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REGULAR ARTICLE

Construction of orthogonal marginally coupled designs

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Abstract
Marginally coupled designs (MCDs) were first introduced by Deng et al. (Stat Sin 25:1567–1581, 2015), as more economical designs than sliced space-filling designs which are the popular choices for computer experiments with both qualitative and quantitative factors. In an MCD, the design for qualitative factors is an orthogonal array, and the one for quantitative factors is a Latin hypercube design (LHD) with its rows corresponding to each level of any qualitative factor also forming a small LHD. As we know, orthogonality is a popular and important property for evaluating LHDs, but was not considered in existing results on MCDs. In this paper, we propose some approaches to constructing a new class of MCDs with orthogonality. In some cases, the designs for quantitative factors also satisfy the two dimensional space-filling property. Besides, the run sizes of the obtained designs are more flexible than the existing ones.

Keywords Computer experiment · Orthogonality · Orthogonal array · Regular design

1 Introduction

Latin hypercube designs (LHDs), proposed by McKay et al. (1979), are widely used in computer experiments. A large number of papers have made efforts to find different variants of LHDs, including orthogonal LHDs and maximin LHDs (see e.g., Lin and Tang 2015; Wang et al. 2018a, b, and the references therein). These designs are constructed for computer experiments with only quantitative factors. For computer experiments with both qualitative and quantitative factors, sliced LHDs (SLHDs) proposed by Qian (2012) are often used. However, if the number of the level-combinations of the qualitative factors is large, the run size of the SLHD will be very large. Recently, Deng et al. (2015) proposed a cost-effective class of designs, called marginally coupled designs (MCDs), which maintain an economic run size with space-filling properties. The construction of MCDs has been studied by Deng et al. (2015), He et al. (2017,

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2019) and He et al. (2017). However, the designs for quantitative factors in these existing papers do not have the orthogonality property. Orthogonality not only is of vital importance for fitting polynomial models, but also can be regarded as a stepping stone to space-filling designs (Bingham et al. 2009). In this paper, we propose several methods to construct MCDs in which the designs for quantitative factors are orthogonal LHDs. In addition, some of the obtained designs have an attractive space-filling property, i.e., the designs for quantitative factors possess stratification in some two-dimensional projections.

The remainder of the paper is organized as follows. In Sect. 2, we present some definitions and notation. Section 3 devotes itself to the construction approaches and examples. Some concluding remarks are given in Sect. 4. All proofs are deferred to the Appendix.

2 Definitions and notation

For any design D, let $D[i, k]$ denote the element in the ith row and kth column of D, $D[i, :]$ denote the ith row of D, $D[i, k : m]$ consist of the kth to mth elements in the ith row of D, and $D[:, k : m]$ consist of the kth to mth columns of D. An $n \times m$ matrix in which the jth column includes s_j equally-spaced levels is called a mixed-level orthogonal array (OA) of strength t, denoted by $OA(n, s_1 s_2 \ldots s_m, t)$, if for each $n \times t$ submatrix, all possible level-combinations occur with the same frequency. When all the s_j's are equal to s, the OA is symmetric and denoted as $OA(n, m, s, t)$. A Latin hypercube design (LHD) with n runs and p factors, denoted as $L(n, p)$, is an $n \times p$ matrix in which each column includes n equally-spaced levels. An LHD is called an orthogonal LHD (OLHD) if the correlation between any two distinct columns is zero. An OLHD with centered levels is called second-order orthogonal if the sum of the elementwise products of any three columns is zero. The correlation between two vectors $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$ is defined as

$$\rho(a, b) = \frac{\sum_{i=1}^{n}(a_i - \bar{a})(b_i - \bar{b})}{\sqrt{\sum_{i=1}^{n}(a_i - \bar{a})^2 \sum_{i=1}^{n}(b_i - \bar{b})^2}},$$

where $\bar{a} = \sum_{i=1}^{n} a_i/n$ and $\bar{b} = \sum_{i=1}^{n} b_i/n$.

For two matrices $\phi = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1k} \\ \vdots & \ddots & \vdots \\ \phi_{n1} & \cdots & \phi_{nk} \end{pmatrix}$ and $\varphi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1m} \\ \vdots & \ddots & \vdots \\ \varphi_{l1} & \cdots & \varphi_{lm} \end{pmatrix}$, define

$$\phi \oplus \varphi = \begin{pmatrix} \phi_{11} + \varphi & \cdots & \phi_{1k} + \varphi \\ \vdots & \ddots & \vdots \\ \phi_{n1} + \varphi & \cdots & \phi_{nk} + \varphi \end{pmatrix}.$$

Furthermore, for $k = m$, define $\phi \oplus_c \varphi = (\phi_1 \oplus \varphi_1, \ldots, \phi_k \oplus \varphi_k)$, where ϕ_i and φ_i are the ith columns of ϕ and φ, respectively.

A design $D = (D_1, D_2)$, where D_1 and D_2 are sub-designs for qualitative and quantitative factors respectively, is called an MCD if (i) D_1 is an OA, (ii) D_2 is an LHD, and (iii) for each level of any factor in D_1, the corresponding rows in D_2 form a small LHD. Furthermore, if D_2 in an MCD is an OLHD, the MCD is called an orthogonal MCD (OMCD).

A design $D = (x_{ij})_{n\times m}$ is called mirror-symmetric if we reverse the level order for all factors, the resulting design is still itself (Tang and Xu 2014). In particular, for a mirror-symmetric design D with centered levels, if x is a row in D, $-x$ is also one of the rows in D.

The notation $B_1\backslash B_0$ represents the array which consists all columns in B_1 but not in B_0.

3 Design construction

This section presents five methods to construct OMCDs. The first four ones are based on the rotation approach, and the fifth one is based on mixed-level OAs. In addition, the low-dimensional projection properties of the proposed designs are also discussed.

3.1 Construction of MCDs using OAs with s^2 runs

Suppose an $OA(s^2, k, s, 2)$ with levels $1, \ldots, s$, denoted as A, is available. Let d and f be positive integers with $k = d + 2f$. Then A can be divided into two parts, denoted as D_1 and C, with d and $2f$ columns respectively.

Let $M = (m_{ij})$ be an $L(s, p)$ with elements $\{-(s-1)/2, -(s-3)/2, \ldots, (s-1)/2\}$. Then the correlation matrix of M is given by

$$R = \left\{\frac{1}{12}s(s^2-1)\right\}^{-1} M^T M.$$

The construction steps are presented in the following algorithm.

Algorithm 1

Step 1 *For $j = 1, \ldots, p$, replace the symbols $1, \ldots, s$ in C by $m_{1j}, \ldots, m_{sj}$, respectively, to obtain an $s^2 \times (2f)$ matrix C_j. Then partition C_j as $C_j = (C_{j1}, \ldots, C_{jf})$, where each of $C_{j1}, \ldots, C_{jf}$ has two columns.*

Step 2 *For $j = 1, \ldots, p$, obtain an $s^2 \times (2f)$ matrix $X_j = (C_{j1}V, \ldots, C_{jf}V)$, where*

$$V = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}.$$

Step 3 *Obtain a matrix $D = (D_1, X)$, where $X = (X_1, \ldots, X_p)$ with order $s^2 \times (2pf)$.*

Lemma 1 (Lin et al. 2009)

(i) *The matrix X is an LHD in the sense that each of its columns is a permutation of* $\{-(s^2-1)/2, -(s^2-3)/2, \ldots, (s^2-1)/2\}$*;*
(ii) *The correlation matrix among the columns of X is given by $\tilde{R} = R \otimes I_{2f}$, where I_u is the identity matrix of order u and $\otimes$ denotes the Kronecker product.*

Theorem 1 *The design D constructed in Algorithm 1 is an MCD. Furthermore, if M is orthogonal, X is orthogonal.*

Next, we discuss the projection property of the proposed designs. Denote the ith column of X_j by $x_{j,i}$.

Corollary 1 *For X constructed above, we have*

(i) *after collapsing the levels of X to s levels, each X_j is an $OA(s^2, 2f, s, 2)$;*
(ii) *$(x_{j,i}, x_{j,i'})$ achieves stratification on $s \times s$ grids for $i \neq i'$ and $j = 1, \ldots, p$;*
(iii) *$(x_{j,i}, x_{j',i'})$ achieves stratification on $s \times s$ grids for $i \neq i'$ and $j \neq j'$.*

Then, an example is given to illustrate the construction.

Example 1 Let A be an $OA(16, 5, 4, 2)$. Divide it into two parts D_1 and C as follows:

$$A = (D_1|C) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \\ \hline 1 & 2 & 3 & 4 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 \\ 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 \end{pmatrix}^T.$$

Let

$$M = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 3 & 1 \\ -1 & 3 \\ -3 & -1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 4 & -1 \\ 1 & 4 \end{pmatrix}.$$

After replacing the symbols of C by $m_{1j}, \ldots, m_{4j}$, C_j is obtained respectively, for $j = 1, 2$.

$$C_1 = \frac{1}{2}\begin{pmatrix} 1 & 3 & -1 & -3 & -1 & -3 & 1 & 3 & -3 & -1 & 3 & 1 & 3 & 1 & -3 & -1 \\ 1 & 3 & -1 & -3 & -3 & -1 & 3 & 1 & 3 & 1 & -3 & -1 & -1 & -3 & 1 & 3 \end{pmatrix}^T;$$

$$C_2 = \frac{1}{2}\begin{pmatrix} -3 & 1 & 3 & -1 & 3 & -1 & -3 & 1 & -1 & 3 & 1 & -3 & 1 & -3 & -1 & 3 \\ -3 & 1 & 3 & -1 & -1 & 3 & 1 & -3 & 1 & -3 & -1 & 3 & 3 & -1 & -3 & 1 \end{pmatrix}^T.$$

Then $X = (X_1, X_2) = (C_1 V, C_2 V)$, where

$$X_1 = \frac{1}{2}\begin{pmatrix} 5 & 15 & -5 & -15 & -7 & -13 & 7 & 13 & -9 & -3 & 9 & 3 & 11 & 1 & -11 & -1 \\ 3 & 9 & -3 & -9 & -11 & -1 & 11 & 1 & 15 & 5 & -15 & -5 & -7 & -13 & 7 & 13 \end{pmatrix}^T;$$

$$X_2 = \frac{1}{2}\begin{pmatrix} -15 & 5 & 15 & -5 & 11 & -1 & -11 & 1 & -3 & 9 & 3 & -9 & 7 & -13 & -7 & 13 \\ -9 & 3 & 9 & -3 & -7 & 13 & 7 & -13 & 5 & -15 & -5 & 15 & 11 & -1 & -11 & 1 \end{pmatrix}^T.$$

Collapsing the levels of X_1 and X_2 by $\lfloor(x + 15/2)/4\rfloor$, we get C_1^* and C_2^* as follows:

$$C_1^* = \begin{pmatrix} 2 & 3 & 1 & 0 & 1 & 0 & 2 & 3 & 0 & 1 & 3 & 2 & 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 & 0 & 1 & 3 & 2 & 3 & 2 & 0 & 1 & 1 & 0 & 2 & 3 \end{pmatrix}^T;$$
$$C_2^* = \begin{pmatrix} 0 & 2 & 3 & 1 & 3 & 1 & 0 & 2 & 1 & 3 & 2 & 0 & 2 & 0 & 1 & 3 \\ 0 & 2 & 3 & 1 & 1 & 3 & 2 & 0 & 2 & 0 & 1 & 3 & 3 & 1 & 0 & 2 \end{pmatrix}^T.$$

Denote the ith column of C_j^* by $c_{j,i}^*$, for $i, j = 1, 2$. It is easy to check that each C_j^* is an $OA(16, 2, 4, 2)$ for $j = 1, 2$, and $(c_{1,1}^*, c_{2,2}^*)$ and $(c_{1,2}^*, c_{2,1}^*)$ are $OA(16, 2, 4, 2)$'s respectively.

3.2 Construction of MCDs using regular factorial designs

This subsection presents three methods for constructing MCDs of s^u runs. The first two methods make use of the Rao–Hamming construction (Hedayat et al. 1999) and some rotation matrices. For an integer $u \geq 2$, a prime power s and $i = 1, \dots, u$, let e_i be an $s^u \times 1$ column vector of s-level with entries from $GF(s)$, the Galois field of order s. Assume that the columns $e_1, \dots, e_u$ are independent. Here the independence means that their linear combination equals the zero vector if and only if the combination coefficients are all equal to zero. $OA(s^u, (s^u - 1)/(s - 1), s, 2)$'s can be constructed using these u independent columns by the Rao–Hamming construction. Then we obtain OA's, $B_0, \dots, B_{s+1}$, in the following way. Let

$$B_0 = (e_1, \dots, e_{u-2})U_0,$$

where U_0 is a $(u - 2) \times [(s^{u-2} - 1)/(s - 1)]$ matrix by collecting all the nonzero column vectors given by $(l_1, l_2, \dots, l_{u-2})^T$ with $l_j \in GF(s)$ for $j = 1, \dots, u - 2$, and the first nonzero entry in $(l_1, l_2, \dots, l_{u-2})^T$ is one. Let

$$B_i = (e_1, \dots, e_{u-2}, w_i)U \text{ for } i = 1, \dots, s + 1,$$

where U is a $(u-1)\times[(s^{u-1}-1)/(s-1)]$ matrix by collecting all the nonzero column vectors given by $(l_1, l_2, \dots, l_{u-1})^T$ with $l_j \in GF(s)$ for $j = 1, \dots, u-1$, and the first nonzero entry in $(l_1, l_2, \dots, l_{u-1})^T$ is one, $w_i = (e_{u-1}, e_u) f_i$ for $i = 1, \dots, s + 1$, $f_i = (1, \alpha_{i-1})^T$ for $i = 1, \dots, s$ and $f_{s+1} = (0, 1)^T$, $\alpha_i \in GF(s) = \{\alpha_0, \dots, \alpha_{s-1}\}$ with $\alpha_0 = 0$. The following lemma discusses the properties of the constructed arrays: $B_0, \dots, B_{s+1}$.

Lemma 2 (He et al. 2017) *For $B_0, \dots, B_{s+1}$ constructed above, we have*

(i) *B_0 is an $OA(s^u, (s^{u-2} - 1)/(s - 1), s, 2)$ consisting of s^2 replicates of $OA(s^{u-2}, (s^{u-2} - 1)/(s - 1), s, 2)$;*

(ii) B_i *is an* $OA(s^u, (s^{u-1} - 1)/(s - 1), s, 2)$ *consisting of* s *replicates of* $OA(s^{u-1}, (s^{u-1} - 1)/(s - 1), s, 2)$*, for* $i = 1, \ldots, s + 1$*;*

(iii) *for* $1 \leq i \leq s + 1$*,* $B_0 \subset B_i$*, and for* $1 \leq i' \neq i \leq s + 1$*, none of the columns in* $B_{i'} \backslash B_0$ *can be generated by any linear combination of columns in* B_i*;*

(iv) *let* $b_1, \ldots, b_{u-1}$ *be any* $u - 1$ *independent columns from* B_i *and* b *be any column of* $B_{i'} \backslash B_0$*, where* $i \neq i'$*, then* $(b_1, \ldots, b_{u-1}, b)$ *is an* $OA(s^u, u, s, u)$*;*

(v) $\{B_0, (B_1\backslash B_0), (B_2\backslash B_0), \ldots, (B_{s+1}\backslash B_0)\}$ *form a disjoint partition of* $OA(s^u, (s^u - 1)/(s - 1), s, 2)$.

Given the OA's $B_0, \ldots, B_{s+1}$, we propose Algorithm 2 below to construct OMCDs. For ease of expression, $B_i\backslash B_0$ is denoted as P_i. In the algorithm, we use the following rotation matrices from Sun and Tang (2017):

$$R_{w1} = \begin{pmatrix} sR_{w0} & -Q_w \\ Q_w & sR_{w0} \end{pmatrix} \text{ and } R_{wv} = \begin{pmatrix} sR_{w(v-1)} & -Q_{w+v-1} \\ Q_{w+v-1} & sR_{w(v-1)} \end{pmatrix} \text{ for } v \geq 2,$$

where

$$R_{10} = \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}, \quad R_{w0} = \begin{pmatrix} s^{2(w-1)}R_{(w-1)0} & -R_{(w-1)0} \\ R_{(w-1)0} & s^{2(w-1)}R_{(w-1)0} \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } Q_w = \begin{pmatrix} Q_{w-1} & 0 \\ 0 & -Q_{w-1} \end{pmatrix}, \text{ for } w \geq 2.$$

Algorithm 2

Step 1 *For a given* k $(1 \leq 2k < s + 1)$ *and* $u = 2^a + 1$ *for some integer* a*, let* $d_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,u-1})$ *consist of* $u - 1$ *independent columns from* P_i*, for* $i = 1, \ldots, 2k$*, and* $O_j = (d_{2j-1}, d_{2j})$ *for* $j = 1, \ldots, k$*.*

Step 2 *Derive an* $s \times q$ *OLHD* $W = (w_{ij})$ *with levels* $\{-(s - 1)/2, -(s - 3)/2, \ldots, (s - 1)/2\}$*.*

Step 3 *For* $l = 1, \ldots, q$*,* $i = 1, \ldots, k$*, obtain an* $s^u \times 2(u - 1)$ *matrix* $O_i^{(l)}$ *from* O_i *by replacing the levels* $0, \ldots, s - 1$ *of* O_i *with* $w_{1l}, \ldots, w_{sl}$*, respectively.*

Step 4 *For* $l = 1, \ldots, q$*,* $i = 1, \ldots, k$*, obtain* $Z_i^{(l)} = (z_{i,1}^{(l)}, \ldots, z_{i,2u-2}^{(l)}) = O_i^{(l)} R_{a1}$*.*

Step 5 *Take* $D_1 = \cup_{i=2k+1}^{s+1} P_i$ *and* $D_2 = (Z_1^{(1)}, \ldots, Z_k^{(1)}, \ldots, Z_1^{(q)}, \ldots, Z_k^{(q)})$*.*

Note that, when $u = 2$, B_0 is an empty set and $B_i\backslash B_0$ for $i = 1, \ldots, s+1$ is the ith column of the $OA(s^2, s + 1, s, 2)$ in Lemma 2(v). Clearly, Algorithm 1 can be seen as a special case of Algorithm 2 if $u = 2$ is chosen in Step 1 of Algorithm 2 and R_{a1} is replaced by R_{10} in Step 4 of Algorithm 2.

Theorem 2 summarizes the properties of D_1 and D_2 constructed above.

Theorem 2 *For* D_1 *and* D_2 *constructed above, we have*

(i) D_1 *is an* $OA(s^u, (s + 1 - 2k)s^{u-2}, s, 2)$ *and* D_2 *is an orthogonal* $L(s^u, 2kq(u - 1))$*;*

(ii) (D_1, D_2) *is an OMCD;*

(iii) $(z_{i,j}^{(l)}, z_{i,j'}^{(l)})$ *achieves stratification on* $s \times s$ *grids for* $1 \le j \ne j' \le 2u - 2$*;*

(iv) $(z_{i,j}^{(l)}, z_{i,j'}^{(l)})$ *and* $(z_{i,h}^{(l)}, z_{i',h'}^{(l)})$ *achieve stratification on* $s \times s^{u-1}$ *and* $s^{u-1} \times s$ *grids in two dimensions, for* $1 \le j \le u - 1$*,* $u \le j' \le 2u - 2$*,* $i \ne i'$*, and* $1 \le h \ne h' \le 2u - 2$*.*

Example 2 Consider the case of $s = 3, u = 3$ and $k = 1$, with $GF(3) = \{\alpha_0, \alpha_1, \alpha_2\} = \{0, 1, 2\}$. Let e_1, e_2 and e_3 be three independent 3-level columns of length 27, and $\omega_1 = e_2, \omega_2 = e_2 + e_3, \omega_3 = e_2 + 2e_3$, and $\omega_4 = e_3$. The arrays $B_0, B_1, \ldots, B_4$ are obtained as $B_0 = \{e_1\}$, $B_i = \{e_1, \omega_i, \omega_i + e_1, \omega_i + 2e_1\}$ for $i = 1, 2, 3, 4$. Then applying the Rao–Hamming construction to e_1, e_2 and e_3, we obtain an $OA(27, 13, 3, 2)$, whose column partition is displayed in Table 1. Here, the jth column of P_i is denoted by $P_{i,j}$. Let $d_1 = (P_{1,2}, P_{1,3})$, $d_2 = (P_{2,1}, P_{2,2})$ and $O_1 = (d_1, d_2)$. Here $s = 3$, the design W only has one column. $O_1^{(1)}$ is obtained from O_1 by replacing the levels 0, 1, 2 with −1, 0, 1 respectively. Take $D_2 = O_1^{(1)} R_{11}$ and $D_1 = (P_3, P_4)$, then (D_1, D_2) is an OMCD, which is presented in Table 1. The stratification properties of D_2 can be seen intuitively in Fig. 1, where $X1$, $X2$, $X3$ and $X4$ denote $z_{1,1}^{(1)}$, $z_{1,2}^{(1)}$, $z_{1,3}^{(1)}$ and $z_{1,4}^{(1)}$ respectively, which are the four columns of D_2. For example, it is easy to see that $(z_{1,1}^{(1)}, z_{1,2}^{(1)})$ achieves stratification on 3×3 grids; $(z_{1,1}^{(1)}, z_{1,3}^{(1)})$ achieves stratification on 3×9 and 9×3 grids in two dimensions.

Corollary 2 *In Algorithm 2, if* $2k < s$*, we can further let* $O_{k+1} = (d_{l_1,u}, \ldots, d_{l_{u-1},u}, d_{2k+1})$ *with* $d_{l_1,u}, \ldots, d_{l_{u-1},u}$ *being* $u-1$ *independent columns from* $P_{l_1} \backslash d_{l_1}, \ldots, P_{l_{u-1}} \backslash d_{l_{u-1}}$ *respectively, and* d_{2k+1} *consisting of* $u - 1$ *independent columns from* P_{2k+1}*, where* $l_1, \ldots, l_{u-1}$ *take values from* $\{1, \ldots, 2k\}$*. Then following the similar steps in Algorithm 2, we can get* $D_1 = \cup_{i=2k+2}^{s+1} P_i$ *and* $D_2 = (Z_1^{(1)}, \ldots, Z_{k+1}^{(1)}, \ldots, Z_1^{(q)}, \ldots, Z_{k+1}^{(q)})$*, and* (D_1, D_2) *is an OMCD.*

Besides orthogonality, second-order orthogonality is a desirable property for LHDs (see e.g., Sun et al. 2009; Wang et al. 2018a). It is easy to see that a mirror-symmetric LHD can guarantee the second-order orthogonality. In the following, OMCDs with second-order orthogonality are constructed via modifying Algorithm 2.

Algorithm 3 (Modified construction of OMCDs)

Step 1 *Permute the levels of* B_i *to obtain a mirror-symmetric design, denoted as* $\widetilde{B}_i$*,* $i = 0, \ldots, s + 1$ *for an odd prime power* s*.*

Step 2 *For a given* k $(1 \le 2k < s + 1)$*, and* $u = 2^a + 1$ *for some integer* a*, let* $\widetilde{d}_i = (\widetilde{d}_{i,1}, \widetilde{d}_{i,2}, \ldots, \widetilde{d}_{i,u-1})$ *consist of* $u-1$ *independent columns from* $\widetilde{B}_i \backslash \widetilde{B}_0$ *(denoted as* $\widetilde{P}_i$*) for* $i = 1, \ldots, 2k$*, and* $O_j = (\widetilde{d}_{2j-1}, \widetilde{d}_{2j})$*, for* $j = 1, \ldots, k$*.*

Step 3 *Derive an* $s \times t$ *mirror-symmetric OLHD* $L = (l_{ij})$ *with levels* $\{-(s - 1)/2, -(s - 3)/2, \ldots, (s - 1)/2\}$*.*

Step 4 *For* $l = 1, \ldots, t$*,* $i = 1, \ldots, k$*, obtain an* $s^u \times 2(u - 1)$ *matrix* $O_i^{(l)}$ *from* O_i *by replacing the levels* $0, \ldots, s - 1$ *of* O_i *with* $l_{1l}, \ldots, l_{sl}$*, respectively.*

Step 5 *For* $j = 1, \ldots, t$*,* $i = 1, \ldots, k$*, obtain* $Z_i^{(l)} = (z_{i,1}^{(l)}, \ldots, z_{i,2u-2}^{(l)}) = O_i^{(l)} R_{a1}$*.*

Step 6 *Take* $D_1 = \cup_{i=2k+1}^{s+1} \widetilde{P}_i$ *and* $D_2 = (Z_1^{(1)}, \ldots, Z_k^{(1)}, \ldots, Z_1^{(t)}, \ldots, Z_k^{(t)})$*.*

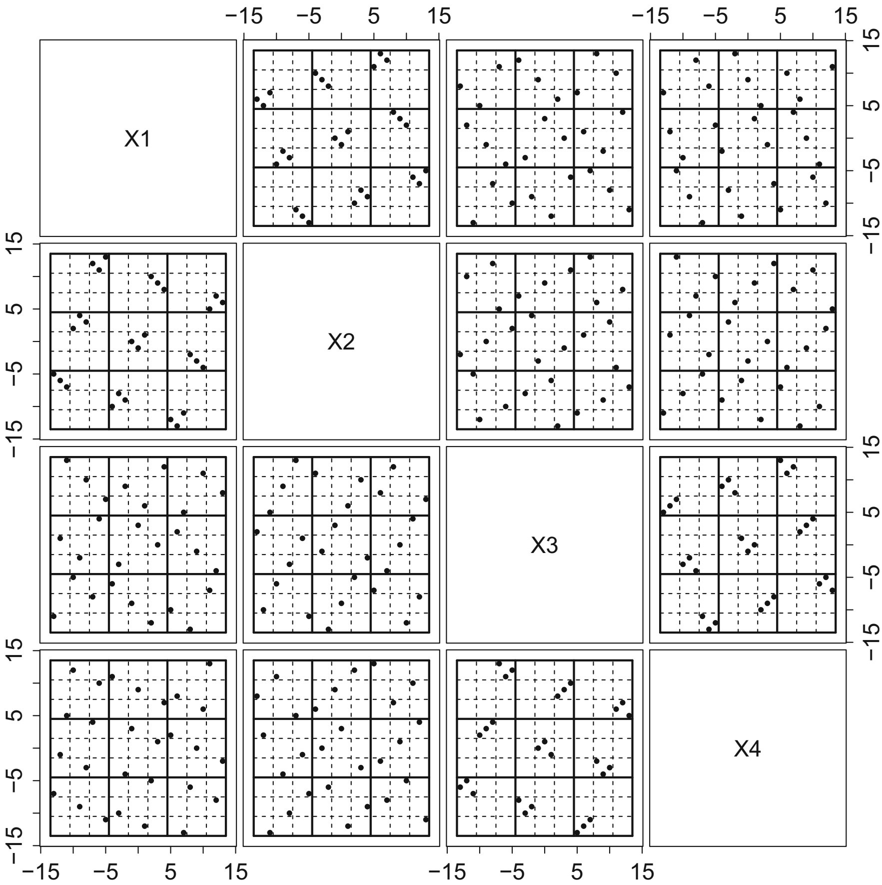

Fig. 1 Bivariate projections among the four columns of D_2 in Table 1

Here B_i is a regular design, and for an odd prime power s, a mirror-symmetric design can be obtained from B_i by permuting the levels which has been studied in Tang and Xu (2014). Theorem 3 summarizes the properties of D_1 and D_2 obtained in Algorithm 3.

Theorem 3 *For D_1 and D_2 constructed above, we have*

(i) *D_1 is an $OA(s^u, (s+1-2k)s^{u-2}, s, 2)$, D_2 is an orthogonal mirror-symmetric $L(s^u, 2kt(u-1))$, and hence D_2 is a second-order orthogonal LHD;*
(ii) *(D_1, D_2) is an OMCD.*

The projection properties of D_2 constructed in Algorithm 3 are the same as that of the D_2 in Theorem 2.

Example 3 (Example 2 continued) First we permute the levels of the $OA(27, 13, 3, 2)$ in Table 1 to derive a mirror-symmetric design which is listed in the left part of

Table 1 $OA(27, 13, 3, 2)$ and OMCD in Example 2

$OA(27, 13, 3, 2)$													OMCD									
B_0	P_1			P_2			P_3			P_4			D_2				$D_1 = (P_3, P_4)$					
0	0	0	0	0	0	0	0	0	0	0	0	0	−13	−5	−11	−7	0	0	0	0	0	0
0	0	0	0	1	1	2	2	2	1	1	1	2	−12	−6	1	−1	2	2	1	1	1	2
0	0	0	0	2	2	1	1	1	2	2	2	1	−11	−7	13	5	1	1	2	2	2	1
0	1	1	2	1	1	2	1	1	2	0	0	0	3	9	0	1	1	1	2	0	0	0
0	1	1	2	2	2	1	0	0	0	1	1	2	4	8	12	7	0	0	0	1	1	2
0	1	1	2	0	0	0	2	2	1	2	2	1	2	10	−12	−5	2	2	1	2	2	1
0	2	2	1	2	2	1	2	2	1	0	0	0	10	−4	11	6	2	2	1	0	0	0
0	2	2	1	0	0	0	1	1	2	1	1	2	8	−2	−13	−6	1	1	2	1	1	2
0	2	2	1	1	1	2	0	0	0	2	2	1	9	−3	−1	0	0	0	0	2	2	1
1	0	1	1	0	1	1	0	1	1	0	1	1	−1	0	−9	3	0	1	1	0	1	1
1	0	1	1	1	2	0	2	0	2	1	2	0	0	−1	3	9	2	0	2	1	2	0
1	0	1	1	2	0	2	1	2	0	2	0	2	1	1	6	−12	1	2	0	2	0	2
1	1	2	0	1	2	0	1	2	0	0	1	1	6	−13	2	8	1	2	0	0	1	1
1	1	2	0	2	0	2	0	1	1	1	2	0	7	−11	5	−13	0	1	1	1	2	0
1	1	2	0	0	1	1	2	0	2	2	0	2	5	−12	−10	2	2	0	2	2	0	2
1	2	0	2	2	0	2	2	0	2	0	1	1	−5	13	7	−11	2	0	2	0	1	1
1	2	0	2	0	1	1	1	2	0	1	2	0	−7	12	−8	4	1	2	0	1	2	0
1	2	0	2	1	2	0	0	1	1	2	0	2	−6	11	4	10	0	1	1	2	0	2
2	0	2	2	0	2	2	0	2	2	0	2	2	11	5	−7	13	0	2	2	0	2	2
2	0	2	2	1	0	1	2	1	0	1	0	1	12	7	−4	−8	2	1	0	1	0	1
2	0	2	2	2	1	0	1	0	1	2	1	0	13	6	8	−2	1	0	1	2	1	0
2	1	0	1	1	0	1	1	0	1	0	2	2	−9	4	−2	−9	1	0	1	0	2	2
2	1	0	1	2	1	0	0	2	2	1	0	1	−8	3	10	−3	0	2	2	1	0	1
2	1	0	1	0	2	2	2	1	0	2	1	0	−10	2	−5	12	2	1	0	2	1	0
2	2	1	0	2	1	0	2	1	0	0	2	2	−2	−9	9	−4	2	1	0	0	2	2
2	2	1	0	0	2	2	1	0	1	1	0	1	−4	−10	−6	11	1	0	1	1	0	1
2	2	1	0	1	0	1	0	2	2	2	1	0	−3	−8	−3	−10	0	2	2	2	1	0

Table 2. Let $\widetilde{d}_1 = (\widetilde{P}_{1,2}, \widetilde{P}_{1,3})$, $\widetilde{d}_2 = (\widetilde{P}_{2,1}, \widetilde{P}_{2,2})$ and $O_1 = (\widetilde{d}_1, \widetilde{d}_2)$. Here $s = 3$, it is impossible to find a mirror-symmetric OLHD with more than one column. Thus, we take $l = t = 1$, and replace the levels 0, 1, 2 of O_1 with $-1, 0, 1$ respectively to obtain $O_1^{(1)}$. After rotating $O_1^{(1)}$, the derived OMCD is shown in the right part of Table 2.

From Theorems 2 and 3, the run sizes of the obtained designs are closely related to the rotation matrices. To construct designs with s^u runs for any integer $u \geq 3$, one way is to find other (orthogonal or nearly orthogonal) rotation matrices through computer search. For example, when $u = 4$, we can use this nearly orthogonal rotation matrix:

Table 2 $OA(27, 13, 3, 2)$ and OMCD in Example 3

$OA(27, 13, 3, 2)$													OMCD									
$\widetilde{B}_0$	$\widetilde{P}_1$			$\widetilde{P}_2$			$\widetilde{P}_3$			$\widetilde{P}_4$			D_2				$D_1 = (\widetilde{P}_3, \widetilde{P}_4)$					
0	0	2	1	2	1	2	1	0	0	0	2	1	10	−3	8	−3	1	0	0	0	2	1
0	0	2	1	0	2	1	0	2	1	1	0	0	8	−4	7	12	0	2	1	1	0	0
0	0	2	1	1	0	0	2	1	2	2	1	2	9	−2	−4	−9	2	1	2	2	1	2
0	1	0	0	0	2	1	2	1	2	0	2	1	−13	−7	−5	11	2	1	2	0	2	1
0	1	0	0	1	0	0	1	0	0	1	0	0	−12	−5	−2	−10	1	0	0	1	0	0
0	1	0	0	2	1	2	0	2	1	2	1	2	−11	−6	10	−4	0	2	1	2	1	2
0	2	1	2	1	0	0	0	2	1	0	2	1	3	10	−3	−8	0	2	1	0	2	1
0	2	1	2	2	1	2	2	1	2	1	0	0	4	9	9	−2	2	1	2	1	0	0
0	2	1	2	0	2	1	1	0	0	2	1	2	2	8	−6	13	1	0	0	2	1	2
1	0	0	2	2	2	0	1	1	1	0	0	2	−5	11	13	7	1	1	1	0	0	2
1	0	0	2	0	0	2	0	0	2	1	1	1	−7	13	−11	−5	0	0	2	1	1	1
1	0	0	2	1	1	1	2	2	0	2	2	0	−6	12	1	1	2	2	0	2	2	0
1	1	1	1	0	0	2	2	2	0	0	0	2	−1	1	−12	−6	2	2	0	0	0	2
1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1
1	1	1	1	2	2	0	0	0	2	2	2	0	1	−1	12	6	0	0	2	2	2	0
1	2	2	0	1	1	1	0	0	2	0	0	2	6	−12	−1	−1	0	0	2	0	0	2
1	2	2	0	2	2	0	2	2	0	1	1	1	7	−13	11	5	2	2	0	1	1	1
1	2	2	0	0	0	2	1	1	1	2	2	0	5	−11	−13	−7	1	1	1	2	2	0
2	0	1	0	2	0	1	1	2	2	0	1	0	−2	−8	6	−13	1	2	2	0	1	0
2	0	1	0	0	1	0	0	1	0	1	2	2	−4	−9	−9	2	0	1	0	1	2	2
2	0	1	0	1	2	2	2	0	1	2	0	1	−3	−10	3	8	2	0	1	2	0	1
2	1	2	2	0	1	0	2	0	1	0	1	0	11	6	−10	4	2	0	1	0	1	0
2	1	2	2	1	2	2	1	2	2	1	2	2	12	5	2	10	1	2	2	1	2	2
2	1	2	2	2	0	1	0	1	0	2	0	1	13	7	5	−11	0	1	0	2	0	1
2	2	0	1	1	2	2	0	1	0	0	1	0	−9	2	4	9	0	1	0	0	1	0
2	2	0	1	2	0	1	2	0	1	1	2	2	−8	4	7	−12	2	0	1	1	2	2
2	2	0	1	0	1	0	1	2	2	2	0	1	−10	3	−8	3	1	2	2	2	0	1

$$R_6 = \begin{pmatrix} s & s & s^3 & -1 & 0 & 0 \\ s^2 & -s^3 & s & 0 & 1 & 0 \\ s^3 & s^2 & -s^2 & 0 & 0 & -1 \\ 1 & 0 & 0 & s & s & s^3 \\ 0 & -1 & 0 & s^2 & -s^3 & s \\ 0 & 0 & 1 & s^3 & s^2 & -s^2 \end{pmatrix}.$$

The resulting designs may be nearly orthogonal, but the numbers of columns will usually be larger than the orthogonal case.

In the following, we present another approach to constructing OMCDs with s^k runs with $k \geq 3$. We recall that the $\oplus$ operator is based on $GF(s) = \{\alpha_0 = 0, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}\}$.

Algorithm 4

Step 1 *Let N be an $s^2 \times 2$ full factorial design with s levels. Two generator matrices are given as*

$$G_1 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{s-1} \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\alpha_i \in GF(s) = \{\alpha_0, \ldots, \alpha_{s-1}\}$ with $\alpha_0 = 0$, $i = 0, \ldots, s-1$. Obtain $D^{(0)} = NG_1$ and $e^{(0)} = NG_2$.

Step 2 *Let $\gamma = (\alpha_0, \alpha_1, \ldots, \alpha_{s-1})^T$, and define $\beta_j = \alpha_j \gamma$ for $j = 0, 1, \ldots, s-1$.*

Step 3 *For $v = 1, 2, \ldots$, create an $s^{2+v} \times s^{1+v}$ matrix as*

$$D^{(v)} = (\beta_0 \oplus D^{(v-1)}, \beta_1 \oplus D^{(v-1)}, \ldots, \beta_{s-1} \oplus D^{(v-1)})$$

and create an $s^{2+v} \times (s-1)^v$ matrix as

$$e^{(v)} = (\beta_1 \oplus e^{(v-1)}, \beta_2 \oplus e^{(v-1)}, \ldots, \beta_{s-1} \oplus e^{(v-1)}).$$

Step 4 *If s is even, divide $D^{(0)}$ into $E_1, \ldots, E_{s/2}$, where each E_i is a full factorial design of two factors. If s is odd, divide $D^{(0)}$ into $E_1, \ldots, E_{\lfloor s/2 \rfloor}$ and one column l.*

Step 5 *For each $i = 1, \ldots, \lfloor s/2 \rfloor$, create $F_{j_1}^{(i)} = (\beta_{2j_1} \oplus E_i, \beta_{2j_1+1} \oplus E_i)$, for each $j_1 = 0, 1, \ldots, \lfloor s/2 \rfloor - 1$; create $F_{j_1 j_2 \cdots j_{v-1}}^{(i)} = (\beta_{2j_{v-1}} \oplus F_{j_1 j_2 \cdots j_{v-2}}^{(i)}, \beta_{2j_{v-1}+1} \oplus F_{j_1 j_2 \cdots j_{v-2}}^{(i)})$ for $j_1, \ldots, j_{v-1} = 0, 1, \ldots, \lfloor s/2 \rfloor - 1$; given p, where $0 \le p \le \lfloor s/2 \rfloor - 1$, create $H_{j_1}^{(i)} = F_{j_1}^{(i)}$ for $j_1 = 0, 1, \ldots, p$ and $H_{j_1 j_2 \cdots j_v}^{(i)} = (\beta_{2j_v} \oplus F_{j_1 j_2 \cdots j_{v-1}}^{(i)}, \beta_{2j_v+1} \oplus F_{j_1 j_2 \cdots j_{v-1}}^{(i)})$ for $v \ge 2$, $j_1, \ldots, j_{v-1} = 0, 1, \ldots, \lfloor s/2 \rfloor - 1$ and $j_v = 0, 1, \ldots, p$. Obtain $H_{j_1 j_2 \cdots j_v}^{(i)*}$ from $H_{j_1 j_2 \cdots j_v}^{(i)}$ via replacing α_j by $j - \frac{s-1}{2}$.*

Step 6 *For $0 \le p \le \lfloor s/2 \rfloor - 1$ and $v = 1, 2, \ldots$, if $2p + 2 \le s - 1$, let $\widetilde{D}_1 = (e^{(v)}, \beta_{2p+2} \oplus D^{(v-1)}, \ldots, \beta_{s-1} \oplus D^{(v-1)})$, and if $2p+2 = s$, let $\widetilde{D}_1 = e^{(v)}$. Obtain D_1 from $\widetilde{D}_1$ by replacing level α_j by level j.*

Step 7 *Let $\widetilde{H}_{j_1 j_2 \cdots j_v}^{(i)} = H_{j_1 j_2 \cdots j_v}^{(i)*} R_{1v}$ for $i = 1, 2, \ldots, \lfloor s/2 \rfloor$, $j_1, \ldots, j_{v-1} = 0, 1, \ldots, \lfloor s/2 \rfloor - 1$ and $j_v = 0, 1, \ldots, p$. Construct D_2 as $D_2 = \left(\widetilde{H}_{j_1 j_2 \cdots j_v}^{(1)}, \ldots, \widetilde{H}_{j_1 j_2 \cdots j_v}^{(\lfloor s/2 \rfloor)} \right)_{j_1 j_2 \cdots j_v}$.*

Theorem 4 summarizes the properties of D_1 and D_2 constructed in Algorithm 4.

Theorem 4 *For D_1 and D_2 obtained in Algorithm 4, we have*

(i) *D_1 is an $OA(s^{2+v}, (s-1)^v + (s-(2p+2))s^v, s, 2)$ and D_2 is an orthogonal $L(s^{2+v}, \lfloor s/2 \rfloor^v (p+1) 2^{1+v})$;*
(ii) *(D_1, D_2) is an OMCD;*
(iii) *D_2 achieves stratification on $s \times s$ grids in any two dimensions.*

The following example provides an illustration for Theorem 4.

Example 4 Consider the case of $s = 3$, $v = 2$ and $p = 0$, with $GF(3) = \{\alpha_0, \alpha_1, \alpha_2\} = \{0, 1, 2\}$. The full factorial design N is

$$N = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}^T.$$

And $D^{(0)}$ and $e^{(0)}$ are obtained as

$$D^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \end{pmatrix}^T \text{ and } e^{(0)} = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}^T.$$

In this case $\beta_0 = (0, 0, 0)^T$, $\beta_1 = (0, 1, 2)^T$, $\beta_2 = (0, 2, 1)^T$, then $D^{(1)}$ and $e^{(1)}$ are obtained as $D^{(1)} = (\beta_0 \oplus D^{(0)}, \beta_1 \oplus D^{(0)}, \beta_2 \oplus D^{(0)})$ and $e^{(1)} = (\beta_1 \oplus e^{(0)}, \beta_2 \oplus e^{(0)})$. Thus for $v = 2$ and $p = 0$, D_1 is obtained as $D_1 = \widetilde{D}_1 = (\beta_1 \oplus e^{(1)}, \beta_2 \oplus e^{(1)}, \beta_2 \oplus D^{(1)})$. Here $s = 3$, then $\lfloor s/2 \rfloor = 1$, so $D^{(0)}$ is divided into E_1 and one column, where

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \end{pmatrix}^T.$$

For $p = 0$, only $F_0^{(1)}$ is obtained as $F_0^{(1)} = (\beta_0 \oplus E_1, \beta_1 \oplus E_1)$ and $F_{00}^{(1)}$ is obtained as $F_{00}^{(1)} = (\beta_0 \oplus F_0^{(1)}, \beta_1 \oplus F_0^{(1)})$. Then for $v = 2$ and $p = 0$, D_2 can be constructed as $D_2 = F_{00}^{(1)*} R_{12} = (F_{00}^{(1)} - J_{81\times 8}) R_{12}$, where $J_{81\times 8}$ is a 81×8 matrix of ones. It is easy to check that $D = (D_1, D_2)$ is an OMCD, which is provided in Tables 5 and 6 in Appendix B. After collapsing the levels of D_2 by $\lfloor (x + 40)/27 \rfloor$, D_2 becomes $F_{00}^{(1)}$ which is an $OA(81, 8, 3, 2)$, then D_2 achieves stratification on 3×3 grids in two dimensions.

Note that if $s = 2$, then D_1 just has one column, which is not desirable, and $D_2 = \widetilde{H}_{00\ldots0}^{(1)}$ is an orthogonal $L(s^{2+v}, 2^{1+v})$. We now present a method to extend the number of columns of D_1 with $s = 2$ up to $2^v + 1$.

Corollary 3 *For $H_{00\ldots0}^{(1)}$ and $\widetilde{H}_{00\ldots0}^{(1)}$ obtained in Algorithm 4 with $s = 2$, let Ψ consist of the first 2^v columns of $H_{00\ldots0}^{(1)}$ and Γ consist of the last 2^v columns of $\widetilde{H}_{00\ldots0}^{(1)}$. Let $\Phi = (e^{(v)}, \Psi)$, then*

(i) *Φ is an $OA(2^{2+v}, 1 + 2^v, 2, 2)$ and Γ is an orthogonal $L(2^{2+v}, 2^v)$;*
(ii) *(Φ, Γ) is an OMCD.*

Note that for $s > 2$, we can extend the number of columns of D_1 up to $(s-1)^v + (s-2)s^v + \lfloor s/2 \rfloor^v 2^v$ similarly as Corollary 3 does. The following example provides an illustration for Corollary 3.

Example 5 Consider the case of $s = 2$ and $v = 3$, with $GF(2) = \{\alpha_0, \alpha_1\} = \{0, 1\}$. The full factorial design N is

$$N = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}^T.$$

And $D^{(0)}$ and $e^{(0)}$ are obtained as

$$D^{(0)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^T \text{ and } e^{(0)} = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T.$$

In this case $\beta_0 = (0, 0)^T$ and $\beta_1 = (0, 1)^T$, then for $v = 3$, $e^{(3)}$ is obtained as

$$e^{(3)} = \beta_1 \oplus (\beta_1 \oplus (\beta_1 \oplus e^{(0)})).$$

Here $s = 2$, so $E_1 = D^{(0)}$. Thus $F_0^{(1)} = (\beta_0 \oplus D^{(0)}, \beta_1 \oplus D^{(0)})$, $F_{00}^{(1)} = (\beta_0 \oplus F_0^{(1)}, \beta_1 \oplus F_0^{(1)})$, $H_{000}^{(1)} = F_{000}^{(1)} = (\beta_0 \oplus F_{00}^{(1)}, \beta_1 \oplus F_{00}^{(1)})$ and $\widetilde{H}_{000}^{(1)} = (H_{000}^{(1)} - (1/2)J_{32\times16})R_{13}$, where $J_{32\times16}$ is a 32×16 matrix of ones. $\Phi = (e^{(3)}, \Psi)$, where Ψ consists of the first eight columns of $H_{000}^{(1)}$, and Γ consists of the last eight columns of $\widetilde{H}_{000}^{(1)}$. It is easy to check that (Φ, Γ) is an OMCD which is given in Table 7 in Appendix B.

If an orthogonal $L(s, m')$ can be constructed, then an OLHD with more columns can be constructed following the idea of Lin et al. (2009). By the similar method, OMCDs with more columns can be constructed as follows.

Corollary 4 *If an orthogonal $L(s, m')$ is available, and an OMCD (D_1, D_2) can be obtained by Algorithm 4, where D_1 is an $OA(s^{2+v}, m_1, s, 2)$ and D_2 is an orthogonal $L(s^{2+v}, m_2)$, then D_2 can be extended to an orthogonal $L(s^{2+v}, m'm_2)$ for $v = 1, 2, \ldots$.*

Note that both Algorithms 2 and 4 can construct OMCDs with s^u runs, but the values of u in the two algorithms are different. Algorithm 2 is suitable for $u = 2^a + 1$ with a being an integer, while Algorithm 4 is appropriate for any integer $u \geq 3$. For $u = 2^a + 1$, Algorithm 2 can construct OMCDs with D_1 having more columns than Algorithm 4.

3.3 Construction of MCDs using mixed-level OAs

The above four algorithms can generate designs with s^u runs for $u \geq 2$. This section introduces another construction for MCDs with n runs, m qualitative factors and k quantitative factors through an $OA(n, s^m(n/s), 2)$ and two LHDs. The construction can get OMCDs with their run sizes n being multiples of s^2. Denote the set $\{i - (r - 1)/2 : i = 0, 1, \ldots, r-1\}$ as $\Omega(r)$. Let T be an $OA(n, s^m(n/s), 2)$, L_1 be an $L(s, k_1)$ with entries from $\Omega(s)$, and L_2 be an $L(n/s, k_2)$ with entries from $\Omega(n/s)$. A new class of MCDs can be constructed as follows.

Algorithm 5

Step 1 If $k_1 = k_2$, let $Y_j = L_1[j, :] \oplus_c (sL_2)$; and if $k_1 < k_2$, let $Y_j = (L_1[j, 1] \oplus (sL_2[:, 1 : k_2 - k_1 + 1]), L_1[j, 2 : k_1] \oplus_c (sL_2[:, (k_2 - k_1 + 2) : k_2]))$, for $j = 1, \ldots, s$.

Step 2 Stack the rows of Y_j for $j = 1, \ldots, s$ together to obtain $Y = (Y_1^T, \ldots, Y_s^T)^T$.

Step 3 Permutate the rows of T to make sure that the last column is $1_s \otimes (1, 2, \ldots, n/s)^T$.

Step 4 Let D_1 be the first m columns of T.

Step 5 Stack the columns of D_1 and Y together, i.e., $D = (D_1, Y)$.

Then we have the following theorem.

Theorem 5 *The design D constructed in Algorithm 5 is an MCD with n runs, m qualitative factors and k_2 quantitative factors.*

In Algorithm 5, the run size n must be a multiple of s^2 and the number of qualitative factors m can be up to n/s according to He et al. (2017). By carefully choosing L_1 and L_2, the resulting design Y in Algorithm 5 can be an OLHD (when $k_1 = k_2$) or nearly OLHD (when $k_1 < k_2$). The theoretical results in the following theorem will justify this.

Theorem 6 *Let L_1, L_2 and Y be the designs in Algorithm 5, we have*

(i) *for $k_1 = k_2$, if L_1 and L_2 are OLHDs, then Y is an OLHD, furthermore, Y is second-order orthogonal if L_1 and L_2 are second-order orthogonal.*

(ii) *for $k_1 < k_2$, let $\lambda = (s^2 - 1)/(n^2 - 1)$ and $\mu = k_2 - k_1$,*

$$\rho_{j_1 j_2}(Y) = \begin{cases} \lambda + (1 - \lambda)\rho_{j_1 j_2}(L_2), \ 1 \le j_1, j_2 \le \mu + 1 \ ; \\ \lambda\rho_{1(j_2-\mu)}(L_1) + (1 - \lambda)\rho_{j_1 j_2}(L_2), \ 1 \le j_1 \le \mu + 1, \mu + 2 \le j_2 \le k_2 \ ; \\ \lambda\rho_{(j_1-\mu)(j_2-\mu)}(L_1) + (1 - \lambda)\rho_{j_1 j_2}(L_2), \ \mu + 2 \le j_1, j_2 \le k_2 \ . \end{cases} \tag{1}$$

Here for any design D, $\rho_{ij}(D) = \rho(d_i, d_j)$ with d_i and d_j being the ith and jth columns of D respectively.

Table 3 presents some orthogonal and nearly orthogonal MCDs constructed via Algorithm 5. The first column lists the mixed-level OAs obtained by He et al. (2017). The second and third columns provide the designs used for the construction of the corresponding Y.

4 Concluding remarks

MCDs proposed by Deng et al. (2015) are cost-effective designs for computer experiments with both quantitative and qualitative factors. Orthogonality is a desirable

Table 3 Some orthogonal and nearly orthogonal MCDs (D_1, Y) with $n \leq 100$ runs constructed by Algorithm 5

T	L_1 [source]	L_2 [source]	D_1	Y
$OA(8, 2^4 4^1, 2)$	$L(2, 2)^*$	$OL(4, 2)$[LB]	$OA(8, 4, 2, 2)$	$NOL(8, 2)$
$OA(16, 2^8 8^1, 2)$	$L(2, 4)^*$	$OL(8, 4)$[LB]	$OA(16, 8, 2, 2)$	$NOL(16, 4)$
$OA(24, 2^{12} 12^1, 2)$	$L(2, 6)^*$	$OL(12, 6)$[LB]	$OA(24, 12, 2, 2)$	$NOL(24, 6)$
$OA(27, 3^9 9^1, 2)$	$L(3, 5)^*$	$OL(9, 5)$[LB]	$OA(27, 9, 3, 2)$	$NOL(27, 5)$
$OA(32, 2^{16} 16^1, 2)$	$L(2, 12)^*$	$OL(16, 12)$[LB]	$OA(32, 16, 2, 2)$	$NOL(32, 12)$
$OA(32, 4^8 8^1, 2)$	$L(4, 4)^*$	$OL(8, 4)$[LB]	$OA(32, 8, 4, 2)$	$NOL(32, 4)$
$OA(32, 4^8 8^1, 2)$	$OL(4, 2)$[LB]	$OL(8, 4)$[LB]	$OA(32, 8, 4, 2)$	$NOL(32, 4)$
$OA(32, 4^8 8^1, 2)$	$OL(4, 2)$[LB]	$OL(8, 2)$[LB]	$OA(32, 8, 4, 2)$	$OL(32, 2)$
$OA(40, 2^{20} 20^1, 2)$	$L(2, 6)^*$	$OL(20, 6)$[LB]	$OA(40, 20, 2, 2)$	$NOL(40, 6)$
$OA(48, 2^{24} 24^1, 2)$	$L(2, 12)^*$	$OL(24, 12)$[GS]	$OA(48, 24, 2, 2)$	$NOL(48, 12)$
$OA(54, 3^{18} 18^1, 2)$	$L(3, 8)^*$	$OL(18, 8)$[GS]	$OA(54, 18, 3, 2)$	$NOL(54, 8)$
$OA(56, 2^{28} 28^1, 2)$	$L(2, 6)^*$	$OL(28, 6)$[LB]	$OA(56, 28, 2, 2)$	$NOL(56, 6)$
$OA(64, 2^{32} 32^1, 2)$	$L(2, 16)^*$	$OL(32, 16)$[GS]	$OA(64, 32, 2, 2)$	$NOL(64, 16)$
$OA(64, 4^{16} 16^1, 2)$	$L(4, 12)^*$	$OL(16, 12)$[LB]	$OA(64, 16, 4, 2)$	$NOL(64, 12)$
$OA(64, 4^{16} 16^1, 2)$	$OL(4, 2)$[LB]	$OL(16, 12)$[LB]	$OA(64, 16, 4, 2)$	$NOL(64, 12)$
$OA(64, 4^{16} 16^1, 2)$	$OL(4, 2)$[LB]	$OL(16, 2)$[LB]	$OA(64, 16, 4, 2)$	$OL(64, 2)$
$OA(72, 2^{36} 36^1, 2)$	$L(2, 6)^*$	$OL(36, 6)$[ST]	$OA(72, 36, 2, 2)$	$NOL(72, 6)$
$OA(80, 2^{40} 40^1, 2)$	$L(2, 20)^*$	$OL(40, 20)$[GS]	$OA(80, 40, 2, 2)$	$NOL(80, 20)$
$OA(81, 3^{27} 27^1, 2)$	$L(3, 12)^*$	$OL(27, 12)$[ST]	$OA(81, 27, 3, 2)$	$NOL(81, 12)$
$OA(88, 2^{44} 44^1, 2)$	$L(2, 6)^*$	$OL(44, 6)$[LB]	$OA(88, 44, 2, 2)$	$NOL(88, 6)$
$OA(96, 2^{48} 48^1, 2)$	$L(2, 24)^*$	$OL(48, 24)$[GS]	$OA(96, 48, 2, 2)$	$NOL(96, 24)$
$OA(100, 5^{20} 20^1, 2)$	$L(5, 6)^*$	$OL(20, 6)$[LB]	$OA(100, 20, 5, 2)$	$NOL(100, 6)$
$OA(100, 5^{20} 20^1, 2)$	$OL(5, 2)$[LB]	$OL(20, 6)$[LB]	$OA(100, 20, 5, 2)$	$NOL(100, 6)$
$OA(100, 5^{20} 20^1, 2)$	$OL(5, 2)$[LB]	$OL(20, 2)$[LB]	$OA(100, 20, 5, 2)$	$OL(100, 2)$

Note: $L(s, k)^*$: $(0, 1, \ldots, s-1)^T 1_k^T$ with 1_k being an $k \times 1$ vector of ones; $OL(n, m)$: orthogonal $L(n, m)$; $NOL(n, m)$: nearly orthogonal $L(n, m)$; GS: Georgiou and Stylianou (2011); LB: Lin et al. (2010); ST: Sun and Tang (2017)

property for designs of computer experiments. To our knowledge, there is no literature on the construction of MCDs with orthogonality.

We provide five methods to construct (nearly) OMCDs, where the designs for the qualitative factors are s-level OAs. The construction methods are easy to be carried out. Besides orthogonality, the MCDs constructed by Algorithms 1 to 4 guarantee some low-dimensional space-filling properties. In addition, the MCDs obtained by Algorithm 3 are second-order orthogonal. The first four approaches can construct MCDs with s^v runs where s is a prime power and $v \geq 2$. The fifth one can construct MCDs with some multiples of s^2 runs. Finally, Table 4 lists some OMCDs that can be constructed by our methods. For the construction of OMCDs with their run sizes being multiples of s^u, this deserves further study.

Table 4 Some resulting OMCDs (D_1, D_2) with $D_1 = OA(n, \mu_1, s, 2)$ and $D_2 = L(n, \mu_2)$

Source	n	μ_1	μ_2	Constraints	Properties
Algorithm 1	s^2	d	$2pf$	$d + 2f \leq s + 1$ and an orthogonal $L(s, p)$, $p \geq 1$, exists	1 and 3
Algorithm 2	s^u	$(s + 1 - 2k)s^{u-2}$	$2kq(u - 1)$	$u = 2^a + 1, a \geq 1, k\ (1 \leq 2k < s + 1)$ and an orthogonal $L(s, q)$, $q \geq 1$, exists	1 and 3
Algorithm 3	s^u	$(s + 1 - 2k)s^{u-2}$	$2kt(u - 1)$	$u = 2^a + 1, a \geq 1, k\ (1 \leq 2k < s + 1)$ and an orthogonal mirror-symmetric $L(s, t)$, $t \geq 1$, exists	1, 2 and 3
Algorithm 4	s^{2+v}	$(s - 1)^v + (s - (2p + 2))s^v$	$\lfloor s/2 \rfloor^v (p + 1)2^{1+v}$	$v \geq 1$ and $0 \leq p \leq \lfloor s/2 \rfloor - 1$	1 and 3
Algorithm 5	n	m	k	$OA(n, s^m(n/s), 2)$ and two (2nd-order) orthogonal LHDs $L(s, k)$ and $L(n/s, k)$ exist	1 and 2

Note: Property 1: orthogonality; Property 2: second-order orthogonality; Property 3: projection property

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Appendix A: Proofs

To prove the theoretical results of this paper, we first give the following two lemmas.

Lemma A.1 *Let ζ be any column of D_1 which is an $OA(s^t, m, s, 2)$ and ξ be any column of D_2 which is an $L(s^t, k)$. And ξ can be represented as $\xi = \pm\eta_1 \pm \eta_2 s \pm \cdots \pm \eta_t s^{t-1}$, where $(\eta_1, \eta_2, \ldots, \eta_t)$ with entries from $\{-(s-1)/2, -(s-3)/2, \ldots, (s-3)/2, (s-1)/2\}$ is an s^t full factorial design. If $(\zeta, \eta_2, \ldots, \eta_t)$ is an s^t full factorial design, then (D_1, D_2) is an MCD.*

Following the symbols in Algorithm 4, let $i_k, j_k = 0, 1, \ldots, \lfloor s/2 \rfloor - 1$, $k = 1, 2, \ldots$. Let $W^{(1)}_{i_1 j_1} = (a^{(1)}_1, a^{(1)}_2, b^{(1)})$, where $a^{(1)}_1$ and $a^{(1)}_2$ are any two columns in $\beta_{i_1} \oplus D^{(0)}$, and $b^{(1)}$ is any column in $\beta_{j_1} \oplus D^{(0)}$, $i_1 \neq j_1$. Create $W^{(2)}_{i_1 j_1 i_2 j_2} = (\beta_{i_2} \oplus W^{(1)}_{i_1 j_1}, b^{(2)})$, where $b^{(2)} \in \beta_{j_2} \oplus D^{(1)}$, $i_2 \neq j_2$. In general, define $W^{(v)}_{i_1 j_1 i_2 j_2 \cdots i_v j_v} = (\beta_{i_v} \oplus W^{(v-1)}_{i_1 j_1 i_2 j_2 \cdots i_{v-1} j_{v-1}}, b^{(v)})$, where $b^{(v)} \in \beta_{j_v} \oplus D^{(v-1)}$, $i_v \neq j_v$.

Lemma A.2 *Suppose $D^{(0)}$, $W^{(v)}_{i_1 j_1 i_2 j_2 \cdots i_v j_v}$ and $e^{(v)}$ for $v = 1, 2, \ldots$ are as defined above, then*

(i) *$W^{(v)}_{i_1 j_1 i_2 j_2 \cdots i_v j_v}$ is a full factorial design with $2 + v$ factors;*

(ii) *for any column $h \in e^{(1)}$, $(a^{(1)}_1, a^{(1)}_2, h)$ is a full factorial design with three factors; furthermore for any column $h \in e^{(v)}$ with $v \geq 2$, $(\beta_{i_v} \oplus W^{(v-1)}_{i_1 j_1 i_2 j_2 \cdots i_{v-1} j_{v-1}}, h)$ is a full factorial design with $2 + v$ factors.*

Proof of Lemma A.1 To make sure (D_1, D_2) is an MCD, it needs to prove that for each level of ζ, the corresponding rows in D_2 form an $L(s^{t-1}, k)$ after level-collapsing. Here, we collapse any level x of ξ by $f(x) = \lfloor \frac{x + (s^t - 1)/2}{s} \rfloor$. After level-collapsing, the levels of ξ are collapsed to the s^{t-1} levels $\{0, 1, s^{t-1} - 1\}$. Let $\lambda_i = \pm\eta_i + ((s-1)/2)1_{s^t}$, $i = 1, \ldots, t$, then the entries of λ_i are all in $\{0, 1, \ldots, s-1\}$, and $\xi + ((s^t - 1)/2)1_{s^t} = \lambda_1 + \lambda_2 s + \cdots + \lambda_t s^{t-1}$. Thus $f(\xi) = \lambda_2 + \lambda_3 s + \cdots + \lambda_t s^{t-2}$. It is easy to see that for each level of ζ, the corresponding rows in $(\lambda_2, \ldots, \lambda_t)$ form an s^{t-1} full factorial design, since $(\zeta, \eta_2, \ldots, \eta_t)$ is an s^t full factorial design. Thus for each level of ζ, the corresponding rows in D_2 form an $L(s^{t-1}, k)$ after level-collapsing. This completes the proof. □

Proof of Lemma A.2 (i) From the construction of $D^{(0)}$, it is easy to see that $D^{(0)}$ is an $OA(s^2, s, s, 2)$, and $(\alpha, \ldots, \alpha)$ is a row in $D^{(0)}$ for any $\alpha \in GF(s)$. Furthermore, the rows of $D^{(0)}$ form a linear space over $GF(s)$. Then, for any $\alpha_i, \alpha_k \in GF(s)$, $\alpha_i \alpha_k J_{s^2 \times s} + D^{(0)}$ can be transformed into $D^{(0)}$ by row permutation. Thus if $\alpha_{j_1} = \alpha_{i_1} + \alpha_{t_1}$, $\alpha_{t_1} \neq 0$, then after row permutation, $\left(\beta_{i_1} \oplus D^{(0)}, \beta_{j_1} \oplus D^{(0)}\right)$ can be transformed into

$$\begin{pmatrix} (D^{(0)})^T & (D^{(0)})^T & \ldots & (D^{(0)})^T \\ (\alpha_{t_1}\alpha_0 J_{s^2\times s} + D^{(0)})^T & (\alpha_{t_1}\alpha_1 J_{s^2\times s} + D^{(0)})^T & \ldots & (\alpha_{t_1}\alpha_{s-1} J_{s^2\times s} + D^{(0)})^T \end{pmatrix}^T.$$

So it is straightforward to obtain that $W^{(1)}_{i_1 j_1}$ is a full factorial design with three factors. Furthermore, for $v = 2, 3, \ldots$, if $\alpha_{j_v} = \alpha_{i_v} + \alpha_{t_v}, \alpha_{t_v} \neq 0$, then after row permutation, $\big(\beta_{i_v} \oplus D^{(v-1)}, \beta_{j_v} \oplus D^{(v-1)}\big)$ can be transformed into

$$\begin{pmatrix} (D^{(v-1)})^T & (D^{(v-1)})^T & \ldots & (D^{(v-1)})^T \\ (\alpha_{t_v}\alpha_0 J_{s^{v+1}\times s^v} + D^{(v-1)})^T & (\alpha_{t_v}\alpha_1 J_{s^{v+1}\times s^v} + D^{(v-1)})^T & \ldots & (\alpha_{t_v}\alpha_{s-1} J_{s^{v+1}\times s^v} + D^{(v-1)})^T \end{pmatrix}^T.$$

Thus $W^{(v)}_{i_1 j_1 i_2 j_2 \cdots i_v j_v}$ is an s-level full factorial design with $2+v$ factors for $v = 2, 3, \ldots$.

(ii) In Algorithm 4, N is a full factorial design which can be written as

$$N = \begin{pmatrix} \gamma^T & \gamma^T & \cdots & \gamma^T \\ \alpha_0 1_s^T & \alpha_1 1_s^T & \cdots & \alpha_{s-1} 1_s^T \end{pmatrix}^T$$

and $G_1 = (1_s, \gamma)^T$, where $\gamma = (\alpha_0, \cdots, \alpha_{s-1})^T$. Then $(D^{(0)}, e^{(0)})$ can be written as

$$(D^{(0)}, e^{(0)}) = \begin{pmatrix} \gamma 1_s^T + \alpha_0 1_s \gamma^T & \alpha_0 1_s \\ \gamma 1_s^T + \alpha_1 1_s \gamma^T & \alpha_1 1_s \\ \vdots & \vdots \\ \gamma 1_s^T + \alpha_{s-1} 1_s \gamma^T & \alpha_{s-1} 1_s \end{pmatrix}.$$

From the definitions, it is obvious to obtain that

$$\begin{aligned} &(\beta_{i_1} \oplus D^{(0)}, \beta_{j_1} \oplus e^{(0)}) \\ &= \begin{pmatrix} \alpha_{i_1}\alpha_0 J_{s^2\times s} + D^{(0)} & \alpha_{j_1}\alpha_0 1_{s^2} + e^{(0)} \\ \vdots & \vdots \\ \alpha_{i_1}\alpha_{s-1} J_{s^2\times s} + D^{(0)} & \alpha_{j_1}\alpha_{s-1} 1_{s^2} + e^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_{i_1}\alpha_0 J_{s\times s} + \gamma 1_s^T + \alpha_0 1_s \gamma^T & \alpha_{j_1}\alpha_0 1_s + \alpha_0 1_s \\ \alpha_{i_1}\alpha_0 J_{s\times s} + \gamma 1_s^T + \alpha_1 1_s \gamma^T & \alpha_{j_1}\alpha_0 1_s + \alpha_1 1_s \\ \vdots & \vdots \\ \alpha_{i_1}\alpha_0 J_{s\times s} + \gamma 1_s^T + \alpha_{s-1} 1_s \gamma^T & \alpha_{j_1}\alpha_0 1_s + \alpha_{s-1} 1_s \\ \vdots & \vdots \\ \vdots & \vdots \\ \alpha_{i_1}\alpha_{s-1} J_{s\times s} + \gamma 1_s^T + \alpha_0 1_s \gamma^T & \alpha_{j_1}\alpha_{s-1} 1_s + \alpha_0 1_s \\ \alpha_{i_1}\alpha_{s-1} J_{s\times s} + \gamma 1_s^T + \alpha_1 1_s \gamma^T & \alpha_{j_1}\alpha_{s-1} 1_s + \alpha_1 1_s \\ \vdots & \vdots \\ \alpha_{i_1}\alpha_{s-1} J_{s\times s} + \gamma 1_s^T + \alpha_{s-1} 1_s \gamma^T & \alpha_{j_1}\alpha_{s-1} 1_s + \alpha_{s-1} 1_s \end{pmatrix}. \end{aligned}$$

Note that $(\alpha_{i_1}\alpha_{k_1}J_{s\times s}+\gamma 1_s^T+\alpha_{k_2}1_s\gamma^T, \alpha_{j_1}\alpha_{k_1}1_s+\alpha_{k_2}1_s)$ can be transformed into $(\gamma 1_s^T+\alpha_{k_2}1_s\gamma^T, \alpha_{j_1}\alpha_{k_1}1_s+\alpha_{k_2}1_s)$ by permuting rows, for any $k_1, k_2 = 0, \dots, s-1$. Then $(\beta_{i_1}\oplus D^{(0)}, \beta_{j_1}\oplus e^{(0)})$ can be transformed into

$$\begin{pmatrix} D^{(0)}, & \alpha_{j_1}\alpha_0 1_{s^2}+e^{(0)} \\ \vdots & \vdots \\ D^{(0)}, & \alpha_{j_1}\alpha_{s-1}1_{s^2}+e^{(0)} \end{pmatrix}.$$

Thus $(a_1^{(1)}, a_2^{(1)}, h)$ is a full factorial design with three factors, where $\alpha_{j_1}\neq 0$. Furthermore, for $v\geq 2$, $(\beta_{i_v}\oplus D^{(v)}, \beta_{j_v}\oplus e^{(v)})$ can be transformed into

$$\begin{pmatrix} D^{(v-1)}, & \alpha_{j_v}\alpha_0 1_{s^2}+e^{(v-1)} \\ \vdots & \vdots \\ D^{(v-1)}, & \alpha_{j_v}\alpha_{s-1}1_{s^2}+e^{(v-1)} \end{pmatrix}.$$

Similarly, for any column $h\in e^{(v)}$ with $v\geq 2$, we can obtain that $(\beta_{i_v}\oplus W^{(v-1)}_{i_1j_1i_2j_2\cdots i_{v-1}j_{v-1}}, h)$ is a full factorial design with $2+v$ factors, where $\alpha_{j_v}\neq 0$. This completes the proof. □

Proof of Theorem 1 From the construction of Algorithm 1, it is easy to obtain that X is an LHD and D_1 is an OA. D being an MCD follows from Lemma A.1 with $t=2$. Any column of X denoted as ξ can be represented as $\xi=\eta_1\pm\eta_2 s$, where (η_1,η_2) is an s^2 full factorial design. From the construction, it is obvious that (ζ,η_2) is an s^2 full factorial design, where ζ is any column of D_1. Then D is an MCD following Lemma A.1. From Lemma 1, if M is orthogonal, then X is orthogonal. □

Proof of Corollary 1 For any two columns in X_j, denoted as ξ_1 and ξ_2, they can be expressed as $\xi_1=\eta_{11}\pm\eta_{12}s$ and $\xi_2=\eta_{21}\pm\eta_{22}s$, respectively. And (η_{11},η_{12}), (η_{21},η_{22}) and (η_{12},η_{22}) are s^2 full factorial designs. Collapse the level x of X_j by $\lfloor (x+(s^2-1)/2)/s\rfloor$ to $\{0,1,\dots,s-1\}$. Then ξ_1 and ξ_2 become $\eta_{12}+((s-1)/2)1_{s^2}$ and $\eta_{22}+((s-1)/2)1_{s^2}$, respectively. As $(\eta_{12}+((s-1)/2)1_{s^2}, \eta_{22}+((s-1)/2)1_{s^2})$ is an s^2 full factorial design, then (i) is correct. The proofs of (ii) and (iii) are similar to that of (i) and thus omitted here. □

Proof of Theorem 2 (i) It is easy to see that D_1 is an OA. From the construction of Algorithm 2, following the idea of Lemma 2, each O_i is an $OA(s^u, 2(u-1), s, u)$. So each $O_i^{(l)}$ is an $OA(s^u, 2(u-1), s, u)$ with levels $\{-(s-1)/2, -(s-3)/2, \dots, (s-1)/2\}$. From the properties of the rotation matrix R_{a1}, $Z_i^{(l)}$ is an OLHD. Then D_2 is an OLHD from Lemma 1.

(ii) Combing the idea of (iv) in Lemma 2 and Lemma A.1, we can obtain that (D_1, D_2) is an OMCD.

(iii) Similarly to the proof of Theorem 1, after the levels of $(z_{i,j}^{(l)}, z_{i,j'}^{(l)})$ are collapsed to $\{0,1,\dots,s-1\}$, they become $(\eta_j+\frac{s-1}{2}1_{s^u}, \eta_{j'}+\frac{s-1}{2}1_{s^u})$, where η_j and $\eta_{j'}$ are the two corresponding columns in $O_i^{(l)}$. Then (iii) can be obtained straightforwardly.

(iv) For $1 \le j \le u-1$ and $u \le j' \le 2u-2$, $z_{i,j}^{(l)}$ and $z_{i,j'}^{(l)}$ can be represented as $z_{i,j}^{(l)} = \varphi d_{2i-1} \pm d_{2i,j}$ and $z_{i,j'}^{(l)} = \varphi' d_{2i} + d_{2i-1,j'}$, where φ and φ' are rows with entries from a signed permutation of $s, s^2, \ldots, s^{u-1}$. After collapsing $z_{i,j}^{(l)}$ and $z_{i,j'}^{(l)}$ to s and s^{u-1} separately, the corresponding columns are $d_{2i-1,\tau}$ and d_{2i} respectively. As $(d_{2i-1,\tau}, d_{2i})$ forms a full factorial design, $(z_{i,j}^{(l)}, z_{i,j'}^{(l)})$ achieves stratification on $s \times s^{u-1}$. The proofs for the other cases are similar.

Proof of Theorem 3 We only need to prove that D_2 is mirror-symmetric as other results follow from Theorem 2. As O_i for $i = 1, \ldots, k$ and L are mirror-symmetric designs, then each $O_i^{(l)}$ is a mirror-symmetric design. So if b is a row of $O_i^{(l)}$, $-b$ is also one of its rows. Then for $Z_i^{(l)}$, if bR_{a1} is one of the rows, $-bR_{a1}$ is also one of its rows. Thus $Z_i^{(l)}$ is mirror-symmetric, furthermore, D_2 is mirror-symmetric. □

Proof of Theorem 4 (i) From the definitions of $D^{(0)}$ and $e^{(0)}$, it is easy to check that D_1 is an $OA(s^{2+v}, (s-1)^v + (s-(2p+2))s^v, s, 2)$. From Theorem 1 of Sun and Tang (2017), we can obtain that D_2 is an orthogonal $L(s^{2+v}, \lfloor s/2 \rfloor^v (p+1)2^{1+v})$.

(ii) Let ζ be any column of D_1, then $\zeta \in e^{(v)}$ or $\zeta \in \beta_i \oplus D^{(v-1)}$ for $i = 2p+2, \ldots, s-1$. Let ξ be any column of D_2, then $\xi \in \widetilde{H}_{j_1 j_2 \cdots j_v}^{(i)}$ for some corresponding $j_1, j_2, \ldots, j_v$. And there exist $W_{i_1 j_1 i_2 j_2 \cdots i_{v-1} j_{v-1}}^{(v-1)}$, β_{i_v} and β_{j_v}, such that ξ can be represented as $\xi = \pm\lambda_1 \pm \lambda_2 s \pm \cdots \pm \lambda_{2+v} s^{2+v-1}$ where $(\lambda_2, \ldots, \lambda_{2+v}) = \beta_{i_v} \oplus W_{i_1 j_1 i_2 j_2 \cdots i_{v-1} j_{v-1}}^{(v-1)}$, and $\lambda_1 \in \beta_{j_v} \oplus D^{(v-1)}$. From Lemma A.2, we can obtain that $(\zeta, \lambda_2, \ldots, \lambda_{2+v})$ is a full factorial design. Then (D_1, D_2) is an MCD from Lemma A.1.

(iii) According to the construction of D_2, it is easy to see that D_2 achieves stratification on $s \times s$ grids in any two dimensions.

□

Proof of Corollary 3 (i) It is clear that Part (i) can be obtained from Theorem 4 and Lemma A.2.

(ii) From Theorem 4, (D_1, Γ) is an OMCD. We only need to prove that (Ψ, Γ) is an OMCD. From the construction in Algorithm 4, we can obtain that

$$H_{00\ldots0}^{(1)} = \left(\beta_0 \oplus H_{00\ldots0 j_{v-1}|j_{v-1}=0}^{(1)}, \beta_1 \oplus H_{00\ldots0 j_{v-1}|j_{v-1}=0}^{(1)}\right).$$

Let ξ be any column of Γ, then $\xi = \pm\eta_1 \pm \eta_2 s \pm \cdots \pm \eta_{2+v} s^{2+v-1}$, where $\eta_1 \in \beta_0 \oplus H_{00\ldots0 j_{v-1}|j_{v-1}=0}^{(1)}$, $\eta_h \in \beta_1 \oplus H_{00\ldots0 j_{v-1}|j_{v-1}=0}^{(1)}$ for $h = 2, \ldots, 2+v$ and $(\eta_1, \eta_2, \ldots, \eta_{2+v})$ is a full factorial design. Let ζ be any column of Ψ, from the definition of Ψ, we can obtain that $(\zeta, \eta_2, \ldots, \eta_{2+v})$ is a full factorial design from Lemma A.2. From Lemma A.1, (Ψ, Γ) is an OMCD. Now Part (ii) can be proved.

□

Proof of Theorem 5 Let $l_i^{(r)}$ denote the ith column of L_r, $r = 1, 2$. Without loss of generality, we only consider the column $l_1^{(1)} \oplus (s l_1^{(2)})$, which is the first column of Y.

Since $l_1^{(1)}$ is a permutation on $\{-(s-1)/2, -(s-3)/2, \ldots, (s-3)/2, (s-1)/2\}$, and $l_1^{(2)}$ is a permutation on $\{-(n/s-1)/2, -(n/s-3)/2, \ldots, (n/s-3)/2, (n/s-1)/2\}$, then $l_1^{(1)} \oplus (sl_1^{(2)})$ is a permutation on $\{-(n-1)/2, -(n-3)/2, \ldots, (n-3)/2, (n-1)/2\}$. Thus Y is an $L(n, k_2)$. It is clear that $\lfloor (Y_j + (n-1)/2)/s \rfloor = L_2 + (n/s-1)/2$ for $j = 1, 2, \ldots, s$, where $L_2 + (n/s-1)/2$ is an $L(n/s, k_2)$ with entries from $\{0, 1, \ldots, n/s-1\}$. So $D = (D_1, Y)$ can be transformed into $(D_1, 1_s \otimes (L_2 + (n/s-1)/2))$ after level-collapsing of Y. Since $(D_1, 1_s \otimes (1, 2, \ldots, n/s)^T)$ is an $OA(n, s^m(n/s), 2)$, $(D_1, 1_s \otimes (l_i^{(2)} + (n/s-1)/2))$ is an $OA(n, s^m(n/s), 2)$ too, $i = 1, 2, \ldots, k_2$. Therefore, D is an MCD with m qualitative factors and k_2 quantitative factors. □

Proof of Theorem 6 For two vectors $a = (a_1, \ldots, a_n)^T$ and $b = (b_1, \ldots, b_n)^T$, define $\odot$ operator as

$$a \odot b = \sum_{i=1}^{n} a_i b_i.$$

Let $l_v^{(r)}$ denote the vth column of L_r for $r = 1, 2$, $l_{uv}^{(r)}$ denote the uth element of $l_v^{(r)}$, and $d_{it} = l_i^{(1)} \oplus (sl_t^{(2)})$, where $i = 1, 2, \ldots, k_1$ and $t = 1, 2, \ldots, k_2$. For $1 \le i_1, i_2 \le k_1$ and $1 \le t_1, t_2 \le k_2$,

$$\begin{aligned} d_{i_1t_1} \odot d_{i_2t_2} &= \sum_{u=1}^{s} (l_{ui_1}^{(1)} \oplus (sl_{t_1}^{(2)})) \odot (l_{ui_2}^{(1)} \oplus (sl_{t_2}^{(2)})) \\ &= (n/s) \sum_{u=1}^{s} l_{ui_1}^{(1)} l_{ui_2}^{(1)} + s^2 \sum_{u=1}^{s} l_{t_1}^{(2)} \odot l_{t_2}^{(2)} \\ &= (n/s) l_{i_1}^{(1)} \odot l_{i_2}^{(1)} + s^3 l_{t_1}^{(2)} \odot l_{t_2}^{(2)}. \end{aligned}$$

From the construction procedure, it is easy to see that

$$\rho(l_{i_1}^{(1)}, l_{i_2}^{(1)}) = \frac{12}{s(s^2-1)} l_{i_1}^{(1)} \odot l_{i_2}^{(1)}, \quad \rho(l_{t_1}^{(2)}, l_{t_2}^{(2)}) = \frac{12s^3}{n(n^2-s^2)} l_{t_1}^{(2)} \odot l_{t_2}^{(2)}$$

and

$$\rho(d_{i_1t_1}, d_{i_2t_2}) = \frac{12}{n(n^2-1)} d_{i_1t_1} \odot d_{i_2t_2}.$$

Then we can obtain that

$$\rho(d_{i_1t_1}, d_{i_2t_2}) = \lambda \rho(l_{i_1}^{(1)}, l_{i_2}^{(1)}) + (1-\lambda) \rho(l_{t_1}^{(2)}, l_{t_2}^{(2)}), \text{ where } \lambda = \frac{s^2-1}{n^2-1}.$$

(i) For $k_1 = k_2$ and $1 \le j_1, j_2 \le k_2$, we can have that $\rho_{j_1j_2}(Y) = \lambda \rho_{j_1j_2}(L_1) + (1-\lambda)\rho_{j_1j_2}(L_2)$. From Corollary 2 of Huang et al. (2014), if L_1 and L_2 are both

OLHDs, then Y is OLHD; furthermore, Y is second-order orthogonal if L_1 and L_2 are both second-order orthogonal.

(ii) It is straightforward to see that (1) is true.

□

Appendix B

Table 5 An OMCD of 81 runs in Example 4

Run	D_1													D_2							
	$e^{(2)}$				$\beta_2 \oplus D^{(1)}$																
1	0	0	0	0	0	0	0	0	0	0	0	0	0	−40	−14	−32	−22	−38	−16	−34	−20
2	1	1	1	1	0	1	2	0	1	2	0	1	2	−31	9	−23	9	−29	9	−25	9
3	2	2	2	2	0	2	1	0	2	1	0	2	1	−22	32	−14	40	−20	34	−16	38
4	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	0	1	2	0	1	2	0	9	23	9	31	9	25	9	29
6	2	2	2	2	1	0	2	1	0	2	1	0	2	−9	−23	−9	−31	−9	−25	−9	−29
7	0	0	0	0	2	2	2	2	2	2	2	2	2	40	14	32	22	38	16	34	20
8	1	1	1	1	2	0	1	2	0	1	2	0	1	22	−32	14	−40	20	−34	16	−38
9	2	2	2	2	2	1	0	2	1	0	2	1	0	31	−9	23	−9	29	−9	25	−9
10	1	2	1	2	0	0	0	1	1	1	2	2	2	−37	−17	3	−3	−35	−19	3	−3
11	2	0	2	0	0	1	2	1	2	0	2	0	1	−28	6	12	28	−26	6	12	26
12	0	1	0	1	0	2	1	1	0	2	2	1	0	−19	38	−6	−25	−17	40	−6	−23
13	1	2	1	2	1	1	1	2	2	2	0	0	0	3	−3	35	19	3	−3	37	17
14	2	0	2	0	1	2	0	2	0	1	0	1	2	12	29	17	−34	12	31	19	−32
15	0	1	0	1	1	0	2	2	1	0	0	2	1	−6	−26	26	−12	−6	−28	28	−12
16	1	2	1	2	2	2	2	0	0	0	1	1	1	34	20	−38	−16	32	22	−40	−14
17	2	0	2	0	2	0	1	0	1	2	1	2	0	16	−35	−29	6	14	−37	−31	6
18	0	1	0	1	2	1	0	0	2	1	1	0	2	25	−12	−20	37	23	−12	−22	35
19	2	1	2	1	0	0	0	2	2	2	1	1	1	−34	−20	38	16	−32	−22	40	14
20	0	2	0	2	0	1	2	2	0	1	1	2	0	−25	12	20	−37	−23	12	22	−35
21	1	0	1	0	0	2	1	2	1	0	1	0	2	−16	35	29	−6	−14	37	31	−6
22	2	1	2	1	1	1	1	0	0	0	2	2	2	−3	3	−35	−19	−3	3	−37	−17
23	0	2	0	2	1	2	0	0	1	2	2	0	1	6	26	−26	12	6	28	−28	12
24	1	0	1	0	1	0	2	0	2	1	2	1	0	−12	−29	−17	34	−12	−31	−19	32
25	2	1	2	1	2	2	2	1	1	1	0	0	0	37	17	−3	3	35	19	−3	3
26	0	2	0	2	2	0	1	1	2	0	0	1	2	19	−38	6	25	17	−40	6	23
27	1	0	1	0	2	1	0	1	0	2	0	2	1	28	−6	−12	−28	26	−6	−12	−26
28	1	1	2	2	2	2	2	2	2	2	2	2	2	−39	−15	−33	−21	1	−1	−1	1
29	2	2	0	0	2	0	1	2	0	1	2	0	1	−30	8	−24	10	10	24	8	30

Table 5 continued

Run	D_1													D_2							
	$e^{(2)}$		$\beta_2 \oplus D^{(1)}$																		
30	0	0	1	1	2	1	0	2	1	0	2	1	0	−21	34	−15	38	−8	−23	−10	−31
31	1	1	2	2	0	0	0	0	0	0	0	0	0	1	−1	−1	1	39	15	33	21
32	2	2	0	0	0	1	2	0	1	2	0	1	2	10	25	8	29	21	−32	15	−40
33	0	0	1	1	0	2	1	0	2	1	0	2	1	−8	−24	−10	−30	30	−10	24	−8
34	1	1	2	2	1	1	1	1	1	1	1	1	1	38	16	34	20	−40	−14	−32	−22
35	2	2	0	0	1	2	0	1	2	0	1	2	0	20	−33	16	−39	−31	8	−23	10
36	0	0	1	1	1	0	2	1	0	2	1	0	2	29	−10	25	−8	−22	33	−14	39
37	2	0	0	1	2	2	2	0	0	0	1	1	1	−36	−18	2	−2	4	−4	36	18
38	0	1	1	2	2	0	1	0	1	2	1	2	0	−27	5	11	26	13	30	18	−34
39	1	2	2	0	2	1	0	0	2	1	1	0	2	−18	40	−7	−24	−5	−26	27	−11
40	2	0	0	1	0	0	0	1	1	1	2	2	2	4	−4	37	17	33	21	−38	−16
41	0	1	1	2	0	1	2	1	2	0	2	0	1	13	31	19	−33	15	−35	−29	7

Table 6 An OMCD of 81 runs in Example 4 (Table 5 continued)

Run	D_1													D_2							
	$e^{(2)}$		$\beta_2 \oplus D^{(1)}$																		
42	1	2	2	0	0	2	1	1	0	2	2	1	0	−5	−27	28	−11	24	−13	−20	36
43	2	0	0	1	1	1	1	2	2	2	0	0	0	32	22	−39	−15	−37	−17	2	−2
44	0	1	1	2	1	2	0	2	0	1	0	1	2	14	−36	−30	7	−28	5	11	27
45	1	2	2	0	1	0	2	2	1	0	0	2	1	23	−13	−21	35	−19	39	−7	−25
46	0	2	1	0	2	2	2	1	1	1	0	0	0	−33	−21	40	14	−2	2	−35	−19
47	1	0	2	1	2	0	1	1	2	0	0	1	2	−24	11	22	−36	7	27	−26	13
48	2	1	0	2	2	1	0	1	0	2	0	2	1	−15	37	31	−5	−11	−29	−17	33
49	0	2	1	0	0	0	0	2	2	2	1	1	1	−2	2	−36	−18	36	18	−4	4
50	1	0	2	1	0	1	2	2	0	1	1	2	0	7	28	−27	13	18	−38	5	24
51	2	1	0	2	0	2	1	2	1	0	1	0	2	−11	−30	−18	32	27	−7	−13	−28
52	0	2	1	0	1	1	1	0	0	0	2	2	2	35	19	−4	4	−34	−20	39	15
53	1	0	2	1	1	2	0	0	1	2	2	0	1	17	−39	5	23	−25	11	21	−37
54	2	1	0	2	1	0	2	0	2	1	2	1	0	26	−7	−13	−27	−16	36	30	−5
55	2	2	1	1	1	1	1	1	1	1	1	1	1	−38	−16	−34	−20	40	14	32	22
56	0	0	2	2	1	2	0	1	2	0	1	2	0	−29	10	−25	8	22	−33	14	−39
57	1	1	0	0	1	0	2	1	0	2	1	0	2	−20	33	−16	39	31	−8	23	−10
58	2	2	1	1	2	2	2	2	2	2	2	2	2	−1	1	1	−1	−39	−15	−33	−21
59	0	0	2	2	2	0	1	2	0	1	2	0	1	8	24	10	30	−30	10	−24	8
60	1	1	0	0	2	1	0	2	1	0	2	1	0	−10	−25	−8	−29	−21	32	−15	40
61	2	2	1	1	0	0	0	0	0	0	0	0	0	39	15	33	21	−1	1	1	−1

Table 6 continued

Run	D_1													D_2							
	$e^{(2)}$				$\beta_2 \oplus D^{(1)}$																
62	0	0	2	2	0	1	2	0	1	2	0	1	2	21	−34	15	−38	8	23	10	31
63	1	1	0	0	0	2	1	0	2	1	0	2	1	30	−8	24	−10	−10	−24	−8	−30
64	0	1	2	0	1	1	1	2	2	2	0	0	0	−35	−19	4	−4	34	20	−39	−15
65	1	2	0	1	1	2	0	2	0	1	0	1	2	−26	7	13	27	16	−36	−30	5
66	2	0	1	2	1	0	2	2	1	0	0	2	1	−17	39	−5	−23	25	−11	−21	37
67	0	1	2	0	2	2	2	0	0	0	1	1	1	2	−2	36	18	−36	−18	4	−4
68	1	2	0	1	2	0	1	0	1	2	1	2	0	11	30	18	−32	−27	7	13	28
69	2	0	1	2	2	1	0	0	2	1	1	0	2	−7	−28	27	−13	−18	38	−5	−24
70	0	1	2	0	0	0	0	1	1	1	2	2	2	33	21	−40	−14	2	−2	35	19
71	1	2	0	1	0	1	2	1	2	0	2	0	1	15	−37	−31	5	11	29	17	−33
72	2	0	1	2	0	2	1	1	0	2	2	1	0	24	−11	−22	36	−7	−27	26	−13
73	1	0	0	2	1	1	1	0	0	0	2	2	2	−32	−22	39	15	37	17	−2	2
74	2	1	1	0	1	2	0	0	1	2	2	0	1	−23	13	21	−35	19	−39	7	25
75	0	2	2	1	1	0	2	0	2	1	2	1	0	−14	36	30	−7	28	−5	−11	−27
76	1	0	0	2	2	2	2	1	1	1	0	0	0	−4	4	−37	−17	−33	−21	38	16
77	2	1	1	0	2	0	1	1	2	0	0	1	2	5	27	−28	11	−24	13	20	−36
78	0	2	2	1	2	1	0	1	0	2	0	2	1	−13	−31	−19	33	−15	35	29	−7
79	1	0	0	2	0	0	0	2	2	2	1	1	1	36	18	−2	2	−4	4	−36	−18
80	2	1	1	0	0	1	2	2	0	1	1	2	0	18	−40	7	24	5	26	−27	11
81	0	2	2	1	0	2	1	2	1	0	1	0	2	27	−5	−11	−26	−13	−30	−18	34

Table 7 An OMCD of 32 runs in Example 5

Run	Φ									Γ							
	$e^{(3)}$	Ψ															
1	0	0	0	0	0	0	0	0	0	−14.5	−1.5	−9.5	−6.5	−13.5	−2.5	−10.5	−5.5
2	1	0	1	0	1	0	1	0	1	−6.5	9.5	−1.5	14.5	−5.5	10.5	−2.5	13.5
3	0	1	1	1	1	1	1	1	1	14.5	1.5	9.5	6.5	13.5	2.5	10.5	5.5
4	1	1	0	1	0	1	0	1	0	6.5	−9.5	1.5	−14.5	5.5	−10.5	2.5	−13.5
5	1	0	0	1	1	0	0	1	1	−10.5	−5.5	13.5	2.5	−9.5	−6.5	14.5	1.5
6	0	0	1	1	0	0	1	1	0	−2.5	13.5	5.5	−10.5	−1.5	14.5	6.5	−9.5
7	1	1	1	0	0	1	1	0	0	10.5	5.5	−13.5	−2.5	9.5	6.5	−14.5	−1.5
8	0	1	0	0	1	1	0	0	1	2.5	−13.5	−5.5	10.5	1.5	−14.5	−6.5	9.5
9	1	0	0	0	0	1	1	1	1	−12.5	−3.5	−11.5	−4.5	15.5	0.5	8.5	7.5
10	0	0	1	0	1	1	0	1	0	−4.5	11.5	−3.5	12.5	7.5	−8.5	0.5	−15.5
11	1	1	1	1	1	0	0	0	0	12.5	3.5	11.5	4.5	−15.5	−0.5	−8.5	−7.5
12	0	1	0	1	0	0	1	0	1	4.5	−11.5	3.5	−12.5	−7.5	8.5	−0.5	15.5

Table 7 continued

Run	Φ									Γ							
	$e^{(3)}$	Ψ															
13	0	0	0	1	1	1	1	0	0	−8.5	−7.5	15.5	0.5	11.5	4.5	−12.5	−3.5
14	1	0	1	1	0	1	0	0	1	−0.5	15.5	7.5	−8.5	3.5	−12.5	−4.5	11.5
15	0	1	1	0	0	0	0	1	1	8.5	7.5	−15.5	−0.5	−11.5	−4.5	12.5	3.5
16	1	1	0	0	1	0	1	1	0	0.5	−15.5	−7.5	8.5	−3.5	12.5	4.5	−11.5
17	1	0	0	0	0	0	0	0	0	15.5	0.5	8.5	7.5	12.5	3.5	11.5	4.5
18	0	0	1	0	1	0	1	0	1	7.5	−8.5	0.5	−15.5	4.5	−11.5	3.5	−12.5
19	1	1	1	1	1	1	1	1	1	−15.5	−0.5	−8.5	−7.5	−12.5	−3.5	−11.5	−4.5
20	0	1	0	1	0	1	0	1	0	−7.5	8.5	−0.5	15.5	−4.5	11.5	−3.5	12.5
21	0	0	0	1	1	0	0	1	1	11.5	4.5	−12.5	−3.5	8.5	7.5	−15.5	−0.5
22	1	0	1	1	0	0	1	1	0	3.5	−12.5	−4.5	11.5	0.5	−15.5	−7.5	8.5
23	0	1	1	0	0	1	1	0	0	−11.5	−4.5	12.5	3.5	−8.5	−7.5	15.5	0.5
24	1	1	0	0	1	1	0	0	1	−3.5	12.5	4.5	−11.5	−0.5	15.5	7.5	−8.5
25	0	0	0	0	0	1	1	1	1	13.5	2.5	10.5	5.5	−14.5	−1.5	−9.5	−6.5
26	1	0	1	0	1	1	0	1	0	5.5	−10.5	2.5	−13.5	−6.5	9.5	−1.5	14.5
27	0	1	1	1	1	0	0	0	0	−13.5	−2.5	−10.5	−5.5	14.5	1.5	9.5	6.5
28	1	1	0	1	0	0	1	0	1	−5.5	10.5	−2.5	13.5	6.5	−9.5	1.5	−14.5
29	1	0	0	1	1	1	1	0	0	9.5	6.5	−14.5	−1.5	−10.5	−5.5	13.5	2.5
30	0	0	1	1	0	1	0	0	1	1.5	−14.5	−6.5	9.5	−2.5	13.5	5.5	−10.5
31	1	1	1	0	0	0	0	1	1	−9.5	−6.5	14.5	1.5	10.5	5.5	−13.5	−2.5
32	0	1	0	0	1	0	1	1	0	−1.5	14.5	6.5	−9.5	2.5	−13.5	−5.5	10.5

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