.* (2003), we also suggest starting monitoring after some

**Table 1.** The $h_{m,t}(\alpha)$ of the EWMA chart for $n = 4$

| | *m* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *10* | | | | *50* | | | |
| | *IC ARL* | | | | *IC ARL* | | | |
| *t* | *100* | *200* | *370* | *500* | *100* | *200* | *370* | *500* |
| 1 | 0.695 | 0.828 | 0.938 | 0.992 | 0.695 | 0.828 | 0.953 | 0.992 |
| 2 | 0.969 | 1.125 | 1.266 | 1.344 | 0.969 | 1.125 | 1.266 | 1.344 |
| 3 | 1.219 | 1.406 | 1.594 | 1.660 | 1.219 | 1.422 | 1.594 | 1.695 |
| 4 | 1.422 | 1.656 | 1.875 | 1.977 | 1.438 | 1.688 | 1.891 | 1.994 |
| 5 | 1.578 | 1.844 | 2.094 | 2.223 | 1.609 | 1.906 | 2.125 | 2.258 |
| 6 | 1.719 | 2.031 | 2.281 | 2.398 | 1.750 | 2.062 | 2.344 | 2.504 |
| 7 | 1.812 | 2.156 | 2.438 | 2.609 | 1.875 | 2.219 | 2.531 | 2.680 |
| 8 | 1.906 | 2.250 | 2.594 | 2.750 | 1.969 | 2.344 | 2.656 | 2.820 |
| 9 | 1.969 | 2.344 | 2.688 | 2.855 | 2.047 | 2.438 | 2.812 | 2.961 |
| 10 | 2.031 | 2.438 | 2.781 | 2.961 | 2.125 | 2.562 | 2.906 | 3.102 |
| 11 | 2.078 | 2.500 | 2.875 | 3.066 | 2.188 | 2.625 | 3.000 | 3.172 |
| 12 | 2.125 | 2.562 | 2.938 | 3.137 | 2.234 | 2.688 | 3.094 | 3.277 |
| 13 | 2.172 | 2.625 | 3.000 | 3.207 | 2.266 | 2.750 | 3.156 | 3.348 |
| 14 | 2.203 | 2.656 | 3.062 | 3.242 | 2.297 | 2.781 | 3.203 | 3.383 |
| 15 | 2.250 | 2.719 | 3.109 | 3.312 | 2.328 | 2.812 | 3.250 | 3.453 |
| 16 | 2.266 | 2.750 | 3.156 | 3.348 | 2.359 | 2.844 | 3.281 | 3.523 |
| 17 | 2.297 | 2.781 | 3.188 | 3.383 | 2.391 | 2.875 | 3.344 | 3.559 |
| 18 | 2.312 | 2.812 | 3.234 | 3.418 | 2.422 | 2.938 | 3.375 | 3.594 |
| 19 | 2.328 | 2.844 | 3.281 | 3.488 | 2.438 | 2.969 | 3.406 | 3.629 |
| 20 | 2.359 | 2.875 | 3.312 | 3.523 | 2.469 | 3.000 | 3.438 | 3.664 |
| 22 | 2.375 | 2.906 | 3.375 | 3.594 | 2.484 | 3.031 | 3.500 | 3.734 |
| 24 | 2.406 | 2.938 | 3.406 | 3.629 | 2.516 | 3.062 | 3.531 | 3.770 |
| 26 | 2.438 | 2.969 | 3.438 | 3.664 | 2.531 | 3.094 | 3.562 | 3.805 |
| 28 | 2.453 | 3.000 | 3.469 | 3.699 | 2.547 | 3.125 | 3.594 | 3.840 |
| 30 | 2.469 | 3.031 | 3.500 | 3.734 | 2.562 | 3.156 | 3.656 | 3.910 |
| 35 | 2.531 | 3.094 | 3.594 | 3.840 | 2.594 | 3.188 | 3.719 | 3.980 |
| 40 | 2.562 | 3.156 | 3.656 | 3.910 | 2.623 | 3.219 | 3.750 | 4.051 |
| 50 | 2.609 | 3.188 | 3.719 | 3.980 | 2.641 | 3.250 | 3.812 | 4.086 |
| 60 | 2.625 | 3.219 | 3.781 | 4.016 | 2.656 | 3.281 | 3.844 | 4.121 |
| 70 | 2.641 | 3.250 | 3.812 | 4.051 | 2.672 | 3.312 | 3.875 | 4.156 |
| 80 | 2.656 | 3.281 | 3.844 | 4.086 | 2.688 | 3.328 | 3.891 | 4.191 |
| 90 | 2.680 | 3.312 | 3.875 | 4.121 | 2.703 | 3.344 | 3.906 | 4.221 |
| 115 | 2.703 | 3.344 | 3.906 | 4.191 | 2.719 | 3.359 | 3.938 | 4.247 |
| 140 | 2.719 | 3.375 | 3.938 | 4.227 | 2.727 | 3.375 | 3.969 | 4.262 |
| 165 | 2.734 | 3.391 | 3.969 | 4.262 | 2.734 | 3.391 | 4.000 | 4.279 |
| 190 | 2.750 | 3.406 | 3.984 | 4.297 | 2.742 | 3.406 | 4.016 | 4.297 |
| 240 | 2.773 | 3.422 | 4.000 | 4.314 | 2.750 | 3.422 | 4.031 | 4.314 |
| 290 | | 3.438 | 4.031 | 4.332 | | 3.438 | 4.039 | 4.332 |
| 390 | | 3.469 | 4.047 | 4.350 | | 3.469 | 4.047 | 4.350 |
| 490 | | | 4.062 | 4.367 | | | 4.062 | 4.367 |

preliminary or historical samples are obtained. In this paper, we generally consider the number of IC samples in the range of $m \geq 10$. Table 1 shows the control limits of the EWMA chart for different combinations of parameters: $n = 4$; $m = 10$ or 50; $t$ is in the range 1-490; and $\alpha$ values are 0.01, 0.005, 0.0027 and 0.002, corresponding to IC ARLs of 100, 200, 370 and 500, respectively. As shown in Table 1, $h_{m,t}(\alpha)$ increases more appreciably when $t$ is small than when $t$ is large. The missing values in Table 1 can be approximated by the last entries in the same column. Our numerous simulation experiments highlight that these control limits perform quite well. We could have come up with a regression fit approximating the control limits in Table 1 and use it as a convenient formula for practitioners, as Hawkins *et al.* (2003) and Hawkins and Zamba (2005a, 2005b) did in their papers. However, it turns out that we could not fit the data in our table well using a simple regression. That is why we chose not to give the regression formula here. Using these tabulated values may not be very convenient for engineers, but the data can be easily incorporated into computer programs, where storing such data is a trivial task.

There is a vital issue remaining to be considered, which is the choice of $h_{m,t}(\alpha)$ for different $m$ and $n$. From Table 1 we observe that the difference between control limits for $m = 10$ and $m = 50$ is very small. The reason for this behavior is that the distribution of $Y_j(m_1, t)$ is approximately the same as that of $Y_j(m_2, t)$ when $m_1$, $m_2$ and $n$ are large enough. It is thus not surprising that the control limits of the EWMA charts for $m_1$ and $m_2$ are close. We investigated the run-length distributions of the EWMA charts (30 000 simulations) for $\alpha = 0.005$, $n = 4$ and $m = 15, 20, \ldots, 45$ using the limits of $h_{10,t}(0.005)$. Since these figures are hardly distinguishable from one another, we do not report them here. From the calculations we can, however, infer that the behavior of the run length for these $m$ values is very close to a geometric distribution. The behavior of the ARLs and of the Standard Deviations of the Run Length (SDRL) supports this observation. The empirical run-length distributions of the EWMA charts (100 000 simulations) for $\alpha = 0.005$, $n = 4$, $m = 100, 200, \ldots, 500$ were also obtained. Even if the control limits obtained from a process with $m = 50$ are used to monitor the process with $m = 500$, the IC ARL, the SDRL, and the distribution of the run length of the control chart are still quite satisfactory. Hence, we suggest $h_{50,t}(\alpha)$ be used as the control limit for $m > 50$ as long as the requirement of IC behavior of run length is not very strict.

The control limits for $n = 5, 6, \ldots, 10$ were also obtained from the same simulations (available from the authors). Because the EWMA statistics $Y_j(m, t)$ with different values of $n$ have approximately the same distribution, we would expect their control limits to be very close. Once again, our simulations verify this conclusion. The ARLs and SDRLs from 100 000 simulations for different values of $n$ are tabulated in Table 2. Apparently, regardless of the choice of $n$, the desired level ($\alpha = 0.005$) of ARLs and SDRLs can almost always be achieved. So, for simplicity, engineers can use the control limits obtained using $n = 4$ or 5 as the control limits for any moderate $n$ ($5 < n < 20$).

In practice, it may be more convenient to use a normalized plot statistic given by $Y_{\max}(m, t)$ divided by the control limit $h_t$. This makes the control limit, of this normalized chart, to be unity, making it easier to construct the chart.

**Table 2.** The ARL and SDRL of the EWMA chart for $n = 4, 5, \ldots, 10$ with $h_{10,t}(0.005)$

| | *n* | | | | | | |
|---|---|---|---|---|---|---|---|
| | *4* | *5* | *6* | *7* | *8* | *9* | *10* |
| ARL | 199.8 | 199.3 | 197.5 | 197.8 | 196.4 | 196.2 | 196.5 |
| SDRL | 199.0 | 198.4 | 195.8 | 197.2 | 194.3 | 195.5 | 194.9 |

### 2.3. *Diagnostic aids and implementation*

In practical applications of quality control, there are two issues that need to be considered. One is detecting if the process is IC, and the other is to point out the position of the shift after the process has indeed shifted. Confirming the existence of change points in a process would help engineers to identify the underlying cause more quickly. In order to aid in identifying the change-point location, the maximum likelihood estimator of the change-point statistic is used. Let us again consider the case where there are $m$ historical IC samples. Once an OC signal is given at the $k$th sample, implying that a shift in parameters must have occurred at the $\tau$th sample, $m \leq \tau < k$, we propose to estimate the change point $\tau$ of a step shift by:

$$\hat{\tau} = \underset{m \leq k_1 < k}{\arg} \max\{slr(k_1 n, kn)\}. \quad (11)$$

Note that this is consistent with Pignatiello and Samuel (2001).

As Kim *et al.* (2003) pointed out, it is very necessary to justify which parameter has shifted after a chart signals a shift. For that purpose, using coded explanatory values, they obtained the following alternative form of the underlying model:

$$y_{ij} = B_0 + B_1 x_i^* + \varepsilon_{ij} \; i = 1, 2, \ldots, n, \quad (12)$$

where $B_0 = A_0 + A_1 \bar{x}$, $B_1 = A_1$ and $x_i^* = (x_i - \bar{x})$. Using Equation (12), the least squares estimators for $B_0$, $B_1$ and $\sigma^2$ are independent. Thus, they proposed to use three independent EWMA charts to detect if the intercept, the slope or the standard deviation, respectively, had changed.

At first glance, our proposed method may appear to analyze many different effects simultaneously, and it may not be easy to diagnose which parameter has shifted. For this type of control chart, diagnostic aids have been proposed and developed in the literature. For example, Hawkins and Zamba (2005b) used two conventional parametric tests: a two-sided $F$-test to detect changes in the variance and an approximate $t$-test to detect changes in the mean. Mahmoud *et al.* (2005) decomposed the likelihood ratio statistic into three parts, each of which reflects the behavior of an individual parameter, so that the decomposed statistics can help pinpoint in which parameter a shift may have occurred. Similar decomposition ideas have also been used in Gulliksen and Wilks (1950) and Fatti and Hawkins (1986). In this paper, similarly to Mahmoud *et al.* (2005), we also decompose the test statistic $lr(\hat{\tau}n, kn)$ in Equation (4) into three components: $I_{lr}(\hat{\tau})$, $S_{lr}(\hat{\tau})$ and $\sigma_{lr}(\hat{\tau})$, which are used, respectively, as the index of the relative contribution from the $Y$-intercept ($B_0$), the slope ($B_1$) and the standard deviation ($\sigma$). The three components are defined as:

$$I_{lr}(\hat{\tau}) = kn \ln\left[1 + \frac{k_1 k_2 (\bar{y}_{k_1 n} - \bar{y}_{k_2 n})^2}{k\left(k_1 \hat{\sigma}^2_{k_1 n} + k_2 \hat{\sigma}^2_{k_2 n}\right)}\right],$$

$$\sigma_{lr}(\hat{\tau}) = kn \ln\left[\frac{k_1 \hat{\sigma}^2_{k_1 n} + k_2 \hat{\sigma}^2_{k_2 n}}{k} \left(\hat{\sigma}^2_{k_1 n}\right)^{-k_1/k} \left(\hat{\sigma}^2_{k_2 n}\right)^{-k_2/k}\right],$$

$$S_{lr}(\hat{\tau}) = kn \ln \left[1 + \frac{k_1 k_2 (1/k_1 S_{xy(k_1 n)} - 1/k_2 S_{xy(k_2 n)})^2}{n S_{xx}\left[k\left(k_1 \hat{\sigma}^2_{k_1 n} + k_2 \hat{\sigma}^2_{k_2 n}\right) + k_1 k_2 (\bar{y}_{k_1 n} - \bar{y}_{k_2 n})^2\right]}\right].$$

To simplify the calculations, we define:

$$U_k = U_{k-1} + \sum_{i=1}^{n} y_{ik},$$

$$V_k = V_{k-1} + \sum_{i=1}^{n} x_i^* y_{ik},$$

$$W_k = W_{k-1} + \sum_{i=1}^{n} y_{ik}^2,$$

where $U_0 = V_0 = W_0 = 0$. Then, the least-squares estimators, $\hat{\sigma}^2_{kn}$, $\hat{\sigma}^2_{k_1 n}$ and $\hat{\sigma}^2_{(k-k_1)n}$ can be rewritten as:

$$\hat{\sigma}^2_{kn} = \frac{1}{kn}\left(W_k - \frac{1}{kn} U_k^2 - \frac{1}{k S_{xx}} V_k^2\right),$$

$$\hat{\sigma}^2_{k_1 n} = \frac{1}{k_1 n}\left(W_{k_1} - \frac{1}{k_1 n} U_{k_1}^2 - \frac{1}{k_1 S_{xx}} V_{k_1}^2\right),$$

$$\hat{\sigma}^2_{(k-k_1)n} = \frac{1}{(k-k_1)n}\left[(W_k - W_{k_1}) - \frac{1}{(k-k_1)n}(U_k - U_{k_1})^2 - \frac{1}{(k-k_1)S_{xx}}(V_k - V_{k_1})^2\right].$$

Using these equations, our proposed EWMA statistic, based on $slr(k_1 n, kn)$, can be calculated easily in a recursive manner.

## 3. An illustrative example

In this section, an example is presented to illustrate the implementation of the proposed EWMA control chart. In this example, the underlying IC linear profile model is $y_{ij} = 3 + 2x_i + \varepsilon_{ij}$, where the $\varepsilon_{ij}$ values are i.i.d Normal random variables with mean zero and unit variance. The same model has also been used by Kang and Albin (2000). The value of the explanatory variable is fixed at 2, 4, 6 or 8. Obviously, $\bar{x} = 5$, $S_{xx} = 20$, $B_0 = 13$ and $B_1 = 2$. There are $m = 10$ IC historical samples, which are the first ten rows in Table 3. Suppose the slope $B_1$ has shifted from 2.0 to 2.25 after the tenth future sample. Given the overall IC ARL = 200, the control limits of the EWMA chart $h_{m,t}(0.005)$ for

**Table 3.** Data for example with a shift in the slope after the 20th sample

| $j$ | $y_{ij}$ | | | | $t$ | $Y_{max}(m,t)$ | $h_{m,t}(0.005)$ | $slr$ | $lr$ | $I_{lr}$ | $S_{lr}$ | $\sigma_{lr}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7.54 | 10.03 | 14.87 | 18.49 | | | | −0.24 | 4.07 | 0.39 | 1.27 | 2.41 |
| 2 | 7.75 | 11.78 | 15.86 | 17.49 | | | | 1.14 | 7.18 | 0.02 | 5.77 | 1.39 |
| 3 | 6.46 | 9.38 | 14.87 | 19.07 | | | | 0.51 | 4.86 | 0.78 | 3.46 | 0.62 |
| 4 | 7.20 | 10.15 | 13.71 | 19.08 | | | | 1.46 | 7.24 | 1.82 | 4.22 | 1.19 |
| 5 | 8.74 | 10.08 | 15.18 | 19.32 | | | | 1.72 | 7.80 | 0.84 | 6.42 | 0.54 |
| 6 | 6.50 | 11.75 | 15.66 | 19.42 | | | | 1.33 | 6.67 | 0.30 | 5.43 | 0.93 |
| 7 | 7.22 | 10.96 | 14.33 | 19.23 | | | | 2.21 | 8.89 | 0.37 | 6.34 | 2.18 |
| 8 | 6.75 | 12.15 | 16.27 | 18.95 | | | | 2.17 | 8.72 | 0.02 | 6.66 | 2.04 |
| 9 | 5.79 | 12.75 | 14.77 | 20.77 | | | | 0.63 | 4.73 | 0.09 | 4.20 | 0.45 |
| 10 | 6.14 | 11.79 | 15.59 | 19.15 | | | | 0.71 | 4.92 | 0.16 | 3.81 | 0.95 |
| 11 | 7.32 | 11.94 | 15.68 | 19.94 | 1 | 0.266 | 0.828 | 1.27 | 6.32 | 0.92 | 3.95 | 1.45 |
| 12 | 7.86 | 11.55 | 14.06 | 19.06 | 2 | 0.000 | 1.125 | 2.42 | 9.20 | 1.09 | 6.41 | 1.70 |
| 13 | 7.98 | 11.31 | 15.89 | 19.30 | 3 | 0.297 | 1.406 | 3.82 | 12.72 | 2.33 | 8.19 | 2.21 |
| 14 | 4.50 | 10.39 | 16.66 | 19.55 | 4 | 0.198 | 1.656 | 1.29 | 6.32 | 1.66 | 4.13 | 0.53 |
| 15 | 7.55 | 12.30 | 14.29 | 18.80 | 5 | 0.017 | 1.844 | 2.56 | 9.53 | 2.18 | 6.93 | 0.42 |
| 16 | 7.31 | 8.83 | 14.88 | 18.08 | 6 | 0.164 | 2.031 | 2.70 | 9.90 | 0.77 | 9.12 | 0.01 |
| 17 | 6.55 | 10.49 | 14.26 | 18.65 | 7 | 0.612 | 2.156 | 3.13 | 11.00 | 0.20 | 10.69 | 0.11 |
| 18 | 7.46 | 10.31 | 16.51 | 19.57 | 8 | 0.084 | 2.250 | 3.40 | 11.73 | 0.67 | 10.94 | 0.11 |
| 19 | 5.88 | 11.35 | 16.02 | 19.89 | 9 | 0.094 | 2.344 | 2.79 | 10.21 | 1.10 | 8.77 | 0.34 |
| 20 | 6.80 | 10.96 | 13.74 | 18.29 | 10 | 0.102 | 2.438 | **3.95** | 13.21 | 0.34 | **12.69** | 0.18 |
| 21 | 5.82 | 10.58 | 14.70 | 19.56 | 11 | 0.475 | 2.500 | 3.54 | 12.24 | 0.07 | 11.47 | 0.69 |
| 22 | 6.21 | 10.03 | 15.82 | 21.38 | 12 | 0.687 | 2.562 | 1.71 | 7.59 | 0.34 | 6.68 | 0.57 |
| 23 | 6.49 | 11.11 | 16.56 | 18.39 | 13 | 0.300 | 2.625 | 2.32 | 9.26 | 0.53 | 8.27 | 0.45 |
| 24 | 5.02 | 9.30 | 15.16 | 19.80 | 14 | 1.409 | 2.656 | 0.71 | 5.13 | 0.00 | 4.65 | 0.48 |
| 25 | 7.05 | 12.65 | 13.16 | 19.36 | 15 | 0.670 | 2.719 | 2.34 | 9.64 | 0.00 | 9.14 | 0.49 |
| 26 | 4.80 | 11.96 | 14.25 | 21.19 | 16 | 1.759 | 2.750 | 1.19 | 6.76 | 0.01 | 4.35 | 2.40 |
| 27 | 5.74 | 11.52 | 16.22 | 19.28 | 17 | 1.835 | 2.781 | 0.92 | 6.50 | 0.15 | 4.01 | 2.35 |
| 28 | 6.08 | 9.43 | 15.79 | 19.85 | 18 | 2.322 | 2.812 | −0.31 | 3.77 | 0.00 | 2.28 | 1.49 |
| 29 | 6.34 | 9.71 | 15.32 | 20.71 | 19 | **2.901** | 2.844 | | | | | |

$t = 1, 2, \ldots, 19$ and the statistic $Y_{\max}(m, t)$ were computed and are listed in Table 3.

Our EWMA chart signals a shift at the 29th sample (i.e., $j = 29$). Then, by looking at the values of $slr(jn, 29n)$ for $j = 10, 11, \ldots, 28$, one can find that its maximum occurs at $j = 20$ with $slr(20n, 29n) = 3.95$. This maximum indicates precisely the change-point location $\tau$ of the shift. Moreover, inspecting the values of $I_{lr}(20)$, $S_{lr}(20)$ and $\sigma_{lr}(20)$, which are 0.34, 12.69 and 0.18, respectively, reveals which parameter underwent a shift. Since $S_{lr}(20) = 12.69$ is much larger than the other two decomposed components, it suggests that there is a shift in the slope $B_1$ after sample 20. It also indicates that our EWMA chart detects the change in parameters after nine OC samples have been collected.

## 4. Performance comparisons

In this section, first, the LRT chart defined by Equation (9) and the EWMA chart defined by Equation (10) are compared. Then, our proposed EWMA chart is compared with the chart proposed by Kim *et al.* (2003). Finally, the performance of our EWMA chart is evaluated for different values of $\tau$.

The underlying IC model is again the same as that of Kang and Albin (2000). Simulation results comparing our proposed LRT and the EWMA chart are shown in Table 4 (50 000 simulations). Note that any series for which a signal occurs before time $\tau$ is discarded.

**Table 4.** The ARL comparisons between the EWMA and LRT charts for $m = 10$ and $\alpha = 0.005$

| | $\tau$ | | | | | |
|---|---|---|---|---|---|---|
| | *10* | | *50* | | *100* | |
| $\delta$ | *EWMA* | *LRT* | *EWMA* | *LRT* | *EWMA* | *LRT* |
| $A_0 + \delta\sigma$ | | | | | | |
| 0.0 | 200.0 | 200.0 | 200.0 | 200.0 | 200.0 | 200.0 |
| 0.2 | 179.8 | 183.8 | 103.2 | 131.8 | 69.8 | 99.2 |
| 0.4 | 125.4 | 139.6 | 22.4 | 31.8 | 17.7 | 22.5 |
| 0.6 | 59.7 | 70.4 | 9.8 | 11.5 | 9.2 | 10.3 |
| 0.8 | 20.5 | 22.5 | 6.2 | 6.7 | 6.0 | 6.4 |
| 1.0 | 8.0 | 8.0 | 4.5 | 4.6 | 4.4 | 4.5 |
| 1.2 | 4.8 | 4.6 | 3.5 | 3.5 | 3.5 | 3.5 |
| 1.4 | 3.5 | 3.3 | 2.9 | 2.8 | 2.9 | 2.8 |
| 1.6 | 2.7 | 2.6 | 2.5 | 2.4 | 2.5 | 2.4 |
| 1.8 | 2.3 | 2.2 | 2.1 | 2.1 | 2.1 | 2.0 |
| 2.0 | 2.0 | 1.9 | 1.9 | 1.8 | 1.9 | 1.8 |

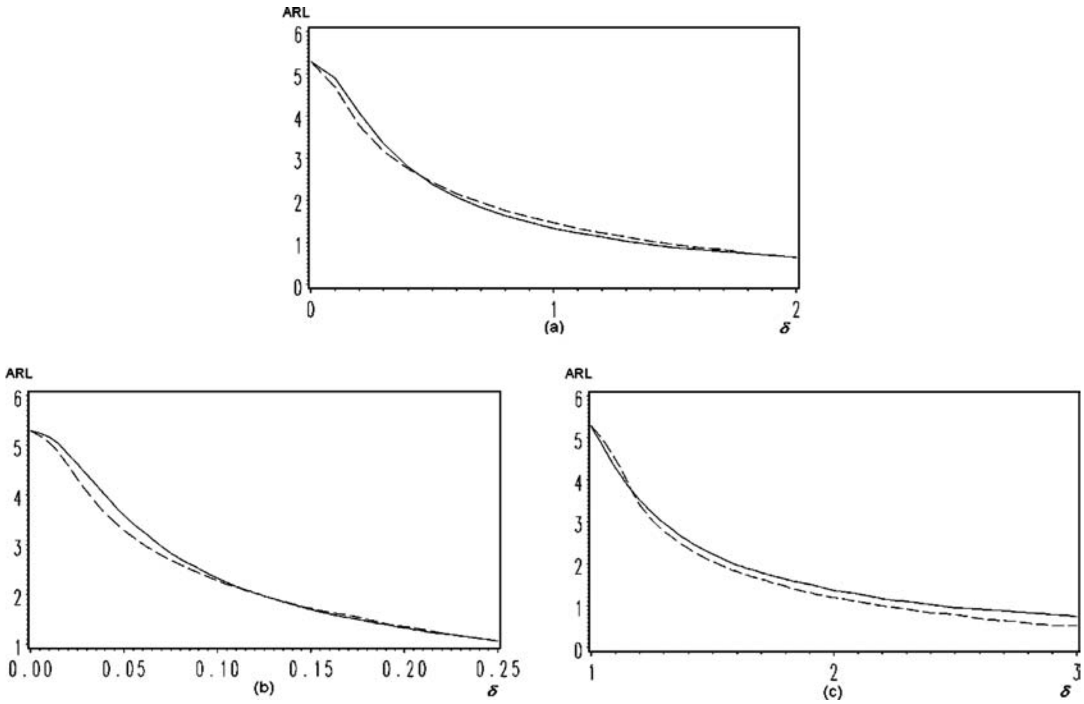


**Fig. 1.** The ARLs for a shift in; (a) the intercept; (b) the slope; and (c) the standard deviation.

From Table 4 we observe that the EWMA chart is a little inferior to the LRT chart for large shifts in the intercept, but it performs significantly better than the LRT chart for small shifts. For example, when $\delta = 0.2$ and $\tau = 100$, the EWMA chart has an ARL of 69.8 which is 0.7 times smaller than that of the LRT chart. Other simulations for different values of $m$, $\alpha$, and shifts in other parameters were also performed by the authors (not reported here but available upon request), and similar results were observed. Based on these observations, we recommend using the EWMA chart. So, in the remainder of this paper, we only consider the EWMA chart for other performance assessments.

For the case of known parameters, Kim *et al.* (2003) showed that the combination of three EWMA charts (denoted by $\mathrm{EWMA}_3$) performed better than the charts in Kang and Albin (2000) in terms of detecting sustained shifts in the parameters. Therefore, we only compare our proposed EWMA chart with $\mathrm{EWMA}_3$. In Kim *et al.* (2003), the control limits $L_\mathrm{I}$, $L_\mathrm{S}$ and $L_\mathrm{E}$ were respectively set to be 3.0156, 3.0109 and 1.3723 for the three EWMA charts. Given this choice of control limits, the overall IC ARL of $\mathrm{EWMA}_3$ is roughly 200 while that for each individual chart is around 584. Moreover, the types of shifts considered in this paper are the same as those in Kim *et al.* (2003).

The OC ARLs of our EWMA charts (dashed line) with $m = 500$ IC historical samples and that of $\mathrm{EWMA}_3$ charts (solid line) with the true values of parameters for detecting the shift in $A_0$, $A_1$ and $\sigma$ are shown in Fig. 1(a–c). Note that the values on the vertical axis of Fig. 1(a–c) are scaled by a logarithm transformation. From this figure, we can draw the following conclusions.

1. Our proposed EWMA chart has a slight disadvantage in detecting moderate shifts in the intercept $A_0$ as compared with the $\mathrm{EWMA}_3$ chart, but the EWMA chart performs better for small shifts.
2. The two charts have almost the same performance in detecting a shift in the slope, for moderate and large shifts. For small shifts, our proposed EWMA performs significantly better than the $\mathrm{EWMA}_3$ chart.
3. The EWMA chart performs better than the $\mathrm{EWMA}_3$ chart, in detecting a shift in the standard deviation, except in cases with very small shifts.

Note that the $\mathrm{EWMA}_3$ chart can detect an upward shift in $\sigma$, but our approach can also detect decreases in variance. The simulated ARLs are listed in Table 5.

Kim *et al.* (2003) showed that their proposed $\mathrm{EWMA}_3$ charts performed very well when there is a step shift

**Table 5.** The ARLs of our proposed EWMA chart for detecting decreases in the variance

| $\delta\sigma$ | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ARL | 2.9 | 3.2 | 3.5 | 3.9 | 4.3 | 4.8 | 5.4 | 6.2 | 7.2 | 8.5 | 10.3 | 13.0 | 17.2 | 24.3 | 37.9 | 69.3 |

**Table 6.** ARL comparisons for $B_1$ to $B_1 + \delta\sigma$ in Equation (12) (IC ARL = 200)

| | $\delta$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | *0.1* | *0.2* | *0.3* | *0.4* | *0.5* | *0.6* | *0.7* | *0.8* | *0.9* | *1.0* |
| $EWMA_3$ | 48.9 | 13.1 | 6.6 | 4.4 | 3.3 | 2.7 | 2.3 | 2.1 | 1.9 | 1.7 |
| EWMA | 37.2 | 13.1 | 7.4 | 5.0 | 3.8 | 3.0 | 2.5 | 2.1 | 1.9 | 1.6 |

**Table 7.** The ARL performance of the EWMA chart for $m = 10$ and $\alpha = 0.005$

| | $\tau$ | | | | |
|---|---|---|---|---|---|
| $\delta$ | *10* | *30* | *50* | *100* | *300* |
| $A_0 + \delta\sigma$ | | | | | |
| 0.2 | 179.8 | 133.2 | 103.2 | 69.8 | 48.7 |
| 0.4 | 125.4 | 35.5 | 22.4 | 17.7 | 15.9 |
| 0.6 | 59.7 | 11.3 | 9.8 | 9.2 | 8.7 |
| 0.8 | 20.5 | 6.6 | 6.2 | 6.0 | 5.9 |
| 1.0 | 8.0 | 4.7 | 4.4 | 4.4 | 4.4 |
| 1.2 | 4.8 | 3.6 | 3.5 | 3.5 | 3.5 |
| 1.4 | 3.5 | 2.9 | 2.9 | 2.9 | 2.9 |
| 1.6 | 2.7 | 2.5 | 2.5 | 2.5 | 2.4 |
| 1.8 | 2.3 | 2.2 | 2.1 | 2.1 | 2.1 |
| 2.0 | 2.0 | 1.9 | 1.9 | 1.9 | 1.9 |
| $A_1 + \delta\sigma$ | | | | | |
| 0.025 | 191.4 | 167.6 | 150.9 | 126.5 | 87.4 |
| 0.05 | 165.6 | 91.6 | 60.1 | 39.2 | 29.0 |
| 0.075 | 125.0 | 33.5 | 21.2 | 17.3 | 15.3 |
| 0.1 | 78.7 | 14.2 | 11.6 | 10.6 | 9.9 |
| 0.125 | 41.1 | 8.7 | 8.0 | 7.5 | 7.3 |
| 0.15 | 19.4 | 6.3 | 6.0 | 5.8 | 5.7 |
| 0.175 | 9.7 | 5.0 | 4.8 | 4.7 | 4.6 |
| 0.2 | 6.1 | 4.1 | 4.1 | 3.9 | 3.9 |
| 0.225 | 4.6 | 3.5 | 3.4 | 3.4 | 3.4 |
| 0.25 | 3.7 | 3.0 | 3.0 | 3.0 | 3.0 |
| $\delta\sigma$ | | | | | |
| 1.2 | 176.7 | 106.3 | 71.3 | 43.8 | 30.3 |
| 1.4 | 123.1 | 20.3 | 13.7 | 11.7 | 10.6 |
| 1.6 | 68.7 | 7.8 | 6.9 | 6.4 | 6.1 |
| 1.8 | 32.2 | 4.9 | 4.6 | 4.4 | 4.3 |
| 2.0 | 13.6 | 3.6 | 3.5 | 3.4 | 3.3 |
| 2.2 | 6.9 | 2.9 | 2.8 | 2.7 | 2.7 |
| 2.4 | 4.2 | 2.4 | 2.4 | 2.3 | 2.3 |
| 2.6 | 3.1 | 2.1 | 2.1 | 2.0 | 2.0 |
| 2.8 | 2.6 | 1.9 | 1.9 | 1.8 | 1.8 |
| 3.0 | 2.2 | 1.7 | 1.7 | 1.7 | 1.7 |
| $B_1 + \delta\sigma$ | | | | | |
| 0.1 | 180.6 | 124.2 | 93.1 | 60.8 | 40.2 |
| 0.2 | 117.7 | 26.7 | 18.0 | 14.9 | 13.3 |
| 0.3 | 48.5 | 9.2 | 8.4 | 7.8 | 7.5 |
| 0.4 | 14.6 | 5.6 | 5.4 | 5.2 | 5.1 |
| 0.5 | 6.0 | 4.0 | 3.9 | 3.8 | 3.8 |
| 0.6 | 3.9 | 3.1 | 3.1 | 3.0 | 3.0 |
| 0.7 | 2.9 | 2.6 | 2.6 | 2.5 | 2.5 |
| 0.8 | 2.3 | 2.2 | 2.2 | 2.2 | 2.1 |
| 0.9 | 2.0 | 1.9 | 1.9 | 1.9 | 1.9 |
| 1.0 | 1.7 | 1.7 | 1.7 | 1.7 | 1.7 |

in the parameter $B_1$ of Equation (12). Table 6 gives a comparison between the EWMA chart with $m = 500$ IC historical samples and the $EWMA_3$ charts with known parameters in this case. Overall, the performance of our approach is comparable to that of the $EWMA_3$ chart in this case.

Finally, the performance of the proposed chart is assessed for different values of $\tau$; the corresponding ARLs are given in Table 7 for $m = 10$ and $\alpha = 0.005$. The values of $\tau$ are chosen to be 10, 30, 50, 100, and 300, so that the performance assessment can be broadly based. From this table, one may notice that the proposed chart performs almost equally well for all values of $\tau$ when detecting a large shift. Naturally, the OC ARL will be affected by the number of reference samples gathered before a shift actually occurs. Yet the benefit is much more obvious in the case of detecting a small or moderate shift than detecting a large shift. Because the EWMA chart updates the parameter estimations with new observations, the more IC future samples one collects, the more sensitive the EWMA chart is to a small or moderate shift.

## 5. Conclusions and considerations

We introduce LRT and EWMA control charts based on LRTs. These control charts are designed to detect shifts in the intercept, slope, and standard deviation of linear profiles. These charts can be designed easily and perform well when process parameters are unknown but some historical samples are available. Through simulations, we have shown that the proposed EWMA chart, which is the EWMA of the likelihood ratio statistic, performs well in detecting process shifts. We believe this EWMA chart can be used as an alternative to the usual LRT chart, because the EWMA chart performs significantly better in detecting small shifts at the expense of being slightly inferior in the detection of large shifts. We also provide two useful diagnostic aids to improve the practicality of the EWMA chart. One provides valuable information to process engineers concerning the time of the change. The other decides which parameter has undergone a change. The proposed approach can be easily generalized to multiple regression profiles. However, it is conceivably more difficult in the case of multiple regression linear profiles to quantify the relative contribution of each parameter to a shift so as to identify which parameter has changed. An ongoing effort of the authors is analyzing recursive residuals for a possible self-starting monitoring of multiple regression linear profiles, an approach similar to that in Quesenberry (1991, 1995).

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## Appendix

### 1. *Limiting distribution of $lr(k_1 n, kn)$ under the null hypothesis $k \to \infty$ (for fixed $k_1$ and $n$)*

Note that:

$$\hat{\sigma}^2_{kn} = \frac{k_1\sigma^2_{k_1 n} + k_2\sigma^2_{k_2 n}}{k} + \frac{k_1 k_2}{k^2}(\bar{y}_{k_1 n} - \bar{y}_{k_2 n})^2 + \frac{k_1 k_2((1/k_1)S_{xy(k_1 n)} - (1/k_2)S_{xy(k_2 n)})^2}{k^2 n S_{xx}},$$

and

$$\hat{\sigma}^2_{kn} \xrightarrow{\mathcal{P}} \sigma^2,\ \hat{\sigma}^2_{k_2 n} \xrightarrow{\mathcal{P}} \sigma^2,\ \bar{y}_{k_2 n} \xrightarrow{\mathcal{P}} A_1\bar{x} + A_0,\ \frac{k_2}{k} \to 1,\ \text{as } k \to \infty.$$

It follows that:

$$\begin{aligned} lr(k_1 n, kn) &= k_1 n\left(\ln \hat{\sigma}^2_{kn} - \ln \hat{\sigma}^2_{k_1 n}\right) + k_2 n\left(\ln \hat{\sigma}^2_{kn} - \ln \hat{\sigma}^2_{k_2 n}\right) \\ &= -k_1 n \ln \frac{\hat{\sigma}^2_{k_1 n}}{\hat{\sigma}^2_{kn}} + k_2 n \ln \left[1 + \frac{k_1}{k}\left(\frac{\hat{\sigma}^2_{k_1 n}}{\hat{\sigma}^2_{kn}} - 1\right)\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{k_1k_2}{k^2}\left(\frac{\bar{y}_{k_1n}-\bar{y}_{k_2n}}{\hat{\sigma}_{k_2n}}\right)^2\\
&+\frac{k_1k_2\big((1/k_1)S_{xy(k_1n)}-1/k_2S_{xy(k_2n)}\big)^2}{k^2nS_{xx}\hat{\sigma}^2_{k_2n}}\Bigg]\\
=&-k_1n\ln\frac{\hat{\sigma}^2_{k_1n}}{\hat{\sigma}^2_{kn}}+k_2n\Bigg[\frac{k_1}{k}\left(\frac{\hat{\sigma}^2_{k_1n}}{\hat{\sigma}^2_{kn}}-1\right)\\
&+\frac{k_1k_2}{k^2}\left(\frac{\bar{y}_{k_1n}-\bar{y}_{k_2n}}{\hat{\sigma}_{k_2n}}\right)^2\\
&+\frac{k_1k_2\big((1/k_1)S_{xy(k_1n)}-(1/k_2)S_{xy(k_2n)}\big)^2}{k^2nS_{xx}\hat{\sigma}^2_{k_2n}}\\
&+o_p(k^{-1})\Bigg]\xrightarrow{\mathcal{P}}-k_1n\ln\frac{\hat{\sigma}^2_{k_1n}}{\sigma^2}+k_1n\frac{\hat{\sigma}^2_{k_1n}}{\sigma^2}\\
&-k_1n+k_1n\left(\frac{\bar{y}_{k_1n}-A_1\bar{x}-A_0}{\sigma}\right)^2\\
&+k_1\left(\frac{(1/k_1)S_{xy(k_1n)}-\sum_1^n(x_i-\bar{x})(A_1x_i+A_0)}{\sigma\sqrt{S_{xx}}}\right)^2\\
&(\text{as } k,k_2\to\infty)\overset{\mathcal{D}}{=}z_1-k_1n\ln\frac{z_1}{k_1n}\\
&\quad+z_2+z_3-k_1n,
\end{aligned}
$$

where

$$
\begin{aligned}
z_1&=k_1n\frac{\hat{\sigma}^2_{k_1n}}{\sigma^2}\sim\chi^2(k_1n-2),\\
z_2&=k_1n\left(\frac{\bar{y}_{k_1n}-A_1\bar{x}-A_0}{\sigma}\right)^2\sim\chi^2(1),\\
z_3&=k_1\left(\frac{(1/k_1)S_{xy(k_1n)}-\sum_1^n(x_i-\bar{x})(A_1x_i+A_0)}{\sigma\sqrt{S_{xx}}}\right)^2\\
&\quad\sim\chi^2(1),
\end{aligned}
$$

and $z_1$, $z_2$ and $z_3$ are independent.

## 2. *In-control expectation and variance of* $lr\,(k_1n,\infty)$

Because the random variables $z_1$, $z_2$ and $z_3$ are independent, the characteristic function of $lr(k_1n,\infty)$ is given by:

$$
\begin{aligned}
\phi(t)&=(1-2it)^{-1}\exp^{(-ik_1nt)}\int_0^\infty\exp^{(it(x-k_1n\ln\frac{x}{k_1n}))}\\
&\quad\times\frac{1}{2^{(k_1n-2/2)}\Gamma(k_1n-2)/2}x^{(k_1n-2)/2-1}e^{-x/2}\mathrm{d}x\\
&=(1-2it)^{-1}\exp^{(-ik_1nt)}\int_0^\infty\exp^{(it-\frac{1}{2})x}\,x^{(k_1n-2)/2-itk_1n-1}\\
&\quad\times\frac{1}{2^{k_1n-2/2}\Gamma(k_1n-2)/2}(k_1n)^{itk_1n}\mathrm{d}x\\
&=(1-2it)^{-1}\exp^{(-ik_1nt)}\,\Gamma\left(\frac{k_1n-2}{2}-itk_1n\right)\\
&\quad\times\left(\frac{2}{1-2it}\right)^{((k_1n-2)/2)-itk_1n}\frac{(k_1n)^{itk_1n}}{2^{((k_1n-2)/2)}\Gamma(k_1n-2)/2}\\
&=\frac{\Gamma((k_1n-2)/2)-itk_1n)}{\Gamma k_1n-2/2}(1-2it)^{itk_1n-(k_1n/2)}\\
&\quad\times\left(\frac{k_1n}{2e}\right)^{itk_1n},
\end{aligned}
$$

where $i$ is the unit of complex number. Therefore, the $r$th moment of $lr(k_1n,\infty)$ is obtained by:

$$E[(lr(k_1n,\infty))^r]=(-i)^r\frac{\mathrm{d}^r\phi(t)}{\mathrm{d}t^r}\Big|_{t=0},$$

from which we get

$$
\begin{aligned}
E[lr(k_1n,\infty)]&=k_1n\left[\ln\left(\frac{k_1n}{2}\right)-\psi_0\left(\frac{k_1n-2}{2}\right)\right],\\
\mathrm{Var}[lr(k_1n,\infty)]&=(k_1n)^2\psi_1\left(\frac{k_1n-2}{2}\right)-2k_1n,
\end{aligned}
$$

where $\psi_0(\cdot)$ and $\psi_1(\cdot)$ are, respectively, the digamma and trigamma function.

## 3. *Calculation of* $\psi_0$ *and* $\psi_1$

The recursive formula for evaluating the digamma function $\psi_0$ and trigamma function $\psi_1$ is given by:

$$
\begin{aligned}
&\psi_0(1)=-\gamma,\quad \psi_0\left(\frac{1}{2}\right)=-\gamma-2\ln 2,\\
&\psi_1(1)=\frac{\pi^2}{6},\quad \psi_1\left(\frac{1}{2}\right)=\frac{\pi^2}{2},\\
&\psi_0(z+1)=\psi_0(z)+\frac{1}{z},\quad \psi_1(z+1)=\psi_1(z)-\frac{1}{z^2},
\end{aligned}
$$

where $\gamma=0.577\,215\,664\ldots$ is the Euler-Mascheroni constant.

Therefore, it is easy to derive the following formula:

$$
\begin{aligned}
&\psi_0(n)=-\gamma+\sum_{k=1}^{n-1}\frac{1}{k},\quad \psi_1(n)=\frac{\pi^2}{6}-\sum_{k=1}^{n-1}\frac{1}{k^2},\\
&\psi_0\left(n+\frac{1}{2}\right)=-\gamma-2\ln 2+2\sum_{k=1}^{n}\frac{1}{2k-1},\\
&\psi_1\left(n+\frac{1}{2}\right)=\frac{\pi^2}{2}-4\sum_{k=1}^{n}\frac{1}{(2k-1)^2},
\end{aligned}
$$

where $n$ is an integer.

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