Individual Word Length Patterns for Fractional Factorial (Split-Plot) Designs*

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Abstract Fractional factorial (FF) designs are commonly used for factorial experiments in many fields. When some prior knowledge has shown that some factors are more likely to be significant than others, Li, et al. (2015) proposed a new pattern, called the individual word length pattern (IWLP), which, defined on a column of the design matrix, measures the aliasing of the effect assigned to this column and effects involving other factors. In this paper, we first investigate the relationships between the IWLP and other popular criteria for regular FF designs. As we know, fractional factorial split-plot (FFSP) designs are important both in theory and practice. So another contribution of this paper is extending the IWLP criterion from FF designs to FFSP designs. We propose the IWLP of a factor from the whole-plot (WP), or sub-plot (SP), denoted by the $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP respectively, in the FFSP design. We further propose combined word length patterns $\mathrm{C_w}$WLP and $\mathrm{C_s}$WLP, in order to select good designs for different cases. The new criteria $\mathrm{C_w}$WLP and $\mathrm{C_s}$WLP apply to the situations that the potential important factors are in WP or SP, respectively. Some examples are presented to illustrate the selected designs based on the criteria established here.

Keywords Effect hierarchy, fractional factorial split-plot, regular design, prior information.

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1 Introduction

Two-level fractional factorial (FF) designs are very important in factor screening experiments and many scientific investigations. A 2^{-m}th fraction of a 2^n factorial design consisting of 2^{n-m} distinct combinations will be referred to as a 2^{n-m} design, which denotes a fraction with 2^{n-m} runs and n two-level factors. The design has $n-m$ independent columns and is determined by m independent defining words. In order to select good 2^{n-m} FF designs, some criteria have been proposed, such as the maximum resolution[1], minimum aberration (MA)[2], clear effects[3], and the general minimum lower-order confounding (GMC)[4]. As we know, the effect hierarchy principle[5] is the most basic one to be followed for selecting good designs in FF designs. It states that, (i) lower-order effects are more likely to be important than higher-order effects, and (ii) effects of the same order are equally likely to be important. So a good design should have as many as possible lower-order effects which are less aliased with each other. Obviously, all the above criteria are based on the effect hierarchy principle.

Recently, Li, et al.[6] proposed a new criterion called the individual word length pattern (IWLP) that measures the degree of aliasing between an individual factor and effects involving other factors. The proposed criterion is more appropriate than other criteria such as the MA, GMC and maximum resolution criterion when the experimenters have any prior knowledge that some factors are particularly important. The critical difference between this criterion and most existing criteria is that it does not treat all factors as being of equal importance. In this paper, we investigate the relationships between the IWLP and other popular criteria for regular FF designs.

In the field of design of experiments, it is usually assumed that the experimental runs can be completely randomized when an experiment is performed. However, this is impractical in practice when the levels of some factors in the experiment are difficult to be changed or controlled. In such a case, the fractional factorial split-plot (FFSP) design, which involves a multi-phase randomization, may represent a practical design option. In general, an FFSP experiment has two types of factors: the factors with hard-to-change levels are called whole-plot (WP) factors, and the factors with relatively-easy-to-change levels are sub-plot (SP) factors. Such an experiment can be arranged by an FFSP design. The factors of an FFSP design have different statuses and the runs are not carried out completely at random, which distinguishes it from the classical FF design. Obviously, when an FFSP design is considered as an FF design, the concepts for an FF design, such as resolution, word length pattern (WLP), MA, clear effects and GMC, are also applicable to the FFSP design. For example, some studies mainly focus on the theory and construction of MA two-level FFSP designs. Interested readers may refer to [7–12]. For the results on FFSP designs under clear effects, one can refer to [13–15]. For the results on mixed-level FFSP designs and nonregular FFSP designs, we refer the readers to [15–20]. For comprehensive discussions on FFSP designs, one can refer to [21, 22], among others.

Although, existing results on FFSP designs focused on constructing optimal designs, there is little work on how to assign columns to factors after a design is selected. Furthermore, existing criteria for FFSP designs, such as these described above, treat all factors as being of equal importance, which is not suitable for the case that some factors are more important than others if we have some prior knowledge about the importance of factors in an experiment. In order to motivate our study, let us consider the following GMC-FFSP design[23] as a toy example. Wei, et al.[23] considered a $2^{(5+4)-(1+3)}$ FFSP design with the defining words:

$$I = ABCDE = ABpq = ACpr = BCps.$$

They showed that the design is a good design for estimating lower-order effects, but they did not provide any information on how to assign columns to factors if some priori knowledge shows that some factors are more important than the others. In practice, the practitioner prefers to an appropriate policy to assign columns rather than a design. With this in mind, this paper proposes new definitions named the IWLPs of WP and SP factors in the FFSP design, denoted by $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP, respectively. And based on these patterns, we propose the new criteria, denoted by $\mathrm{C_w}$WLP and $\mathrm{C_s}$WLP, to select good designs according to different prior information. Actually, the $\mathrm{C_w}$WLP and $\mathrm{C_s}$WLP are the linear combination of the $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP, respectively, and they apply to the situations where the potential significant factors belong to WP or SP. In general, if the backgrounds do not show the certain information on the potential important factors belonging to which plot or how many important factors are in the design, we suggest the criterion CWLP for selecting good designs. This criterion in fact is the proportional combination of $\mathrm{C_w}$WLP and $\mathrm{C_s}$WLP.

The paper is organized as follows. Section 2 introduces some preliminaries. In Section 3, we develop the relationships between the IWLP and other popular criteria. Some new criteria that measure the degree of aliasing between certain individual factors (i.e., WP and SP factors) and effects involving other factors in an FFSP design will be introduced in Section 4. Section 5 is devoted to selecting effective designs using the new criteria, and some examples are provided for illustrating the theoretical results. General results for the IWLP and concluding remarks are given in Section 6.

2 Preliminaries

In this section, we recall some important definitions and notation used in the sequel. A 2^{n-m} FF design F is defined by m independent words $W_1, \ldots, W_m$, and the defining contrast subgroup of F consists of $2^m - 1$ nonzero words formed by all possible products of independent words. Let $A_i(F)$ be the number of distinct defining words of length i in the defining contrast subgroup of F. The vector $W(F) = (A_1(F), \ldots, A_n(F))$ is called the WLP of F. The resolution R is the smallest i with $A_i(F) > 0$. A design has maximum resolution if no other design has a larger resolution. In general, a design with a positive A_1 or A_2 would be useless, so we only consider the design with resolution $R \geq$ III. For two 2^{n-m} designs F_1 and F_2 with $W(F_1)$ and $W(F_2)$, respectively, if there exists an $r(3 \leq r \leq n)$, such that $A_r(F_1) < A_r(F_2)$ and $A_j(F_1) = A_j(F_2)$ for $j = 3, \ldots, r-1$ in these two WLPs, then F_1 is said to have less aberration than F_2, and F_1 is said to have MA if there is no other design having less aberration[2]. For a column γ in F, let $A_j(\gamma)$ denote the number of length j defining words involving γ in the defining relation of F, then the IWLP of γ is defined to be $W(\gamma) = (A_3(\gamma), A_4(\gamma), \ldots, A_n(\gamma))$[6].

Frankly speaking, an FFSP design can be considered as a usual 2^{n-m} FF design but written as a $2^{(n_1+n_2)-(m_1+m_2)}$ design, where n_1 is the number of WP factors denoted by the capital letters $A, B, C, \ldots$ in this paper, and n_2 is the number of SP factors denoted by lowercase letters $p, q, r, \ldots$. In such a design, there are m_1 WP and m_2 ($m_2 = m - m_1$) SP defining words, respectively. A WP defining word means that there is no SP factor in the word, and an SP defining word contains at least one SP factor. From the nature of an FFSP design, a necessary requirement is that, an SP defining word can contain any number of WP factors, but it is not allowed that an SP defining word contains only one SP factor since if so the split-plot nature of this kind of experiments will be destroyed[24]. Accordingly, the two-factor interactions (2FIs) in an FFSP design can be divided into three types: WP2FI, SP2FI and WS2FI. Specially, a WP2FI is the interaction of two WP factors, an SP2FI is the interaction of two SP factors, and a WS2FI is the interaction of a WP factor and an SP factor. Similarly, a WP3FI and an SP3FI are the interaction of three WP factors and of three SP factors, respectively. In FFSP designs, effects involving only WP factors are called WP-type effects, and effects involving at least one SP factor are called SP-type effects. Correspondingly, an alias set is said to be of WP-type if it contains at least one WP-type effect, or of SP-type otherwise. As the WP level error is typically larger than the SP level error[25], a good FFSP design should have lower-order SP-type effects not aliased with WP effects as many as possible. Then, an FFSP design obeys the following rules

(i) there is no defining word that contains only one SP factor;

(ii) the number of lower-order SP-type effects which are not aliased with WP effects is as large as possible.

The concepts of clear main effect and 2FI are introduced as follows.

Definition 2.1 A main effect is said to be clear if it is not aliased with any other main effect or any 2FI. A 2FI is said to be clear if it is not aliased with any main effect or any other 2FI.

Proposition 2.2 *For any column α in a design F, the main effect for α is clear if and only if $A_3(\alpha) = 0$. All the 2FIs involving α are clear if and only if $A_3(\alpha) = A_4(\alpha) = 0$.*

3 Relationships Between IWLP and Other Criteria for FF Designs

In this section, we investigate the relationships between the IWLP and other criteria for FF designs. Throughout the section, we consider the designs with resolution III or higher.

3.1 Relationship Between IWLP and MA

In general, we suppose that all three-factor and higher order interactions are negligible. In this case, $A_3(F)$ and $A_4(F)$ are always paid more attention in the literature and practice. $A_3(F)$ can be used to measure the aliasing relationship between main effects and 2FIs, and

$A_4(F)$ implies the aliasing among 2FIs of a 2^{n-m} design. For a column γ in F, the following lemma from [26] is obvious.

Lemma 3.1 *For a 2^{n-m} design F, we have the following relationship*

$$A_3(F) = \frac{\sum_\gamma A_3(\gamma)}{3} \quad \text{and} \quad A_4(F) = \frac{\sum_\gamma A_4(\gamma)}{4}.$$

Clearly, for a given column γ in F, $A_3(F)$ and $A_4(F)$ are functions of $A_3(\gamma)$ and $A_4(\gamma)$, respectively. More generally, we can directly conclude that $A_k(F)$ is a function of $A_k(\gamma)$. The corresponding expression is

$$A_k(F) = \frac{\sum_\gamma A_k(\gamma)}{k}, \tag{1}$$

where k=3, 4, $\ldots$. Compared with the WLP, IWLP has an elaborate expression with an important factor. If an experimenter does not have any such prior knowledge, perhaps an MA design is more suitable. But if the experimenter has some prior knowledge about the ordering of the factors according to their importance in the experiment, IWLP should be a more suitable choice, since more information can be provided to clearly estimate the differently important effects via an orderly identification of the important factors.

3.2 Relationship Between IWLP and GMC

For a 2^{n-m} design, Zhang, et al.[4] introduced the sequence

$$^{\#}C = ({}_1^{\#}C_2,\ {}_2^{\#}C_1,\ {}_2^{\#}C_2,\ {}_0^{\#}C_3,\ {}_1^{\#}C_3,\ {}_2^{\#}C_3,\ {}_3^{\#}C_1,\ {}_3^{\#}C_2,\ {}_3^{\#}C_3, \ldots), \tag{2}$$

which is called the aliased effect-number pattern (AENP) of the design, where ${}_i^{\#}C_j = ({}_i^{\#}C_j^{(0)}$, ${}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)})$, ${}_i^{\#}C_j^{(k)}$ is the number of ith-order effects aliased with k jth-order effects and $K_j = \binom{n}{j}$. The design sequentially maximizing the components of the AENP is called a GMC design. Zhang, et al.[27] gave the simplified version of (2) as

$$^{\#}C = ({}_1^{\#}C_2,\ {}_2^{\#}C_2,\ {}_1^{\#}C_3,\ {}_2^{\#}C_3,\ {}_3^{\#}C_2,\ {}_3^{\#}C_3, \ldots)$$

for designs with resolution III or higher.

The following result shows the relationship between IWLP and GMC in a 2^{n-m} design.

Theorem 3.2 *Given a 2^{n-m} design with $R \geq III$, for $i = 3, 4, \ldots, n$, we have the following relationships*

$${}_i^{\#}C_0^{(0)} = \binom{n}{i} - \frac{\sum_\gamma A_i(\gamma)}{i}.$$

Proof According to the definition of components in the AENP, we can easily get that ${}_i^{\#}C_0^{(0)}$ means the number of clear i-order effects in a design. So from [4], we can get the relationship between WLP and ${}_i^{\#}C_0^{(0)}$ as follows

$${}_i^{\#}C_0^{(0)} = \binom{n}{i} - A_i.$$

Then according to equation (1), we get the conclusion.

This theorem tells us that we can get the degree of aliasing between an individual factor and effects involving other factors according to the form of IWLP.

3.3 Relationship Between IWLP and Clear Effects

For a design F, let $C_1(F)$ and $C_2(F)$ denote the numbers of clear main effects and 2FIs respectively. According to the clear effects criterion[12], we should sequentially maximize $C_1(F)$ and $C_2(F)$ to select good designs.

Let

$$a = \#\{\beta\gamma : A_4(\beta\gamma) = 0\} \text{ and } b = \#\{\alpha : A_3(\alpha) = 1\}$$

denote the number of 2FIs that do not appear in any defining word with length 4, and the number of main effects that only appear in one defining word with length 3, respectively. Here the α in $A_3(\alpha)$ denotes one factor (or column) and, similarly, the β and γ in the $A_4(\beta\gamma)$ denote two factors (or columns). The relationship between IWLP and clear effects criterion is shown as follows.

Theorem 3.3 *For a 2^{n-m} design F, we have the following relationship*

$$C_1(F) = \sum_{\gamma} I\{A_3(\gamma) = 0\} \quad \text{and} \quad C_2(F) = a - b,$$

where $I\{\cdot\}$ is the indicator function which takes the value 1 or 0 depending on whether the condition $\{\cdot\}$ is true or not.

Proof The first equality is obvious. For the second equality, C_2 is the number of clear 2FIs, which equals the number of all 2FIs minus the number of 2FIs that are aliased with any main effect or any other 2FI. Inspired by [4], we have

$$C_2(F) =_2^{\#} C_2^{(0)} -_1^{\#} C_2^{(1)} = a - b.$$

This completes the proof.

4 IWLPs for FFSP Designs

4.1 $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP for FFSP Designs

According to rules (i) and (ii) in Section 2, we can respectively define the IWLPs of a WP and an SP factor for an FFSP design for comparing designs.

Definition 4.1 For a column α in an FFSP design F, let $A_j^{iw}(\alpha)$ denote the number of length j defining words involving α with i WP factors and $(j-i)$ SP factors and $A_j^{ks}(\alpha)$ denote the number of length j defining words involving α with k SP factors and $j-k$ WP factors.

It is obvious that $A_j^{iw}(\alpha) = A_j^{(j-i)s}(\alpha)$ for a given FFSP design d. Throughout this paper, we use the notation of $A_j^{iw}(\alpha)$ to illustrate the results. From rule (i), the defining word that contains only one SP factor is forbidden in F, so we have $A_j^{(j-1)w}(\alpha) = A_j^{1s}(\alpha) = 0, j \geq 3$.

We first consider the case of column α being a WP factor. It is obvious that, $A_3^{1w}(\alpha) \neq 0$ means that column α is aliased with at least one SP2FI, $A_3^{3w}(\alpha) \neq 0$ means that column α is aliased with at least one WP2FI. For $A_3^{1w}(\alpha)$ and $A_3^{3w}(\alpha)$, we need care more about $A_3^{1w}(\alpha)$ since rule (ii) tells us that we should consider the SP defining words involving WP factors at first. As for $A_4^{1w}(\alpha), A_4^{2w}(\alpha)$ and $A_4^{4w}(\alpha)$, $A_4^{1w}(\alpha) \neq 0$ means that column α is aliased with at least one SP3FI, $A_4^{2w}(\alpha) \neq 0$ means that column α is aliased with at least one WS3FI or an SP2FI is aliased with at least one WP2FI, $A_4^{4w}(\alpha) \neq 0$ means that α is in at least one WP defining word with length 4. From rule (ii), $A_4^{2w}(\alpha) \neq 0$ means an SP2FI is included in at least one WP alias set and $A_4^{1w} \neq 0$ means an SP3FI is included in at least one WP alias set. From the effect hierarchy principle[5], the lower-order effects are more likely to be important than higher order effects, so when we consider the order of SP effects, $A_4^{2w}(\alpha)$ is more important than $A_4^{1w}(\alpha)$. Hence, we rank the order of defining words with length 4 involving column α into the following sequence

$$A_4(\alpha) = (A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha)),$$

and sequentially minimize this sequence. Similarly, we have the rank of length 5 defining words containing column α as follows

$$A_5(\alpha) = (A_5^{3w}(\alpha), A_5^{2w}(\alpha), A_5^{1w}(\alpha), A_5^{5w}(\alpha)).$$

In general, we have

$$A_k(\alpha) = (A_k^{(k-2)w}(\alpha), A_k^{(k-3)w}(\alpha), \ldots, A_k^{1w}(\alpha), A_k^{kw}(\alpha)).$$

From the effect hierarchy principle and Definition 4.1, for a column α from the WP part of an FFSP design F, we define the vector

$$\mathrm{I_wWLP}(F,\alpha) = (\underbrace{A_3^{1w}(\alpha), A_3^{3w}(\alpha)}_{A_3(\alpha)}, \underbrace{A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha)}_{A_4(\alpha)}, \ldots, \underbrace{A_k^{(k-2)w}(\alpha), \ldots, A_k^{kw}(\alpha)}_{A_k(\alpha)}, \ldots) \tag{3}$$

as the IWLP of α for F. Since $A_j^{iw}(\alpha) = A_j^{(j-i)s}(\alpha)$, expression (3) is equivalent to the following vector

$$(A_3^{2s}(\alpha), A_3^{0s}(\alpha), A_4^{2s}(\alpha), A_4^{3s}(\alpha), A_4^{0s}(\alpha), \ldots, A_k^{2s}(\alpha), \ldots, A_k^{0s}(\alpha), \ldots). \tag{4}$$

Next, we consider any column α from the SP part of an FFSP design F. Upon using the similar method and principle, we define the vector

$$\mathrm{I_sWLP}(F,\alpha) = (A_3^{1w}(\alpha), A_3^{0w}(\alpha), A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{0w}(\alpha), \ldots, A_k^{(k-2)w}(\alpha), \ldots, A_k^{0w}(\alpha), \ldots) \tag{5}$$

as the IWLP of α for F. Since $A_j^{iw}(\alpha) = A_j^{(j-i)s}(\alpha)$, expression (5) is equivalent to the following vector

$$(A_3^{2s}(\alpha), A_3^{3s}(\alpha), A_4^{2s}(\alpha), A_4^{3s}(\alpha), A_4^{4s}(\alpha), \ldots, A_k^{2s}(\alpha), \ldots, A_k^{ks}(\alpha), \ldots). \tag{6}$$

So, a good design is to minimize the sequences (3) and (5) in turn, or equivalently, to minimize the sequences (4) and (6) in turn.

For a given FFSP design, $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP provide useful measures for ranking the columns. If we have some prior information on which factors being more likely to be significant than others, we should assign those factors to the columns that minimize the aliasing between those important factors and the others. In the following, we give an example to illustrate how to rank the columns.

Example 4.2 Consider the $2^{(5+4)-(1+3)}$ design d_1 with defining words

$$I = ABCDE = ABpq = ACpr = BCps.$$

According to the definition, the detailed values of $\mathrm{I_s}$WLP and $\mathrm{I_w}$WLP can be calculated and listed in Table 1.

Table 1: The $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP of d_1

	A_4^{2w}	A_4^{1w}	A_4^{4w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{5w}	$\mathbf{0}_5$	$\mathbf{0}_6$	$\mathbf{0}_7$	A_9^{7w}	A_9^{6w}	A_9^{5w}	A_9^{4w}	A_9^{3w}	A_9^{2w}	A_9^{1w}	A_9^{9w}
A	4	0	0	2	0	0	1				0	0	1	0	0	0	0	0
B	4	0	0	2	0	0	1				0	0	1	0	0	0	0	0
C	4	0	0	2	0	0	1				0	0	1	0	0	0	0	0
D	0	0	0	6	0	0	1				0	0	1	0	0	0	0	0
E	0	0	0	6	0	0	1				0	0	1	0	0	0	0	0
	A_4^{2w}	A_4^{1w}	A_4^{0w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{0w}	$\mathbf{0}_5$	$\mathbf{0}_6$	$\mathbf{0}_7$	A_9^{7w}	A_9^{6w}	A_9^{5w}	A_9^{4w}	A_9^{3w}	A_9^{2w}	A_9^{1w}	A_9^{0w}
p	3	0	1	3	0	0	0				0	0	1	0	0	0	0	0
q	3	0	1	3	0	0	0				0	0	1	0	0	0	0	0
r	3	0	1	3	0	0	0				0	0	1	0	0	0	0	0
s	3	0	1	3	0	0	0				0	0	1	0	0	0	0	0

* $\mathbf{0}_k$ denotes that the k components of vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{(k+1)w})$ or vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{0w})$ are all zeros, here k=5, 6, 7.

Table 1 indicates that we should assign the important WP factors on column D or E or columns D and E if some priori knowledge shows that one or two WP factors are more important than other WP factors.

4.2 Statistical Justifications of $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP for FFSP Designs

For an $N \times n$ FFSP design, suppose we have the prior information that the factor assigned to column α is particularly important such that all effects involving α are of primary interest, then the true model can be written as

$$Y = \beta_0 I + X_1\beta_1 + \sum_{j=2}^{m-1} (X_j(\alpha)\beta_j(\alpha) + X_j(\overline{\alpha})\beta_j(\overline{\alpha})) + X_m(\alpha)\beta_m(\alpha) + \varepsilon, \tag{7}$$

where Y denotes the vector of n observations, β_0 is the grand mean and I is an $n \times 1$ column vector with all elements unity, X_1 is the original design matrix D, β_1 is the vector of all main effects, $\beta_j(r)$ denotes the $\binom{m-1}{j-1}$ j-factor interactions involving α, $\beta_j(\overline{\alpha})$ denotes the other j-factor interactions not involving α, $X_j(\alpha)$ and $X_j(\overline{\alpha})$ represent the corresponding model matrices, and ε is the vector of random errors. For an orthogonal design, if we only fit the main effects model

$$Y = \beta_0 I + X_1\beta_1 + \varepsilon, \tag{8}$$

the least squares estimate of β_1 is $\widehat{\beta}_1 = (X_1^T X_1)^{-1} X_1^T Y = n^{-1} X_1^T Y$. Under the true model (7), we have

$$E(\widehat{\beta}_1) = \beta_1 + \sum_{j=2}^{m-1} (C_j(\alpha)\beta_j(\alpha) + C_j(\overline{\alpha})\beta_j(\overline{\alpha})) + C_m(\alpha)\beta_m(\alpha),$$

where $C_j(\alpha) = n^{-1} X_1^T X_j(\alpha)$ and $C_j(\overline{\alpha}) = n^{-1} X_1^T X_j(\overline{\alpha})$ for $j \geq 2$. For an FF design, [26] showed that sequentially minimizing $|C_2(\alpha)|^2, |C_3(\alpha)|^2, \ldots, |C_{m-1}(\alpha)|^2$ is equivalent to sequentially minimizing $A_3(\alpha), A_4(\alpha), \ldots, A_m(\alpha)$, in other words, the column with the best IWLP sequentially minimizes the number of j-factor interactions involving α aliased with the grand mean and main effects, in the order given by $j = 2, 3, \ldots, m$.

Motivated by [26], we now discuss the relationship between $|C_j(\alpha)|^2$ and $\mathrm{I_w}$WLP or $\mathrm{I_s}$WLP for FFSP designs as follows. The proof is trivial and is omitted here.

Lemma 4.3 *For an FFSP design F, let columns α_1 and α_2 be a WP and an SP factor, respectively. Then for $2 \leq j \leq m-1$, we have the following relationships*

$$\begin{aligned}
|C_j(\alpha_1)|^2 = & j \left\{ A_{j+1}^{1w}(\alpha_1) + \cdots + A_{j+1}^{(j-1)w}(\alpha_1) + A_{j+1}^{(j+1)w}(\alpha_1) \right\} \\
& + \left\{ A_j^{1w}(\alpha_1) + A_j^{2w}(\alpha_1) + \cdots + A_j^{(j-2)w}(\alpha_1) + A_j^{(j)w}(\alpha_1) \right\} \\
& + (m-j) \left\{ A_{j-1}^{1w}(\alpha_1) + \cdots + A_{j-1}^{(j-3)w}(\alpha_1) + A_{j-1}^{(j-1)w}(\alpha_1) \right\} \\
& + \left\{ A_{j-1}^{1w} + \cdots + A_{j-1}^{(j-3)w} + A_{j-1}^{(j-1)w} \right\}; \text{ and} \\
|C_j(\alpha_2)|^2 = & j \left\{ A_{j+1}^{2s}(\alpha_2) + \cdots + A_{j+1}^{(j)s}(\alpha_2) + A_{j+1}^{(j+1)s}(\alpha_2) \right\} \\
& + \left\{ A_j^{2s}(\alpha_2) + A_j^{3s}(\alpha_2) + \cdots + A_j^{(j-1)s}(\alpha_2) + A_j^{(j)s}(\alpha_2) \right\} \\
& + (m-j) \left\{ A_{j-1}^{2s}(\alpha_2) + \cdots + A_{j-1}^{(j-2)s}(\alpha_2) + A_{j-1}^{(j-1)s}(\alpha_2) \right\} \\
& + \left\{ A_{j-1}^{2s} + \cdots + A_{j-1}^{(j-2)s} + A_{j-1}^{(j-1)s} \right\}.
\end{aligned}$$

For an FFSP design, let us consider the bias contributed by the j-factor interactions when we estimate β_1. From the effect hierarchy principle and Section 4.1, sequentially minimizing $(|C_2(\alpha)|^2, |C_3(\alpha)|^2, \ldots, |C_{m-1}(\alpha)|^2)$ is equivalent to sequentially minimizing

$$(\underbrace{A_3^{1w}(\alpha), A_3^{3w}(\alpha)}_{A_3(\alpha)}, \underbrace{A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha)}_{A_4(\alpha)}, \ldots, \underbrace{A_m^{(m-2)w}(\alpha), \ldots, A_m^{mw}(\alpha)}_{A_m(\alpha)}).$$

From the above conclusions, we have the following result.

Theorem 4.4 *For an FFSP design F, assigning a factor to the column with the best* $\mathrm{I_w}$WLP *or* $\mathrm{I_s}$WLP *is equivalent to sequentially minimizing the contamination of interactions involving this factor on the estimation of grand mean and main effects.*

5 Combined WLPs with Examples

Apart from ranking columns for a given design, $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP can also assist in selecting designs. From some different designs, the new patterns would help to select the better one. The following example illustrates how to select the better design.

Example 5.1 (Example 4.2 continued) Following Example 4.2, we consider another two $2^{(5+4)-(1+3)}$ designs d_2 with

$$I = ABCDE = ABDpq = ACDpr = BCDps,$$

and d_3 with

$$I = ABCDE = ABpq = ACpr = ADps.$$

Note that the d_1 in Example 4.2 is a GMC-FFSP design. Here d_2 is an MA-MSA-FFSP design[23] and d_3 is an MA design[28]. According to the definitions, Tables 1, 2 and 3 list the $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP for all factors in the above three designs, respectively.

Table 2: The $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP of d_2

	A_4^{2w}	A_4^{1w}	A_4^{4w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{5w}	$\mathbf{0}_5$	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{8w}	$\mathbf{0}_8$
A	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
B	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
C	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
D	0	0	0	6	0	1	1			0	0	0	0	0	0	0	
E	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
	A_4^{2w}	A_4^{1w}	A_4^{0w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{0w}	$\mathbf{0}_5$	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{0w}	$\mathbf{0}_8$
p	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
q	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
r	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
s	3	0	0	3	0	1	0			0	0	1	0	0	0	0	

* $\mathbf{0}_k$ denotes that the k components of vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{(k+1)w})$ or vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{0w})$ are all zeros, here k=5, 6, 8.

Table 3: The $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP of d_3

	A_4^{2w}	A_4^{1w}	A_4^{4w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{5w}	$\mathbf{0}_5$	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{8w}	$\mathbf{0}_8$
A	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
B	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
C	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
D	3	0	0	3	0	0	1			0	0	1	0	0	0	0	
E	0	0	0	6	0	1	1			0	0	0	0	0	0	0	
	A_4^{2w}	A_4^{1w}	A_4^{0w}	A_5^{3w}	A_5^{2w}	A_5^{1w}	A_5^{0w}	$\mathbf{0}_5$	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{0w}	$\mathbf{0}_8$
p	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
q	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
r	3	0	0	3	0	1	0			0	0	1	0	0	0	0	
s	3	0	0	3	0	1	0			0	0	1	0	0	0	0	

* $\mathbf{0}_k$ denotes that the k components of vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{(k+1)w})$ or vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{0w})$ are all zeros, here k=5, 6, 8.

It can be observed that the types and numbers of $\mathrm{I_wWLP}$s and $\mathrm{I_sWLP}$s of all factors in the designs d_2 and d_3 are the same. This means that the choices of d_2 and d_3 are equivalent. So it is not necessary to discuss d_3 in the following. For d_1 and d_2, we can see that the factors D and E in Table 1 have the same best $\mathrm{I_wWLP}$, i.e. both are $(\mathbf{0}_3, 6, 0, 0, 1, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, 0, 0, 1, 0, 0, 0, 0, 0)$, while the factor D in Table 2 has the best $\mathrm{I_wWLP}$, which is $(\mathbf{0}_3, 6, 0, 1, 1, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, \mathbf{0}_8)$. So we can conclude that we will select design d_1 if some priori knowledge shows that one or two WP factors are more important than other WP factors. If some priori knowledge shows that i $(i = 1, 2, 3, 4)$ SP factors are more important than others, we will select design d_2. That is because the $\mathrm{I_sWLP}$ of all factors in d_2 are $(3, 0, 0, 3, 0, 1, 0, \mathbf{0}_5, \mathbf{0}_6, 0, 0, 1, 0, 0, 0, 0, \mathbf{0}_8)$, which are better than those in d_1 with $(3, 0, 1, 3, 0, 0, 0, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, 0, 0, 1, 0, 0, 0, 0, 0)$.

For some given designs, one can calculate the $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$ of each factor for each design accordingly. The experimenter could select the design whose factors have the best $\mathrm{I_wWLP}$ or (and) $\mathrm{I_sWLP}$ if the prior information shows that the number of potential important factors is less than the number of the factors with the best $\mathrm{I_wWLP}$ or (and) $\mathrm{I_sWLP}$. When the number of important factors according to the prior information is larger than the number of the factors with the best $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$, we consider the following combined WLP(CWLP).

For an FFSP design, it is usual to see that there are some factors with the same $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$, such as $\mathrm{I_wWLP} = (3, 0, 0, 3, 0, 0, 1, \mathbf{0}_5, \mathbf{0}_6, 0, 0, 1, 0, 0, 0, 0, \mathbf{0}_8)$ for factors A, B, C, E in design d_2 in Example 5.1. Accordingly, for an FFSP design represented by an $N \times n$ matrix, let k_1 denote the number of distinct $\mathrm{I_wWLP}$s, $(\mathrm{I_wWLP})_i$ denote the ith type of $\mathrm{I_wWLP}$, λ_i denote the occurrence frequency of $(\mathrm{I_wWLP})_i$, and m_1 denote the number of WP factors. Based on the definition of $\mathrm{I_wWLP}$ in Section 4.1, we can gain a combined WLP($\mathrm{C_wWLP}$) for an FFSP design F as

$$\mathrm{C_wWLP}(F) = \sum_{i=1}^{k_1} \omega_i (\mathrm{I_wWLP})_i,$$

where the weight ω_i is often taken to be the occurrence probability of $(\mathrm{I_wWLP})_i$, i.e. $\omega_i = \lambda_i/m_1$. Similarly, we have $\mathrm{C_sWLP}$ for an FFSP design as

$$\mathrm{C_sWLP}(F) = \sum_{i=1}^{k_2} \upsilon_i (\mathrm{I_sWLP})_i,$$

where k_2 denotes the number of distinct $\mathrm{I_sWLP}$s, $(\mathrm{I_sWLP})_i$ denotes the ith type of $\mathrm{I_sWLP}$, μ_i denotes the occurrence frequency of $(\mathrm{I_sWLP})_i$, m_2 denotes the number of SP factors, the weight υ_i is often taken to be the occurrence probability of $(\mathrm{I_sWLP})_i$, i.e. $\upsilon_i = \mu_i/m_2$.

As discussed above, for some given designs, we first calculate the $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$ of each factor for each design accordingly, and then select the best design according to the following rules:

(i) if the priori knowledge shows that the number of potential important WP (or SP) factors is less than the number of factors with the best $\mathrm{I_wWLP}$ (or $\mathrm{I_sWLP}$), we will select the design according to the $\mathrm{I_wWLP}$ (or $\mathrm{I_sWLP}$) criterion;

(ii) otherwise, we select the best design according to the $\mathrm{C_wWLP}$ (or $\mathrm{C_sWLP}$) when the prior information shows that the WP (or SP) factors are more important than SP (or WP) factors.

In other case, if we get the same result for two or more designs according to the above rules, we can consider the following CWLP for an FFSP design F as

$$\begin{aligned}\mathrm{CWLP}(F) &= \frac{m_1}{m}\mathrm{C_wWLP}(F) + \frac{m_2}{m}\mathrm{C_sWLP}(F) \\ &= \sum_{i=1}^{k_1}\omega_i^{(1)}(\mathrm{I_wWLP})_i + \sum_{j=1}^{k_2}\omega_j^{(2)}(\mathrm{I_sWLP})_j,\end{aligned}$$

where the weight $\omega_i^{(1)}$ is often taken to be the occurrence probability of $(\mathrm{I_wWLP})_i$ in the whole design, i.e. $\omega_i^{(1)} = \lambda_i/m$, the weight $\omega_j^{(2)}$ is often taken to be the occurrence probability of $(\mathrm{I_sWLP})_j$ in the whole design, i.e. $\omega_j^{(2)} = \mu_j/m$, and $m = m_1 + m_2$ denotes the number of all factors.

In order to illustrate how to select the best design under the CWLP criterion, let us see the following example.

Example 5.2 (Example 5.1 continued) When the number of important WP factors exceeds two according to the prior information, we can use the $\mathrm{C_wWLP}$ and CWLP criteria to select the best design. For designs d_1 and d_2, we have

$$\begin{aligned}\mathrm{C_wWLP}(d_1) =& \sum_{i=1}^{2}\omega_i(\mathrm{I_wWLP})_i \\ =& \frac{2}{5}(0,0,0,6,0,0,1,\mathbf{0}_5,\mathbf{0}_6,\mathbf{0}_7,0,0,1,0,0,0,0,0) \\ &+ \frac{3}{5}(4,0,0,2,0,0,1,\mathbf{0}_5,\mathbf{0}_6,\mathbf{0}_7,0,0,1,0,0,0,0,0) \\ =& \left(\frac{12}{5},0,0,\frac{18}{5},0,0,1,\mathbf{0}_5,\mathbf{0}_6,\mathbf{0}_7,0,0,1,0,0,0,0,0\right), \quad \text{and} \\ \mathrm{C_wWLP}(d_2) =& \sum_{i=1}^{2}\omega_i(\mathrm{I_wWLP})_i \\ =& \frac{1}{5}(0,0,0,6,0,1,1,\mathbf{0}_5,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8) \\ &+ \frac{4}{5}(3,0,0,3,0,0,1,\mathbf{0}_5,\mathbf{0}_6,0,0,1,0,0,0,0,\mathbf{0}_8) \\ =& \left(\frac{12}{5},0,0,\frac{18}{5},0,\frac{1}{5},1,\mathbf{0}_5,\mathbf{0}_6,0,0,\frac{4}{5},0,0,0,0,\mathbf{0}_8\right).\end{aligned}$$

According to the $\mathrm{C_wWLP}$, we prefer design d_1 when we have $l \geq 3$ important WP factors. In other case, if we do not have any idea on selecting the design according to the above methods, we can use the CWLP by combining $\mathrm{C_sWLP}$ and $\mathrm{C_wWLP}$ together. Here

$$\begin{aligned}
\mathrm{CWLP}(d_1) &= \frac{5}{9}\mathrm{C_wWLP}(d_1) + \frac{4}{9}\mathrm{C_sWLP}(d_1) \\
&= \frac{5}{9}\left(\frac{12}{5}, 0, 0, \frac{18}{5}, 0, 0, 1, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, 0, 0, 1, 0, 0, 0, 0, 0\right), \\
&\quad + \frac{4}{9}(3, 0, 1, 3, 0, 0, 0, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, 0, 0, 1, 0, 0, 0, 0, 0) \\
&= \left(\frac{8}{3}, 0, \frac{4}{9}, \frac{10}{3}, 0, 0, \frac{5}{9}, \mathbf{0}_5, \mathbf{0}_6, \mathbf{0}_7, 0, 0, 1, 0, 0, 0, 0, 0\right), \quad \text{and} \\
\mathrm{CWLP}(d_2) &= \frac{5}{9}\mathrm{C_wWLP}(d_2) + \frac{4}{9}\mathrm{C_sWLP}(d_2) \\
&= \frac{5}{9}\left(\frac{12}{5}, 0, 0, \frac{18}{5}, 0, \frac{1}{5}, 1, \mathbf{0}_5, \mathbf{0}_6, 0, 0, \frac{4}{5}, 0, 0, 0, 0, \mathbf{0}_8\right) \\
&\quad + \frac{4}{9}(3, 0, 0, 3, 0, 1, 0, \mathbf{0}_5, \mathbf{0}_6, 0, 0, 1, 0, 0, 0, 0, \mathbf{0}_8) \\
&= \left(\frac{8}{3}, 0, 0, \frac{10}{3}, 0, \frac{5}{9}, \frac{5}{9}, \mathbf{0}_5, \mathbf{0}_6, 0, 0, \frac{8}{9}, 0, 0, 0, 0, \mathbf{0}_8\right).
\end{aligned}$$

According to the CWLP, we can select d_2 for designing the experiment.

Next, we give an example to explain all kinds of situations for selecting the best design.

Example 5.3 Consider the following two $2^{(4+6)-(0+5)}$ designs

$$d_4 : I = BDpq = ABpr = CDps = ABCDpt = ACpu, \quad \text{and}$$
$$d_5 : I = BDpq = BCpr = ADps = CDpt = ABpu,$$

where design d_4 is an MA-MSA-FFSP design and d_5 is a GMC-FFSP design[23]. According to the definitions, we have Tables 4 and 5 listing the $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ for the factors in both designs, respectively.

Table 4: The $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ of d_4

	A_4^{2w}	A_4^{1w}	A_4^{4w}	$\mathbf{0}_4$	A_6^{4w}	A_6^{3w}	A_6^{2w}	A_6^{1w}	A_6^{6w}	$\mathbf{0}_6$	$\mathbf{0}_7$	$\mathbf{0}_8$	A_{10}^{8w}	A_{10}^{7w}	A_{10}^{6w}	A_{10}^{5w}	A_{10}^{4w}	A_{10}^{3w}	A_{10}^{2w}	A_{10}^{1w}	A_{10}^{10w}
A	6	0	0		3	0	6	0	0				0	0	0	0	1	0	0	0	0
B	6	0	0		3	0	6	0	0				0	0	0	0	1	0	0	0	0
C	6	0	0		3	0	6	0	0				0	0	0	0	1	0	0	0	0
D	6	0	0		3	0	6	0	0				0	0	0	0	1	0	0	0	0
	A_4^{2w}	A_4^{1w}	A_4^{0w}	$\mathbf{0}_4$	A_6^{4w}	A_6^{3w}	A_6^{2w}	A_6^{1w}	A_6^{6w}	$\mathbf{0}_6$	$\mathbf{0}_7$	$\mathbf{0}_8$	A_{10}^{8w}	A_{10}^{7w}	A_{10}^{6w}	A_{10}^{5w}	A_{10}^{4w}	A_{10}^{3w}	A_{10}^{2w}	A_{10}^{1w}	A_{10}^{0w}
p	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0
q	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0
r	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0
s	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0
t	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0
u	4	0	2		1	0	8	0	0				0	0	0	0	1	0	0	0	0

* $\mathbf{0}_k$ denotes that the k components of vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{(k+1)w})$ or vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{0w})$ are all zeros, here k=4, 6, 7, 8.

Table 5: The $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP of d_5

	A_4^{2w}	A_4^{1w}	A_4^{4w}	$\mathbf{0}_4$	A_6^{4w}	A_6^{3w}	A_6^{2w}	A_6^{1w}	A_6^{6w}	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{8w}	$\mathbf{0}_8$	$\mathbf{0}_9$
A	6	0	0		2	0	6	0	0		0	0	2	0	0	0	0		
B	7	0	0		2	0	4	0	0		0	0	2	0	1	0	0		
C	6	0	0		2	0	6	0	0		0	0	2	0	0	0	0		
D	7	0	0		2	0	4	0	0		0	0	2	0	1	0	0		
	A_4^{2w}	A_4^{1w}	A_4^{0w}	$\mathbf{0}_4$	A_6^{4w}	A_6^{3w}	A_6^{2w}	A_6^{1w}	A_6^{0w}	$\mathbf{0}_6$	A_8^{6w}	A_8^{5w}	A_8^{4w}	A_8^{3w}	A_8^{2w}	A_8^{1w}	A_8^{0w}	$\mathbf{0}_8$	$\mathbf{0}_9$
p	5	0	2		0	0	6	0	0		0	0	2	0	1	0	0		
q	5	0	2		0	0	6	0	0		0	0	2	0	1	0	0		
r	4	0	2		1	0	7	0	0		0	0	1	0	1	0	0		
s	4	0	2		1	0	7	0	0		0	0	1	0	1	0	0		
t	4	0	2		1	0	7	0	0		0	0	1	0	1	0	0		
u	4	0	2		1	0	7	0	0		0	0	1	0	1	0	0		

* $\mathbf{0}_k$ denotes that the k components of vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{(k+1)w})$ or vector $(A_{k+1}^{(k-1)w}, A_{k+1}^{(k-2)w}, \ldots, A_{k+1}^{1w}, A_{k+1}^{0w})$ are all zeros, here k=4, 6, 8, 9.

From these two tables, we can conclude that if some priori knowledge shows one or two WP factors or at most four SP factors are important, we will select design d_5. For other situations, we can use the weighted criteria, i.e. $\mathrm{C_w}$WLP, $\mathrm{C_s}$WLP, or even CWLP.

In this example, we have

$$\begin{aligned}
\mathrm{C_wWLP}(d_4) &= (6,0,0,\mathbf{0}_4,3,0,6,0,0,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8,0,0,0,0,1,0,0,0,0), \quad \text{and} \\
\mathrm{C_wWLP}(d_5) &= \frac{2}{4}(6,0,0,\mathbf{0}_4,2,0,6,0,0,\mathbf{0}_6,0,0,2,0,0,0,0,\mathbf{0}_8,\mathbf{0}_9) \\
&\quad + \frac{2}{4}(7,0,0,\mathbf{0}_4,2,0,4,0,0,\mathbf{0}_6,0,0,2,0,1,0,0,\mathbf{0}_8,\mathbf{0}_9) \\
&= \left(\frac{13}{2},0,0,\mathbf{0}_4,2,0,5,0,0,\mathbf{0}_6,0,0,2,0,\frac{1}{2},0,0,\mathbf{0}_8,\mathbf{0}_9\right).
\end{aligned}$$

According to these two $\mathrm{C_w}$WLPs, if we know that most of the WP factors are potentially important, we can select d_4 as a better one. Similarly, if most of the SP factors are important, we will also take d_4 as a better one, since

$$\begin{aligned}
\mathrm{C_sWLP}(d_4) &= (4,0,2,\mathbf{0}_4,1,0,8,0,0,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8,0,0,0,0,1,0,0,0,0), \quad \text{and} \\
\mathrm{C_sWLP}(d_5) &= \frac{2}{6}(5,0,2,\mathbf{0}_4,0,0,6,0,0,\mathbf{0}_6,0,0,2,0,1,0,0,\mathbf{0}_8,\mathbf{0}_9) \\
&\quad + \frac{4}{6}(4,0,2,\mathbf{0}_4,1,0,7,0,0,\mathbf{0}_6,0,0,1,0,1,0,0,\mathbf{0}_8,\mathbf{0}_9) \\
&= \left(\frac{13}{3},0,2,\mathbf{0}_4,\frac{2}{3},0,\frac{20}{3},0,0,\mathbf{0}_6,0,0,\frac{4}{3},0,1,0,0,\mathbf{0}_8,\mathbf{0}_9\right).
\end{aligned}$$

Furthermore, if we have no much priori knowledge, we can use the CWLP for choosing a better design. Here

$$\begin{aligned}
\mathrm{CWLP}(d_4) &= \frac{4}{10}(6,0,0,\mathbf{0}_4,3,0,6,0,0,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8,0,0,0,0,1,0,0,0,0) \\
&\quad + \frac{6}{10}(4,0,2,\mathbf{0}_4,1,0,8,0,0,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8,0,0,0,0,1,0,0,0,0) \\
&= \left(\frac{24}{5},0,\frac{6}{5},\mathbf{0}_4,\frac{9}{5},0,\frac{36}{5},0,0,\mathbf{0}_6,\mathbf{0}_7,\mathbf{0}_8,0,0,0,0,1,0,0,0,0\right), \quad \text{and} \\
\mathrm{CWLP}(d_5) &= \frac{4}{10}\mathrm{C_wWLP}(d_5) + \frac{6}{10}\mathrm{C_sWLP}(d_5) \\
&= \frac{4}{10}\left(\frac{13}{2},0,0,\mathbf{0}_4,2,0,5,0,0,\mathbf{0}_6,0,0,2,0,\frac{1}{2},0,0,\mathbf{0}_8,\mathbf{0}_9\right) \\
&\quad + \frac{6}{10}\left(\frac{13}{3},0,2,\mathbf{0}_4,\frac{2}{3},0,\frac{20}{3},0,0,\mathbf{0}_6,0,0,\frac{4}{3},0,1,0,0,\mathbf{0}_8,\mathbf{0}_9\right) \\
&= \left(\frac{26}{5},0,\frac{6}{5},\mathbf{0}_4,\frac{6}{5},0,6,0,0,\mathbf{0}_6,0,0,\frac{8}{5},0,\frac{4}{5},0,0,\mathbf{0}_8,\mathbf{0}_9\right).
\end{aligned}$$

Accordingly, we can select d_3 as a better design.

6 Further Results and Concluding Remarks

In this paper, we first established the relationships between the IWLP criterion and other popular criteria for regular FF designs. Then we extended the IWLP criterion to the FFSP design and proposed the corresponding criteria, such as $\mathrm{I_wWLP}$, $\mathrm{I_sWLP}$, $\mathrm{C_wWLP}$ and $\mathrm{C_sWLP}$. The $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ criteria are simple tools to assist practitioners in assigning factors to the appropriate design columns. By sequentially minimizing $\mathrm{I_wWLP}$ (or $\mathrm{I_sWLP}$), we can use the prior information on factors' importance to reduce the estimation bias caused by model misspecification.

Section 5 shows that $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ are not only useful for assigning factors to design columns but also helpful for selecting designs. We also consider different cases in practice for selecting the designs under the $\mathrm{I_wWLP}$, $\mathrm{I_sWLP}$, $\mathrm{C_wWLP}$ and $\mathrm{C_sWLP}$. According to the examples, we tabulate some tables to illustrate the results. Tables 6 and 7 summarize the results on the $\mathrm{I_wWLP}$s, $\mathrm{I_sWLP}$s, $\mathrm{C_wWLP}$s and $\mathrm{C_sWLP}$s, of designs d_1, d_2, d_4 and d_5, respectively. If we need to compare the CWLPs of these designs, Table 8 shows us the results. Table 9 provides some design recommendations for different situations.

It is worth mentioning that the critical difference between the proposed criteria and the most existing criteria for FFSP designs is that the statuses of factors may not be the same. When it is known in advance that some factors are more important than others from some prior information, the proposed criteria are more appropriate than the other existing criteria for selecting good designs. For simplicity, we focus on two-level designs here. However, the proposed criteria can be easily extended to designs with higher-level factors.

Table 6: The $\mathrm{I_w}$WLPs and $\mathrm{C_w}$WLPs of designs d_1, d_2, d_4 and d_5

Design	Column (α)	$\mathrm{I_w}$WLP*	$\mathrm{C_w}$WLP*
d_1	A, B, C	(4, 0, 0, 2, 0, 0, 1)	$(\frac{12}{5}, 0, 0, \frac{18}{5}, 0, 0, 1)$
	D, E	(0, 0, 0, 6, 0, 0, 1)	
d_2	A, B, C, E	(3, 0, 0, 3, 0, 0, 1)	$(\frac{12}{5}, 0, 0, \frac{18}{5}, 0, \frac{1}{5}, 1)$
	D	(0, 0, 0, 6, 0, 1, 1)	
Design	**Column (α)**	**$\mathrm{I_w}$WLP$^\#$**	**$\mathrm{C_w}$WLP$^\#$**
d_4	A, B, C, D	(6, 0, 0, $\mathbf{0}_4$, 3, 0, 6, 0, 0)	(6, 0, 0, $\mathbf{0}_4$, 3, 0, 6, 0, 0)
d_5	A, C	(6, 0, 0, $\mathbf{0}_4$, 2, 0, 6, 0, 0)	$(\frac{13}{2}$, 0, 0, $\mathbf{0}_4$, 2, 0, 5, 0, 0)
	B, D	(7, 0, 0, $\mathbf{0}_4$, 2, 0, 4, 0, 0)	

$*$: the pattern is $(A_4^{2w}, A_4^{1w}, A_4^{4w}, A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{5w})$.

#: the pattern is $(A_4^{2w}, A_4^{1w}, A_4^{4w}, \mathbf{0}_4, A_6^{4w}, A_6^{3w}, A_6^{2w}, A_6^{1w}, A_6^{6w})$.

$\mathbf{0}_4$ denotes that the 4 components of vector $(A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{5w})$ are zeros.

Table 7: The $\mathrm{I_s}$WLPs and $\mathrm{C_s}$WLPs of designs d_1, d_2, d_4 and d_5

Design	Column (β)	$\mathrm{I_s}$WLP*	$\mathrm{C_s}$WLP*
d_1	p, q, r, s	(3, 0, 1, 3, 0, 0, 0)	(3, 0, 1, 3, 0, 0, 0)
d_2	p, q, r, s	(3, 0, 0, 3, 0, 1, 0)	(3, 0, 0, 3, 0, 1, 0)
Design	**Column (α)**	**$\mathrm{I_s}$WLP$^\#$**	**$\mathrm{C_s}$WLP$^\#$**
d_4	p, q, r, s, t, u	(4, 0, 2, $\mathbf{0}_4$, 1, 0, 8, 0, 0)	(4, 0, 2, $\mathbf{0}_4$, 1, 0, 8, 0, 0)
d_5	p, q	(5, 0, 2, $\mathbf{0}_4$, 0, 0, 6, 0, 0)	$(\frac{13}{3}$, 0, 2, $\mathbf{0}_4$, $\frac{2}{3}$, 0, $\frac{20}{3}$, 0, 0)
	r, s, t, u	(4, 0, 2, $\mathbf{0}_4$, 1, 0, 7, 0, 0)	

$*$: the pattern is $(A_4^{2w}, A_4^{1w}, A_4^{0w}, A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{0w})$.

#: the pattern is $(A_4^{2w}, A_4^{1w}, A_4^{0w}, \mathbf{0}_4, A_6^{4w}, A_6^{3w}, A_6^{2w}, A_6^{1w}, A_6^{0w})$.

$\mathbf{0}_4$ denotes that the 4 components of vector $(A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{0w})$ are zeros.

Table 8: The CWLPs of designs

	CWLP($*$)		CWLP($*$)
d_1	$(\frac{8}{3}, 0, \frac{4}{9}, \frac{10}{3}, 0, 0, \frac{5}{9})$	d_4	$(\frac{24}{5}, 0, \frac{6}{5}, \mathbf{0}_4, \frac{9}{5}, 0, \frac{36}{5}, 0, 0)$
d_2	$(\frac{8}{3}, 0, 0, \frac{10}{3}, 0, \frac{5}{9}, \frac{5}{9})$	d_5	$(\frac{26}{5}, 0, \frac{6}{5}, \mathbf{0}_4, \frac{6}{5}, 0, 6, 0, 0)$

$*$ denotes $(A_4^{2w}, A_4^{1w}, A_4^{0w}, A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{0w}))$.

# denotes $(A_4^{2w}, A_4^{1w}, A_4^{0w}, \mathbf{0}_4, A_6^{4w}, A_6^{3w}, A_6^{2w}, A_6^{1w}, A_6^{0w})$.

$\mathbf{0}_4$ denotes that the 4 components of vector $(A_5^{3w}, A_5^{2w}, A_5^{1w}, A_5^{0w})$ are zeros.

Table 9: Suggested designs for different situations

1 or 2 WP factors	3, 4, 5 WP factors	1, 2, 3, 4 SP factors	5, 6 SP factors	no priori knowledge
d_1	d_1	d_2	\	d_2
d_5	d_4	d_5	d_4	d_4

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