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Optimal split-plot designs under individual word length patterns

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ABSTRACT

For multi-factor experiments that cannot run all the factors in a completely random order, fractional factorial split-plot (FFSP) designs are often used in practice. When some prior knowledge has shown that some factors are more likely to be significant than others, Han et al. (2023) proposed the individual word length patterns (IWLPs) of whole-plot (WP) and sub-plot (SP), denoted by the $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ respectively, in the FFSP design. In this paper, we propose a construction method for optimal FFSP designs based on these two criteria, where the key of the method is to construct generating matrices for different FFSP designs from the generating matrix of a fractional factorial design, and hence we get a class of effective FFSP designs. These designs are more applicable in many situations. The results for 16-run two-level FFSP designs are tabulated in the supplementary material for possible use by practitioners.

1. Introduction

Two-level fractional factorial (FF) designs are commonly used for factorial experiments. A 2^{-k}th fraction of a 2^n factorial design consisting of 2^{n-k} distinct combinations is referred to as a 2^{n-k} design, which denotes a fraction with 2^{n-k} runs and n two-level factors. The design has $n-k$ independent columns and is determined by k independent defining words. In order to compare 2^{n-k} FF designs, many criteria have been proposed, such as maximum resolution (Box and Hunter, 1961), minimum aberration (Fries and Hunter, 1980), clear effects (Wu and Chen, 1992), general minimum lower-order confounding (Zhang et al., 2008) and so on.

When the experimenters have any prior knowledge that some factors are particularly more significant than others, Li et al. (2015) proposed a new criterion called the individual word length pattern (IWLP) that measures the degree of aliasing between an individual factor and the effects involving other factors. The critical difference between this criterion and most existing criteria for 2^{n-k} FF designs is that it does not treat all factors as being of equal importance.

When an experiment is performed, it is a natural assumption that the experimental runs can be completely randomized. However, this is impractical in practice when it is difficult to change the levels of some factors in the experiment. In such a case, a fractional factorial split-plot (FFSP) design may represent a practical and popularly used experimental strategy. In general, an FFSP experiment has two types of factors: the factors with hard-to-change levels which are called whole-plot (WP) factors, and the factors with relatively-easy-to-change levels which are sub-plot (SP) factors. Such an experiment can be arranged by an FFSP design. The concepts for these criteria such as minimum aberration (MA) and clear effects can be easily extended to FFSP designs, see e.g. Huang et al. (1998), Bingham and Sitter (1999a,b), Mukerjee and Fang (2002), Ai and Zhang (2004), Cheng and Tsai (2009) for researches on MA two-level FFSP designs, and Yang et al. (2006), Zhao and Chen (2012), Zhao and Zhao (2015), Han et al. (2020a,b) for researches on FFSP designs under the clear effects criterion.

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As we know, the criteria mentioned above consider all factors as equally important, which is not suit for the situation that some factors are more significant than others if we have some prior information about the importance of factors in an experiment. Compared with these criteria, Han et al. (2022) proposed new criteria named the IWLPs of WP and SP factors in the FFSP design, denoted by $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$, respectively. These two criteria are very useful for us to assign columns to factors after an FFSP design is selected. In this way, constructing the optimal designs under these two criteria is meaningful. Note that Huang et al. (1998) first established the relation between FFSP designs and FF designs. This relation was best seen through the generating matrices by which FFSP designs were well represented. Then, they used this relation and the concept of MA for FF designs to study FFSP designs. In the end, they explored methods for constructing two-level MA FFSP designs.

With this in mind, this paper utilizes the way described above to construct optimal designs under $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$, based on Table 2 in Li et al. (2015) which tabulates the IWLPs of all non-isomorphic 16-run regular FF designs.

The paper is organized as follows. Section 2 introduces some preliminaries and the way we use to construct optimal designs. In Section 3, we propose a construction method for FFSP designs from FF designs, where the method is mainly based on constructing generating matrices for different FFSP designs from the generating matrix of an FF design. Then we obtain optimal FFSP designs based on all regular non-isomorphic 16-run FF designs, where the detailed optimal FFSP designs with 16 runs are tabulated in the Supplementary Material. We also propose effective designs when the focus is on all possible number of particularly important factors. Concluding remarks are given in Section 4.

2. Preliminaries

In this section, we first recall criteria of $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$, and then we show generating matrices corresponding to the FF and FFSP designs.

Based on the natural features of an FFSP design, Han et al. (2022) proposed criteria of $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$. Before introducing the criteria, we recall some important notation and definitions. A 2^{n-k} FF design d is defined by k independent words $W_1, \ldots, W_k$, and the defining contrast subgroup of d consists of 2^k-1 nonzero words formed by all possible products of the independent words. Let $A_i(d)$ be the number of distinct defining words of length i in the defining contrast subgroup of d. The vector $W(d) = (A_1(d), \ldots, A_n(d))$ is called the wordlength pattern (WLP) of d. For an FFSP design, it can be considered as a usual 2^{n-k} FF design but written as a $2^{(n_1+n_2)-(k_1+k_2)}$ design, where n_1 is the number of WP factors denoted by capital letters $A, B, C, \ldots$ in this paper, and n_2 is the number of SP factors denoted by lowercase letters $p, q, r, \ldots$. In such a design, there are k_1 WP and k_2 ($k_2 = k - k_1$) SP defining words, respectively. Similarly to the WLP for FF designs, we can respectively define the IWLPs of a WP and an SP factor for an FFSP design. For a column α in an FFSP design, let $A_j^{iw}(\alpha)$ denote the number of length j defining words involving α with i WP factors and $(j-i)$ SP factors.

An FFSP design obeys the following two rules: there is no defining word that contains only one SP factor, and the number of lower-order SP-type effects which are not aliased with WP effects is as large as possible. Consider the case of column α being a WP factor. It is obvious that, $A_3^{1w}(\alpha) \neq 0$ means that column α is aliased with at least one interaction of two SP factors, $A_3^{3w}(\alpha) \neq 0$ means that column α is aliased with at least one interaction of two WP factors. For $A_3^{1w}(\alpha)$ and $A_3^{3w}(\alpha)$, we need care more about $A_3^{1w}(\alpha)$ since the second rule tells us that we should consider the SP defining words involving WP factors at first. As for $A_4^{1w}(\alpha)$, $A_4^{2w}(\alpha)$ and $A_4^{4w}(\alpha)$, $A_4^{1w}(\alpha) \neq 0$ means that column α is aliased with at least one interaction of three SP factors, $A_4^{2w}(\alpha) \neq 0$ means that column α is aliased with at least one interaction of one WP factor and two SP factors, $A_4^{4w}(\alpha) \neq 0$ means that α is in at least one WP defining word of length 4. From the second rule, $A_4^{2w}(\alpha) \neq 0$ means an interaction of two SP factors is included in at least one WP alias set, and $A_4^{1w}(\alpha) \neq 0$ means an interaction of three SP factors is included in at least one WP alias set. From the effect hierarchy principle (Wu and Hamada, 2021), the lower-order effects are more likely to be important than higher order effects, so when we consider the order of SP effects, $A_4^{2w}(\alpha)$ is more important than $A_4^{1w}(\alpha)$. Hence, we rank the order of defining words of length 4 involving column α into the sequence $\{A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha)\}$ and sequentially minimize this sequence. In general, we have

$$\mathrm{I_wWLP}(f,\alpha) = (\underbrace{A_3^{1w}(\alpha), A_3^{3w}(\alpha)}_{A_3(\alpha)}, \underbrace{A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha)}_{A_4(\alpha)}, \ldots, \underbrace{A_{n_1+n_2}^{(n_1+n_2-2)w}(\alpha), \ldots, A_{n_1+n_2}^{1w}(\alpha), A_{n_1+n_2}^{(n_1+n_2)w}(\alpha)}_{A_{n_1+n_2}(\alpha)}).$$

When the factor β is an SP factor, the principle is similar. Based on the above discussion, Han et al. (2022) proposed $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ as follows.

For columns α and β from the WP and SP parts of a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design f, respectively, define the vectors

$$\mathrm{I_wWLP}(f,\alpha) = (A_3^{1w}(\alpha), A_3^{3w}(\alpha), A_4^{2w}(\alpha), A_4^{1w}(\alpha), A_4^{4w}(\alpha), \ldots, A_{n_1+n_2}^{(n_1+n_2-2)w}(\alpha), \ldots, A_{n_1+n_2}^{1w}(\alpha), A_{n_1+n_2}^{(n_1+n_2)w}(\alpha)) \quad (1)$$

and

$$\mathrm{I_sWLP}(f,\beta) = (A_3^{1w}(\beta), A_3^{0w}(\beta), A_4^{2w}(\beta), A_4^{1w}(\beta), A_4^{0w}(\beta), \ldots, A_{n_1+n_2}^{(n_1+n_2-2)w}(\beta), \ldots, A_{n_1+n_2}^{0w}(\beta)) \quad (2)$$

as the IWLPs of α and β for f, respectively. A good design is to minimize the sequences (1) and (2) in turn.

According to the definitions of $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$, we can have the corresponding criteria for finding the best $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ of an FFSP design.

Definition 1. Suppose that f is a $2^{(n_1+n_2)-(k_1+k_2)}$ design. For every WP or SP factor in f, there is a corresponding $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$. For columns α_1 and α_2 from the WP of f with $\mathrm{I_wWLP}(f,\alpha_1)$ and $\mathrm{I_wWLP}(f,\alpha_2)$ as defined in (1). Let r be the smallest i such that $A_i^{jw}(\alpha_1) \neq A_i^{jw}(\alpha_2)$. Then α_1 is better than α_2 if $A_r^{jw}(\alpha_1) < A_r^{jw}(\alpha_2)$. α_1 is said to be the best WP factor of f if no other α_l $(l = 2, 3, \dots, n_1)$ is better than it, and $\mathrm{I_wWLP}(f,\alpha_1)$ is called the best $\mathrm{I_wWLP}$ of f. Similarly, we can define the best SP factor and best $\mathrm{I_sWLP}$ of f.

Based on Definition 1, we can get the best $\mathrm{I_wWLP}$ and $\mathrm{I_sWLP}$ of each FFSP design. For an FFSP design, some different WP or SP factors may correspond to the same $\mathrm{I_wWLP}$ or $\mathrm{I_sWLP}$. For different $2^{(n_1+n_2)-(k_1+k_2)}$ designs, their best $\mathrm{I_wWLPs}$ or $\mathrm{I_sWLPs}$ may not be the same. From the above discussion, we can get the process of selecting the optimal FFSP design. Firstly, calculate the $\mathrm{I_sWLPs}$ and $\mathrm{I_wWLPs}$ of all FFSP designs with given parameters, and obtain the best $\mathrm{I_sWLP}$ and $\mathrm{I_wWLP}$ after comparison. Secondly, record the respective numbers of columns with the best $\mathrm{I_sWLP}$ and $\mathrm{I_wWLP}$ in each design, and the design containing the maximum number of columns with the best $\mathrm{I_sWLP}$ or $\mathrm{I_wWLP}$ is the optimal FFSP design. It is worth noting that the optimal designs containing the respective maximum numbers of columns with the best $\mathrm{I_sWLP}$ and best $\mathrm{I_wWLP}$ may not be the same design, and this phenomenon is quite normal. At this point, researchers can choose the proper design according to their actual needs.

Here we present a practical example to illustrate the application of the proposed criteria. Consider an experiment to improve a heat treatment process on truck leaf springs reported in Pignatiello and Ramberg (1985). The heat treatment that forms the camber (or curvature) in leaf springs consists of heating in a high-temperature furnace, processing by a forming machine, and quenching in an oil bath. The height of an unloaded spring, known as free height, is an important quality characteristic whose target value is 8.0 inches. An experimental goal is to make the variation about the target 8.0 as small as possible. The five factors were chosen across the various stages of the process: furnace temperature (B) and heating time (C) are from the heating stage, transfer time (D) is the time it takes the conveyer to transport the springs from the furnace to the forming machine, hold-down time (E) is the time that the springs are held in a high-pressure press to form the camber, and quench oil temperature (Q) is from the quenching stage. These factors are all quantitative. Furnace temperature (B) and quench oil temperature (Q) are difficult to change and can be used as WP factors. The other three factors can be used as SP factors. This constitutes an FFSP design. Suppose we have prior information that factor B is a significant factor, then we need to choose a design that has the fewest interactions aliased with factor B. In this case, an optimal FFSP design under the $\mathrm{I_wWLP}$ criterion is the most suitable one.

In the following, we introduce generating matrices (Franklin, 1984) corresponding to the FF and FFSP designs. In fact, each FF or FFSP design corresponds to a generating matrix. The generating matrix can be transformed to a standard form through matrix operations, so that we can find the corresponding design more easily. There are three steps for this process.

The first step is about the correspondence between an FF design and the generating matrix. The defining contrast subgroup of a 2^{n-k} FF design d can be represented by the generating matrix g. Without loss of generality, the generating matrix g can be written in the form of

$$g = (\mathrm{I}\ \mathrm{C}), \tag{3}$$

where I is the $k \times k$ identity matrix and C is a $k \times (n-k)$ matrix with its elements equal to 0 or 1, in which every row must contain at least one 1. In fact, every row of g corresponds to a defining word of design f.

The second step is about the correspondence between an FFSP design and the generating matrix. Similarly to the first step, one can apply generating matrix g to two-level FFSP designs. Suppose that there are n_1 factors in the WP part with fractionation element k_1, and n_2 factors in the SP part with fractionation element k_2. Then there are $2^{n_1-k_1}$ treatment combinations in the WP part and $2^{(n_1+n_2)-(k_1+k_2)}$ treatment combinations for the total FFSP design. Let g_1 be the generating matrix of a $2^{n_1-k_1}$ design d_1 for the WP part, then g_1 can be denoted as

$$g_1 = (\mathrm{I}_1\ \mathrm{C}_1),$$

where I_1 is the $k_1 \times k_1$ identity matrix and C_1 is a $k_1 \times (n_1-k_1)$ matrix. Let g_2 be the generating matrix of a $2^{n_2-k_2}$ design d_2 for the SP part. If the fractionation in the SP part depends only on the SP factors, then

$$g_2 = (\mathrm{I}_2\ \mathrm{C}_2),$$

where I_2 is the $k_2 \times k_2$ identity matrix, and C_2 is a $k_2 \times (n_2-k_2)$ matrix. The design matrix for the total FFSP design can be represented by the generating matrix

$$g = \begin{pmatrix} g_1 & \mathrm{O} \\ \mathrm{O} & g_2 \end{pmatrix} = \begin{array}{c@{}c} \begin{array}{cccc} k_1 & n_1-k_1 & k_2 & n_2-k_2 \end{array} & \\ \begin{pmatrix} \mathrm{I}_1 & \mathrm{C}_1 & \mathrm{O}_1 & \mathrm{O}_2 \\ \mathrm{O}_3 & \mathrm{O}_4 & \mathrm{I}_2 & \mathrm{C}_2 \end{pmatrix} & \begin{array}{c} k_1 \\ k_2 \end{array} \end{array}, \tag{4}$$

where all the elements in $\mathrm{O}, \mathrm{O}_1, \mathrm{O}_2, \mathrm{O}_3$ and O_4 are zeros. In general, letters from the WP part are allowed to appear in the generators for the SP part. So the generating matrix g of such a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design will now have the form of

$$g = \begin{array}{c@{}c} \begin{array}{cccc} k_1 & n_1-k_1 & k_2 & n_2-k_2 \end{array} & \\ \begin{pmatrix} \mathrm{I}_1 & \mathrm{C}_1 & \mathrm{O}_1 & \mathrm{O}_2 \\ \mathrm{B}_1 & \mathrm{B}_2 & \mathrm{I}_2 & \mathrm{C}_2 \end{pmatrix} & \begin{array}{c} k_1 \\ k_2 \end{array} \end{array}. \tag{5}$$

Here, $\mathrm{I}_1, \mathrm{I}_2, \mathrm{C}_1, \mathrm{C}_2, \mathrm{O}_1$ and O_2 are the matrices with the same structures as above; B_1 and B_2 are matrices with elements 0 and 1; $(\mathrm{I}_1\ \mathrm{C}_1)$, denoted by g_1, represents the generating matrix of the WP part; and $(\mathrm{B}_1\ \mathrm{B}_2\ \mathrm{I}_2\ \mathrm{C}_2)$ represents the generating matrix of the SP part. In fact, the g in (5) is a natural generalization of the g in (4).

The third step is to convert the form in (5) to a standard form for the generating matrix g of an FFSP design by performing operations between rows and switching some columns. The purpose of this step is to make it clear which FFSP designs can be obtained from FF designs. A standard form for the generating matrix g is as follows

$$g = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} k_1 & k_2 & n_1-k_1 & n_2-k_2 \\ \end{array}$$
$$g = \begin{pmatrix} \mathrm{I}_1 & \mathrm{O}_1 & \mathrm{C}_1 & \mathrm{O}_2 \\ \mathrm{O}_3 & \mathrm{I}_2 & \mathrm{B}_3 & \mathrm{C}_2 \end{pmatrix} \begin{array}{c} k_1 \\ k_2 \end{array}. \tag{6}$$

3. Optimal designs based on $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP

In this section, we first introduce how to use the generating matrix g of an FF design to construct generating matrices for different FFSP designs and thus generate those FFSP designs. Then we obtain and tabulate the effective designs for non-isomorphic 16-run FFSP designs using the construction method of generating matrix.

One can get different FFSP designs by different permutations between rows or columns of g. The essence of the difference is that the dimension of O_2 in form (6) varies with different values of k_1 and k_2. Each type of dimension of O_2 corresponds to an FFSP design, and the parameters of n_1, n_2, k_1 and k_2 can be got from the dimensions of those parts in the standard form. That is the way we use to construct optimal designs.

The following example shows how to get $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs from a 2^{9-5} FF design. The strategy here is to rearrange the generating matrix g of a 2^{9-5} design to be of the form (6), whose two key features are as follows:

1. The left part of k columns is the identity matrix of dimension $k \times k$, here k=5.
2. The upper right corner is a $k_1 \times (n_2 - k_2)$ matrix of zeros.

Example 1. Suppose there is a 2^{9-5} FF design with generating relation $I = ABE = ACF = ADG = BCDH = ABCDJ$. The generating matrix of this design is

$$\begin{array}{c} \begin{array}{ccccccccc} A & B & C & D & E & F & G & H & J \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{7}$$

The following cases are the FFSP designs that we can construct through this 2^{9-5} design.

Case 1: In contrast to form (6), we convert the matrix to

$$\begin{array}{cc} & \begin{array}{ccccccccc} E & F & G & H & J & A & B & C & D \end{array} \\ g = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = (\mathrm{I\ C}) \\ & \begin{array}{ccccccccc} A & B & p & q & r & C & D & E & s \end{array} \leftarrow \textit{Case } 1 \\ & \begin{array}{ccccccccc} A & p & q & r & s & B & C & t & u \end{array} \leftarrow \textit{Case } 2 \\ & \begin{array}{ccccccccc} p & q & r & s & t & u & v & A & B \end{array} \leftarrow \textit{Case } 4[1] \\ & \begin{array}{ccccccccc} p & q & r & s & t & u & A & v & B \end{array} \leftarrow \textit{Case } 4[2] \\ & \begin{array}{ccccccccc} p & q & r & s & t & u & A & B & v \end{array} \leftarrow \textit{Case } 4[3] \end{array}. \tag{8}$$

Thus, comparing (I C) with form (6), we can obtain $k_1 = 2$, $k_2 = 3$, $n_2 - k_2 = 1$ ($n_2 = 1 + 3 = 4$), and $n_1 - k_1 = 3$ ($n_1 = 3 + 2 = 5$). Relabel A, B, C, D, E, F, G, H and J to C, D, E, s, A, B, p, q and r, respectively, where the new A, B, C, D and E are WP factors, and p, q, r and s are SP factors. See (8) for the corresponding factors before and after relabeling. Then we can get a 2^{5-2} WP design with factors A, B, C, D and E and generating relation $I = ACD = BCE$, similarly, we have a 2^{4-3} SP design with factors p, q, r and s and generating relation $I = Cps = DEqs = CDErs$.

Case 2: Another FFSP design can also be got from the above matrix with $k_1 = 1$, $k_2 = 4$, $n_2 - k_2 = 2$ ($n_2 = 2 + 4 = 6$), and $n_1 - k_1 = 2$ ($n_1 = 2+1 = 3$). And this design contains a 2^{3-1} WP design with factors A, B and C and generating relation $I = ABC$, and a 2^{6-4} SP design with factors p, q, r, s, t and u and generating relation $I = Bpt = Bqu = Crtu = BCstu$. See (8) for the corresponding factors before and after relabeling.

Case 3: Another FFSP design can be derived from the preceding 2^{9-5} design. Observe that the column under factor A in (7) has one zero element. We want to move it to the upper right corner. First we switch columns to get

$$\begin{array}{ccccccccc} E & F & G & H & J & B & C & D & A \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{9}$$

then switch runs to get

$$\begin{array}{ccccccccc} E & F & G & H & J & B & C & D & A \end{array}$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \tag{10}$$

in the end, switch columns to get

$$\begin{array}{ccccccccc} H & F & G & E & J & B & C & D & A \end{array}$$
$$g' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \tag{11}$$
$$\begin{array}{ccccccccc} A & p & q & r & s & B & C & D & t \end{array} \quad \leftarrow Case\ 3$$

Comparing g' with form (6), we have $k_1 = 1$, $k_2 = 4$, $n_2 - k_2 = 1$ ($n_2 = 1+4 = 5$), and $n_1 - k_1 = 3$ ($n_1 = 3+1 = 4$). Relabel A, B, C, D, E, F, G, H and J to t, B, C, D, r, p, q, A and s, respectively, where the new A, B, C and D are WP factors, and p, q, r, s and t are SP factors. Then we can get a 2^{4-1} WP design with factors A, B, C and D and generating relation $I = ABCD$, and a 2^{5-4} SP design with factors p, q, r, s and t and generating relation $I = Brt = Cpt = Dqt = BCDst$. See (11) for the corresponding factors before and after relabeling.

Case 4: In the end, we construct designs with $k_1 = 0$. In this case, the generating matrix must have the form of $g = (\mathrm{I}_2\ \mathrm{B}_3\ \mathrm{C}_2)$ in (6). Note that C_2 is a $k_2 \times (n_2 - k_2)$ (i.e. $5 \times (n_2 - 5)$) matrix, and at least one element of each row is 1. So we cannot get a design with $n_2 - k_2 = 1$ as g does not contain a column with all elements being 1. Next we construct designs with $k_1 = 0$, $k_2 = 5$, $n_2 - k_2 = 2$ ($n_2 = 2+5 = 7$) and $n_1 - k_1 = 2$ ($n_1 = 2+0 = 2$). For this case, relabel E, F, G, H and J to p, q, r, s and t, respectively, as SP factors. For the remaining factors, there are three scenarios. The first scenario is relabeling A, B, C and D to u, v, A and B, where the new A and B are WP factors and u and v are SP factors. The generating relation is $I = Aqu = Bur = puv = ABsv = ABtuv$. The second scenario is relabeling A, B, C and D to u, A, v and B. The generating relation is $I = Apu = Bru = quv = ABsv = ABtuv$. And the last scenario is relabeling A, B, C and D to u, A, B and v. The generating relation is $I = Apu = Bqu = ruv = ABsv = ABtuv$. These are all the possible FFSP designs with $k_1 = 0$ we can construct. See (8) for the corresponding factors before and after relabeling for these three scenarios.

For ease of understanding the construction of FFSP designs from the FF design above, we simplify the procedure as Algorithm 1.

From the construction method of generating matrix introduced above, we can get the FFSP designs listed in Table 1 in the Supplementary Material from FF designs. This table displays the optimal designs obtained by computing the $\mathrm{I_w}$WLPs and $\mathrm{I_s}$WLPs of all columns in the FFSP designs which are generated from all non-isomorphic 16-run FF designs. For designs of the same parameters with multiple possibilities, the final $\mathrm{I_w}$WLPs, $\mathrm{I_s}$WLPs and the corresponding numbers of factors may be the same. Since the ultimate goal of our construction is to provide more designs with different factor arrangements, we keep one design in Table 1 in the Supplementary Material for this case. Non-isomorphic FF designs may lead to isomorphic FFSP designs. For example, a $2^{(6+3)-(3+2)}$ design can be constructed from two 2^{9-5} FF designs with WLP= $(A_3, A_4) = (8, 10)$ and $(6, 10)$, respectively. These two FF designs are non-isomorphic, but the resulting $2^{(6+3)-(3+2)}$ designs are isomorphic. For this case, we only keep one FFSP design. It is worth mentioning that, we have listed an extensive (but likely incomplete) set of such designs for 16 runs by using the construction method of generating matrix. We have tried to find all such designs for $2^{n-k} = 16$, with $n = 5$ to 15, $k_2 \geq 1$, and Resolution at least III. In Table 1 in the Supplementary Material, the notation "$*$" after a $\mathrm{I_w}$WLP or $\mathrm{I_s}$WLP means the corresponding $\mathrm{I_w}$WLP or $\mathrm{I_s}$WLP is the best one of the designs with the same n_1, n_2, k_1 and k_2.

When we need to arrange the specific number of significant WP or SP factors in a design, we can look for a design with the $\mathrm{I_w}$WLP or $\mathrm{I_s}$WLP being marked by "$*$" where the number of WP or SP factors preceding it in the table being at least equal to the specific number. Here are two examples to illustrate the use of the tables. For example, we can construct three non-isomorphic $2^{(4+2)-(1+1)}$ designs through three 2^{6-2} designs. By comparing the IWLPs of all factors in these three designs, we get the best $\mathrm{I_w}$WLP being (0,0,0,0,0) and the best $\mathrm{I_s}$WLP being (0,0,1,0,0). Obviously, the third $2^{(4+2)-(1+1)}$ design contains a WP factor C with the best $\mathrm{I_w}$WLP, and the second $2^{(4+2)-(1+1)}$ design contains two SP factors p and q with the best $\mathrm{I_s}$WLP. From the table, we can find

Algorithm 1 Construction of FFSP designs in Example 1

Step 1: According to the generating relation of the 2^{9-5} FF design, obtain matrix (7). Then write it in the form of a matrix (6) by column permutation, see matrix (8).

Step 2: By the different dimensions of the zero matrices in the upper right corner of matrix (8), we can get two O_2 matrices and thus two FFSP designs. The first O_2 is taken to be a 2×1 zero matrix, we get $k_1 = 2$, $k_2 = 3$, $n_2 - k_2 = 1$ ($n_2 = 1 + 3 = 4$), and $n_1 - k_1 = 3$ ($n_1 = 3 + 2 = 5$). Then we obtain a $2^{(5+4)-(2+3)}$ design, that is Case 1. The second O_2 is taken to be a 1×2 zero matrix, we get $k_1 = 1$, $k_2 = 4$, $n_2 - k_2 = 2$ ($n_2 = 2 + 4 = 6$), and $n_1 - k_1 = 2$ ($n_1 = 2 + 1 = 3$). Then we have a $2^{(3+6)-(1+4)}$ design, that is Case 2.

Step 3: By switching rows and columns of matrix (8), we can make the dimension of the zero matrix in the upper right corner different from that of Cases 1 and 2. See matrix (11), now we get a 1×1 zero matrix in the upper right corner of matrix (11). Then we have $k_1 = 1$, $k_2 = 4$, $n_2 - k_2 = 1$ ($n_2 = 1 + 4 = 5$), and $n_1 - k_1 = 3$ ($n_1 = 3 + 1 = 4$). In this way, we get a $2^{(4+5)-(1+4)}$ design, and this is Case 3.

Step 4: In the end, we construct designs with $k_1 = 0$. By the form of matrix (8), we only construct designs with $k_1 = 0$, $k_2 = 5$, $n_2 - k_2 = 2$ ($n_2 = 2 + 5 = 7$) and $n_1 - k_1 = 2$ ($n_1 = 2 + 0 = 2$) under the requirement that the resulting matrix must have the form of $g = (\mathrm{I}_2\ \mathrm{B}_3\ \mathrm{C}_2)$ in matrix (6). Then we construct three $2^{(2+7)-(0+5)}$ designs, since the WP and SP factors can be arranged differently, this is Case 4.

the $\mathrm{I_w}$WLP of the third design and the $\mathrm{I_s}$WLP of the second design that are marked by "$*$", respectively. Therefore, when we need a design that can arrange a potentially important WP factor, we choose the third design. Similarly, when we need a design that can arrange two potentially important SP factors, we choose the second design. Let us see another example. We can construct four $2^{(5+4)-(2+3)}$ FFSP designs from four 2^{9-5} FF designs. By calculating and comparing the IWLPs of all factors in these four designs, we get the best $\mathrm{I_w}$WLP being (0,1,4,0,1) and the best $\mathrm{I_s}$WLP being (1,0,5,0,0). Obviously, the first $2^{(5+4)-(2+3)}$ design has both the best $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP, as marked by "$*$" in Table 1 in the Supplementary Material. If we want to arrange a potentially important WP factor and/or a potentially important SP factor in a $2^{(5+4)-(2+3)}$ design, we can choose this design for planning the experiment. It is worth mentioning that some designs are suitable for situations where there are multiple significant factors in it. These designs will have greater practical applicability. To facilitate the calculation of $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP, we simplify the procedure as Algorithm 2.

Algorithm 2 Calculation of $\mathrm{I_w}$WLP and $\mathrm{I_s}$WLP

Input: The number of WP factors n_1, the number of SP factors n_2 and k independent defining words.

Output: $\mathrm{I_w}$WLP for each WP factor and $\mathrm{I_s}$WLP for each SP factor.

Step 1: Let $n = n_1 + n_2$. Obtain all $2^k - 1$ defining words by k independent defining words.

Step 2: For each WP factor α, find all defining words including α. Then calculate the values of $A_3^{1w}(\alpha)$, $A_3^{3w}(\alpha)$, $A_4^{2w}(\alpha)$, $A_4^{1w}(\alpha)$, $A_4^{4w}(\alpha)$, …, $A_{n_1+n_2}^{(n_1+n_2-2)w}(\alpha)$, …, $A_{n_1+n_2}^{1w}(\alpha)$, $A_{n_1+n_2}^{(n_1+n_2)w}(\alpha)$ in turn.

Step 3: For each SP factor β, find all defining words including β. Then calculate the values of $A_3^{1w}(\beta)$, $A_3^{0w}(\beta)$, $A_4^{2w}(\beta)$, $A_4^{1w}(\beta)$, $A_4^{0w}(\beta)$, …, $A_{n_1+n_2}^{(n_1+n_2-2)w}(\beta)$, …, $A_{n_1+n_2}^{0w}(\beta)$ in turn.

4. Concluding remarks

In this paper, we use generating matrix to construct FFSP designs with different parameters as many as possible from a given FF design, and provide optimal designs with different numbers of significant factors. The advantage of the resulting designs is that when practitioners already know the number of significant factors in advance, they can choose an appropriate design to carry out the experiment according to the actual situation. It should be mentioned that the construction methods presented in this paper are all algebraic methods without any computer search, so it is of great value to explore more complex FFSP designs from higher-level or mixed-level FF designs. The methods can be easily generalized to two-level regular designs with more runs, e.g. 32 runs, 64 runs and so on. The resulting FFSP designs will be more numerous and more complex. But the calculation process is the same as the process for choosing the 16-run designs. If we want a two-level effective design with more than 16 runs, we can perform the calculations in the same way as in this paper, and then choose the appropriate design. If we want a two-level optimal 16-run design, but cannot find it in Table 1 in the Supplementary Material, further calculations can be made according to the criteria of C_wWLP and C_sWLP proposed by Han et al. (2022) to obtain the optimal design.

The difficulty with this work is that not all of the FFSP designs can be constructed through FF designs, so the 'optimal' design mentioned here is only the best design selected from those that can be constructed. Further work is needed in this area. For simplicity, we focus on two-level designs. And for higher-level and mixed-level designs, we leave for future research.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2024.110311.

Data availability

No data was used for the research described in the article.

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