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Column-orthogonal designs with multi-dimensional stratifications

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Abstract The orthogonal Latin hypercube design and its relaxation, and column-orthogonal design, are two kinds of orthogonal designs for computer experiments. However, they usually do not achieve maximum stratifications in multi-dimensional margins. In this paper, we propose some methods to construct column-orthogonal designs with multi-dimensional stratifications by rotating symmetric and asymmetric orthogonal arrays. The newly constructed column-orthogonal designs ensure that the estimates of all linear effects are uncorrelated with each other and even uncorrelated with the estimates of all second-order effects (quadratic effects and bi-linear effects) when the rotated orthogonal arrays have strength larger than two. Besides orthogonality, the resulting designs also preserve better space-filling properties than those constructed by using the existing methods. In addition, we provide a method to construct a new class of orthogonal Latin hypercube designs with multi-dimensional stratifications by rotating regular factorial designs. Some newly constructed orthogonal Latin hypercube designs are tabulated for practical use.

Keywords computer experiment, Latin hypercube design, orthogonal array, orthogonality, rotation matrix

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1 Introduction

Latin hypercube designs (LHDs) were first proposed by McKay et al. [13] for computer experiments. Ever since then, LHDs have become a class of most popular designs because of their maximum stratification when projected onto any univariate dimension (see [4, 7, 14, 17, 18, 20, 23–25] and among others). It is well known that orthogonality can be viewed as a stepping stone to space-filling designs [3]. Many efforts have been made to construct orthogonal LHDs by rotating factorial designs. Steinberg and Lin [16] rotated grouped two-level regular fractional factorial designs to construct orthogonal LHDs with 2^m runs and $\lfloor (2^m-1)/m \rfloor m$ factors, where m itself is a power of two and $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Pang et al. [14] extended [16]'s method to obtain orthogonal LHDs with p^m runs and $(p^m-1)/(p-1)$ factors, where p is a prime and m is a power of two. By using the subfield theory, Ai et

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al. [1] provided a general partition scheme to divide regular saturated factorial designs of strength two into groups each being a full factorial design, but the method does not work for designs of strength three or more. They obtained many new orthogonal LHDs. Recently, Wang et al. [21] also used the rotation method to construct a kind of orthogonal LHDs. However, all orthogonal LHDs constructed above do not consider stratifications when projected onto multiple dimensions.

Tang [20] provided a kind of orthogonal array (OA)-based LHDs which achieve stratifications on t-dimensional margins if OAs of strength t are employed. A new class of arrays called strong OAs (SOAs) was introduced and constructed by He and Tang [8]. Since an SOA of strength t achieves uniformity on finer grids in all g-dimensional projections for any $g \leqslant t-1$, the LHD constructed from an SOA of strength t has better space-filling properties than OA-based LHDs of the same size in all g-dimensional projections for any $2 \leqslant g \leqslant t-1$ (see [8]). For computer experiments, Bingham et al. [3] pointed out that practical experiments have revealed that designs with many levels are desirable, but it is not essential to restrict the number of levels of each factor to be the run size. Then Sun et al. [19] relaxed the restriction of LHDs and provided some methods to construct column-orthogonal designs (CODs) and nearly CODs by rotating OAs. The newly constructed designs preserve the geometric configuration of the original OAs in each rotation part, and thus they have good space-filling properties. Recently, Liu and Liu [12] proposed some approaches to constructing column-orthogonal and nearly column-orthogonal SOAs by rotating symmetric OAs. Zhou and Tang [26] provided column-orthogonal SOAs of strength two plus and three minus.

In this paper, we provide some new methods for constructing CODs with multi-dimensional stratifications by rotating symmetric and asymmetric OAs. By rotating regular fractional factorial designs with any prime power number of levels, we can construct orthogonal and second-order orthogonal LHDs with multi-dimensional stratifications.

The rest of this paper is organized as follows. Section 2 presents some useful notation and definitions. Section 3 provides a method for constructing CODs with multi-dimensional stratifications using symmetric OAs. Section 4 proposes an approach to constructing CODs with multi-dimensional stratifications by rotating asymmetric OAs. Orthogonal LHDs with multi-dimensional stratifications obtained by rotating grouped regular factorial designs are given in Section 5. Some concluding remarks are provided in Section 6. All proofs are deferred to Appendix A.

2 Definitions and notation

This section provides some useful notation and definitions. Let $D(n; q_1 q_2 \cdots q_m)$ be the design of n runs and m factors each of $q_1, q_2, \ldots, q_m$ levels, respectively, which is represented by an $n \times m$ matrix

$$D = (d_1, d_2, \ldots, d_m).$$

For convenience, the q_j levels of the j-th column are taken to be $-(q_j-1), -(q_j-3), \ldots, (q_j-3), (q_j-1)$, where $j = 1, 2, \ldots, m$. The q_j's are not necessarily distinct; for example, a $D(n; (q_1)^{m_1}(q_2)^{m_2} \cdots (q_u)^{m_u})$ is a design that has m_j factors of q_j levels, where $m = \sum_{i=1}^{u} m_i$. A design $D(n; q_1 q_2 \cdots q_m)$ is called an OA of strength e, denoted by $\mathrm{OA}(n; m, q_1 q_2 \cdots q_m, e)$, if all possible level-combinations for any e columns occur with the same frequency. When all q_j's are equal to q, the array is symmetric and denoted by $\mathrm{OA}(n; q^m, e)$. Here, this orthogonality is called combinatorial orthogonality. If all levels are equally replicated in each column and the inner product of any two distinct columns of a design $D(n; q_1 q_2 \cdots q_m)$ is zero, then this design is called a COD, denoted by $\mathrm{COD}(n; q_1 q_2 \cdots q_m)$. For a first-order model, orthogonal designs ensure independent estimates of linear effects. For a second-order model, however, an orthogonal design requires (a) each design column is orthogonal to all the others, and (b) the sum of elementwise products of any three columns (whether they are distinct or not) is zero. A design is called second-order orthogonal if it satisfies Properties (a) and (b) (see [3]). If each factor includes n uniformly spaced levels, then the design $D(n; n^m)$ is called an LHD, denoted by $L(n, m)$.

3 Construction of CODs with multi-dimensional stratifications from symmetric OAs

This section provides a method for constructing CODs with multi-dimensional stratifications by rotating symmetric OAs.

Let

$$A = (A_1, \dots, A_v, A^*) \tag{3.1}$$

be an $\mathrm{OA}(n; q^m, e)$ $(e \geqslant 2)$ with A_i being an $\mathrm{OA}(n; q^{f_i}, t_i)$ for $i = 1, 2, \dots, v$ and A^* being an $\mathrm{OA}(n; q^r, t_{v+1})$, where

$$m = \sum_{i=1}^{v} f_i + r, \quad t_i \geqslant e, \quad t_{v+1} < e$$

and

$$f_i = \begin{cases} t_i, & t_i \text{ is even}, \\ t_i + 1, & t_i \text{ is odd} \end{cases}$$

for $i = 1, 2, \dots, v$. Define

$$X_1 = R^q_{(t_1, t_2, \dots, t_v, 1)} = \mathrm{diag}\{H^q_{t_1}, H^q_{t_2}, \dots, H^q_{t_v}, I_r\}, \tag{3.2}$$

where $H^q_{t_i}$ is an $f_i \times 2$ column-orthogonal matrix with the form

$$\begin{pmatrix} 1 & q & \cdots & q^{\frac{t_i-3}{2}} & q^{\frac{t_i-1}{2}} & q^{\frac{t_i+1}{2}} & \cdots & q^{t_i-1} & 0 \\ q^{t_i-1} & q^{t_i-2} & \cdots & q^{\frac{t_i+1}{2}} & 0 & -q^{\frac{t_i-3}{2}} & \cdots & -1 & q^{\frac{t_i-1}{2}} \end{pmatrix}^{\mathrm{T}} \tag{3.3}$$

for odd t_i,

$$\begin{pmatrix} 1 & q & \cdots & q^{\frac{t_i}{2}-1} & q^{\frac{t_i}{2}} & \cdots & q^{t_i-1} \\ q^{t_i-1} & q^{t_i-2} & \cdots & q^{\frac{t_i}{2}} & -q^{\frac{t_i}{2}-1} & \cdots & -1 \end{pmatrix}^{\mathrm{T}} \tag{3.4}$$

for even t_i, and I_r is the identity matrix of order r. The following theorem provides a construction and the property of stratifications of the resulting designs.

Theorem 3.1. *For the A in* (3.1) *and X_1 in* (3.2)*, define $D = AX_1$.*

(i) *If $e \geqslant 3$, then D is a second-order orthogonal* $\mathrm{COD}(n; (q^{t_1})^2 (q^{t_2})^2 \cdots (q^{t_v})^2 q^r)$ *which achieves stratifications on $q^{u_1} \times q^{u_2} \times \cdots \times q^{u_g}$ grids in g-dimensional projections, where $g \leqslant e$ and $\sum_{l=1}^{g} u_l = e$. In particular, the two q^{t_i}-level columns achieve stratifications on $q^{u_1} \times q^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = t_i$, $i = 1, 2, \dots, v$.*

(ii) *If $e = 2$, then D is a* $\mathrm{COD}(n; (q^{t_1})^2 (q^{t_2})^2 \cdots (q^{t_v})^2 q^r)$ *in which the two q^{t_i}-level columns achieve stratifications on $q^{u_1} \times q^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = t_i$, $i = 1, 2, \dots, v$.*

The designs constructed above have flexible stratifications which cannot be guaranteed by other CODs from existing methods. Also, the parameter settings (the run size n, the number of factors m and the number of levels q) in the resulting designs are very flexible, since we have no strict requirement for the original OA A in (3.1). The following two examples illustrate how to construct CODs with multi-dimensional stratifications by rotating symmetric OAs.

Example 3.2. Suppose A is an $\mathrm{OA}(16; 2^8, 3)$ with the strength of the first four columns being four and the strength of the next four columns being three. From Theorem 3.1, $D = AR^2_{(4,3)}$ is a second-order orthogonal $\mathrm{COD}(16; (16)^2 8^2)$ in which the sum of the elementwise products of any three columns is zero, where

$$R^2_{(4,3)} = \mathrm{diag}\{H^2_4, H^2_3\}, \quad H^2_4 = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 4 & -2 & -1 \end{pmatrix}^{\mathrm{T}}$$

Table 1 OA$(16;2^8,3)$ and COD$(16;(16)^2 8^2)$ in Example 3.2

Run	OA$(16;2^8,3)$								COD$(16;(16)^2 8^2)$			
1	−1	−1	−1	−1	−1	−1	−1	−1	−15	−9	−7	−5
2	−1	−1	1	−1	1	1	1	−1	−7	−13	7	1
3	−1	1	−1	1	−1	1	1	−1	5	−3	5	−7
4	−1	1	1	1	1	−1	−1	−1	13	−7	−5	3
5	1	−1	−1	1	1	−1	1	−1	3	5	3	1
6	1	−1	1	1	−1	1	−1	−1	11	1	−3	−5
7	1	1	−1	−1	1	1	−1	−1	−9	15	−1	3
8	1	1	1	−1	−1	−1	1	−1	−1	11	1	−7
9	1	1	1	1	1	1	1	1	15	9	7	5
10	1	1	−1	1	−1	−1	−1	1	7	13	−7	−1
11	1	−1	1	−1	1	−1	−1	1	−5	3	−5	7
12	1	−1	−1	−1	−1	1	1	1	−13	7	5	−3
13	−1	1	1	−1	−1	1	−1	1	−3	−5	−3	−1
14	−1	1	−1	−1	1	−1	1	1	−11	−1	3	5
15	−1	−1	1	1	−1	−1	1	1	9	−15	1	−3
16	−1	−1	−1	1	1	1	−1	1	1	−11	−1	7

and

$$H_3^2 = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 4 & 0 & -1 & 2 \end{pmatrix}^{\mathrm{T}}.$$

The OA$(16;2^8,3)$ and COD$(16;(16)^2 8^2)$ are listed in Table 1. In the generated design, the two 16-level columns achieve stratifications on 8×2, 2×8 and 4×4 grids in a two-dimensional projection; the two 8-level columns achieve stratifications on 4×2 and 2×4 grids in a two-dimensional projection; and the whole design achieves stratifications on 4×2 and 2×4 grids in two-dimensional projections and achieves stratifications on $2\times 2\times 2$ grids in three-dimensional projections.

Example 3.3. Suppose A is an OA$(16;2^{15},2)$, where the columns 1–4, 5–8 and 9–12 form three full 2^4 factorial designs, respectively. Let A_1 consist of the first 14 columns of A, A_2 consist of the first 12 columns of A, and

$$H_4^2 = \begin{pmatrix} 1 & 2 & 4 & 8 \\ 8 & 4 & -2 & -1 \end{pmatrix}^{\mathrm{T}} \quad \text{and} \quad H_2^2 = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Then $D = AR^2_{(4,4,4,2,1)}$, $D_1 = A_1 R^2_{(4,4,4,2)}$ and $D_2 = A_2 R^2_{(4,4,4)}$ are COD$(16;(16)^6 4^2 2^1)$, COD$(16;(16)^6 4^2)$ and COD$(16;(16)^6)$, respectively, where

$$R^2_{(4,4,4,2)} = \mathrm{diag}\{H_4^2, H_4^2, H_4^2, H_2^2\}, \quad R^2_{(4,4,4)} = \mathrm{diag}\{H_4^2, H_4^2, H_4^2\}$$

and

$$R^2_{(4,4,4,2,1)} = \begin{pmatrix} H_4^2 & & & & \\ & H_4^2 & & & \\ & & H_4^2 & & \\ & & & H_2^2 & \\ & & & & 1 \end{pmatrix}.$$

The OA$(16;2^{15},2)$ and COD$(16;(16)^6 4^2 2^1)$ are listed in Table 2. Furthermore, any two columns of D, D_1 and D_2 are orthogonal and the two 16-level columns obtained by rotating a full 2^4 factorial design achieve stratifications on 2×8, 8×2 and 4×4 grids in a two-dimensional projection.

Table 2 OA$(16; 2^{15}, 2)$ and COD$(16; (16)^6 4^2 2^1)$ in Example 3.3

Run	OA$(16; 2^{15}, 2)$															COD$(16; (16)^6 4^2 2^1)$								
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	15	9	15	9	15	9	3	1	1
2	1	1	1	−1	1	1	1	−1	1	−1	−1	−1	−1	−1	−1	−1	11	−1	11	−13	7	−3	−1	−1
3	1	1	−1	1	1	−1	−1	1	−1	1	1	−1	−1	−1	−1	7	13	3	5	−3	−5	−3	−1	−1
4	1	1	−1	−1	1	−1	−1	−1	−1	−1	−1	1	1	1	1	−9	15	−13	7	1	−11	3	1	1
5	1	−1	1	1	−1	1	−1	1	−1	−1	−1	−1	1	1	−1	11	1	5	−3	−15	−9	3	1	−1
6	1	−1	1	−1	−1	1	−1	−1	−1	1	1	1	−1	−1	1	−5	3	−11	−1	13	−7	−3	−1	1
7	1	−1	−1	1	−1	−1	1	1	1	−1	−1	1	−1	−1	1	3	5	9	−15	3	5	−3	−1	1
8	1	−1	−1	−1	−1	−1	1	−1	1	1	1	−1	1	1	−1	−13	7	−7	−13	−1	11	3	1	−1
9	−1	1	1	1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1	13	−7	−15	−9	7	13	−1	3	−1
10	−1	1	1	−1	−1	−1	−1	1	1	−1	1	−1	−1	1	1	−3	−5	1	−11	−5	3	1	−3	1
11	−1	1	−1	1	−1	1	1	−1	−1	1	−1	−1	−1	1	1	5	−3	−3	−5	−11	−1	1	−3	1
12	−1	1	−1	−1	−1	1	1	1	−1	−1	1	1	1	−1	−1	−11	−1	13	−7	9	−15	−1	3	−1
13	−1	−1	1	1	1	−1	1	−1	−1	−1	1	−1	1	−1	1	9	−15	−5	3	−7	−13	−1	3	1
14	−1	−1	1	−1	1	−1	1	1	−1	1	−1	1	−1	1	−1	−7	−13	11	1	5	−3	1	−3	−1
15	−1	−1	−1	1	1	1	−1	−1	1	−1	1	1	−1	1	−1	1	−11	−9	15	11	1	1	−3	−1
16	−1	−1	−1	−1	1	1	−1	1	1	1	−1	−1	1	−1	1	−15	−9	7	13	−9	15	−1	3	1

Note. The first 14 columns of OA$(16; 2^{15}, 2)$ can generate a COD$(16; (16)^6 4^2)$ and the first 12 columns can generate a COD$(16; (16)^6)$.

4 Construction of CODs with multi-dimensional stratifications from asymmetric OAs

This section provides a method to construct CODs with multi-dimensional stratifications by rotating asymmetric OAs. Suppose B is an OA$(n; m, (q_1)^{m_1}(q_2)^{m_2}\cdots(q_u)^{m_u}, e)$ with $e \geqslant 2$ and $m = \sum_{i=1}^{u} m_i$. Without loss of generality, we only consider the situation of $u = 2$ (i.e., OA$(n; m, (q_1)^{m_1}(q_2)^{m_2}, e)$) in this section. The results can be straightly extended to the case of $u \geqslant 3$.

Let

$$B = (B_1, B_2) \tag{4.1}$$

be an OA$(n; m, (q_1)^{m_1}(q_2)^{m_2}, e)$ with $B_i = (B_i^1, \ldots, B_i^{v_i}, B_i^*)$, where for $i = 1, 2$ and $j = 1, 2, \ldots, v_i$, B_i^j is an OA$(n; (q_i)^{f_j^i}, t_j^i)$, B_i^* is an OA$(n; (q_i)^{r_i}, t_{v_i+1}^i)$, $t_j^i \geqslant e$, $t_{v_i+1}^i < e$ and

$$f_j^i = \begin{cases} t_j^i, & t_j^i \text{ is even}, \\ t_j^i + 1, & t_j^i \text{ is odd}. \end{cases}$$

Define X_2 as

$$X_2 = R^{q_1 q_2}_{(t_1^1, \ldots, t_{v_1}^1, 1_{r_1}, t_1^2, \ldots, t_{v_2}^2, 1_{r_2})} = \mathrm{diag}\{H^{q_1}_{t_1^1}, \ldots, H^{q_1}_{t_{v_1}^1}, I_{r_1}, H^{q_2}_{t_1^2}, \ldots, H^{q_2}_{t_{v_2}^2}, I_{r_2}\}, \tag{4.2}$$

where $H^{q_i}_{t_j^i}$ has the same form as in (3.3) or (3.4).

Theorem 4.1. *For the B in* (4.1) *and X_2 in* (4.2)*, define $D = BX_2$.*

(i) *If $e \geqslant 3$, then D is a second-order orthogonal* COD$(n; (s_1)^2 \cdots (s_{v_1})^2 (s_{v_1+1})^2 \cdots (s_{v_1+v_2})^2 (q_1)^{r_1}$ $(q_2)^{r_2})$ *which achieves stratifications on $q_i^{u_1} \times \cdots \times q_{i'}^{u_g}$ $(i, i' \in \{1, 2\})$ grids in g-dimensional projections, where $s_j = (q_1)^{t_j^1}$ for $1 \leqslant j \leqslant v_1$ and $s_{v_1+j} = (q_2)^{t_j^2}$ for $1 \leqslant j \leqslant v_2$, $g \leqslant e$, $\sum_{l=1}^{g} u_l = e$. The two $(q_i)^{t_j^i}$-level columns achieve stratifications on $q_i^{u_1} \times q_i^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = t_j^i$, $i = 1, 2$, $j = 1, 2, \ldots, v_i$.*

(ii) *If $e = 2$, then D is a* COD$(n; (s_1)^2 \cdots (s_{v_1})^2 (s_{v_1+1})^2 \cdots (s_{v_1+v_2})^2 (q_1)^{r_1} (q_2)^{r_2})$*, where $s_j = (q_1)^{t_j^1}$ for $1 \leqslant j \leqslant v_1$ and $s_{v_1+j} = (q_2)^{t_j^2}$ for $1 \leqslant j \leqslant v_2$. The two $(q_i)^{t_j^i}$-level columns achieve stratifications on $q_i^{u_1} \times q_i^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = t_j^i$, $i = 1, 2$, $j = 1, 2, \ldots, v_i$.*

The following example is used to illustrate the construction.

Example 4.2. Suppose B is an $\mathrm{OA}(64;10,4^4 2^6,3)$, which is shown in Table 3. Let

$$R^{4,2}_{(3,3,1_2)} = \begin{pmatrix} H_3^4 & & \\ & H_3^2 & \\ & & I_2 \end{pmatrix},$$

where

$$H_3^4 = \begin{pmatrix} 1 & 4 & 16 & 0 \\ 16 & 0 & -1 & 4 \end{pmatrix}^{\mathrm{T}} \quad \text{and} \quad H_3^2 = \begin{pmatrix} 1 & 2 & 4 & 0 \\ 4 & 0 & -1 & 2 \end{pmatrix}^{\mathrm{T}}.$$

Following Theorem 4.1, $D = BR^{4,2}_{(3,3,1_2)}$ is a second-order orthogonal $\mathrm{COD}(64;(64)^2 8^2 2^2)$ design, as shown in Table 4, and it possesses the following property of stratifications: (i) the two 64-level columns achieve stratifications on $4^{u_1} \times 4^{u_2}$ grids in a two-dimensional projection for $u_1 + u_2 = 3$; (ii) the two 8-level columns achieve stratifications on $2^{u_1} \times 2^{u_2}$ grids in a two-dimensional projection for $u_1 + u_2 = 3$; (iii) any two columns with different levels (e.g., a 64-level column and an 8-level column) achieve stratifications on $4^{u_1} \times 2^{u_2}$ grids in a two-dimensional projection for $u_1 + u_2 = 3$; and (iv) any three columns achieve a stratification on a $2 \times 2 \times 2$ grid in a three-dimensional projection.

Table 3 $\mathrm{OA}(64;10,4^4 2^6,3)$ in Example 4.2

Run	1	2	3	4	5	6	7	8	9	10	Run	1	2	3	4	5	6	7	8	9	10
1	−3	−3	−3	−3	−1	−1	−1	−1	−1	−1	33	1	−3	−3	1	1	−1	1	1	−1	1
2	−3	−3	−1	−1	−1	−1	1	1	1	1	34	1	−3	−1	3	1	−1	−1	−1	1	−1
3	−3	−3	1	1	1	1	−1	−1	1	1	35	1	−3	1	−1	−1	1	−1	1	−1	−1
4	−3	−3	3	3	1	1	1	1	−1	−1	36	1	−3	3	−3	−1	1	1	−1	1	1
5	−3	−1	−3	−1	1	1	−1	1	−1	1	37	1	−1	−3	3	−1	1	1	−1	−1	−1
6	−3	−1	−1	−3	1	1	1	−1	1	−1	38	1	−1	−1	1	−1	1	−1	1	1	1
7	−3	−1	1	3	−1	−1	−1	1	1	−1	39	1	−1	1	−3	1	−1	−1	−1	−1	1
8	−3	−1	3	1	−1	−1	1	−1	−1	1	40	1	−1	3	−1	1	−1	1	1	1	−1
9	−3	1	−3	1	−1	1	1	1	1	−1	41	1	1	−3	−1	−1	−1	−1	−1	1	1
10	−3	1	−1	3	−1	1	−1	−1	−1	1	42	1	1	−1	−3	−1	−1	1	1	−1	−1
11	−3	1	1	−1	1	−1	1	−1	−1	−1	43	1	1	1	3	1	1	1	1	1	1
12	−3	1	3	−3	1	−1	−1	1	1	1	44	1	1	3	1	1	1	−1	−1	−1	−1
13	−3	3	−3	3	1	−1	1	−1	1	1	45	1	3	−3	−3	1	1	−1	1	1	−1
14	−3	3	−1	1	1	−1	−1	1	−1	−1	46	1	3	−1	−1	1	1	1	−1	−1	1
15	−3	3	1	−3	−1	1	1	1	−1	1	47	1	3	1	1	−1	−1	1	−1	1	−1
16	−3	3	3	−1	−1	1	−1	−1	1	−1	48	1	3	3	3	−1	−1	−1	1	−1	1
17	−1	−3	−3	−1	1	1	1	−1	1	−1	49	3	−3	−3	3	−1	1	−1	1	1	1
18	−1	−3	−1	−3	1	1	−1	1	−1	1	50	3	−3	−1	1	−1	1	1	−1	−1	−1
19	−1	−3	1	3	−1	−1	1	−1	−1	1	51	3	−3	1	−3	1	−1	1	1	1	−1
20	−1	−3	3	1	−1	−1	−1	1	1	−1	52	3	−3	3	−1	1	−1	−1	−1	−1	1
21	−1	−1	−3	−3	−1	−1	1	1	1	1	53	3	−1	−3	1	1	−1	−1	−1	1	−1
22	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	54	3	−1	−1	3	1	−1	1	1	−1	1
23	−1	−1	1	1	1	1	1	1	−1	−1	55	3	−1	1	−1	−1	1	1	−1	1	1
24	−1	−1	3	3	1	1	−1	−1	1	1	56	3	−1	3	−3	−1	1	−1	1	−1	−1
25	−1	1	−3	3	1	−1	−1	1	−1	−1	57	3	1	−3	−3	1	1	1	−1	−1	1
26	−1	1	−1	1	1	−1	1	−1	1	1	58	3	1	−1	−1	1	1	−1	1	1	−1
27	−1	1	1	−3	−1	1	−1	−1	1	−1	59	3	1	1	1	−1	−1	−1	1	−1	1
28	−1	1	3	−1	−1	1	1	1	−1	1	60	3	1	3	3	−1	−1	1	−1	1	−1
29	−1	3	−3	1	−1	1	−1	−1	−1	1	61	3	3	−3	−1	−1	−1	1	1	−1	−1
30	−1	3	−1	3	−1	1	1	1	1	−1	62	3	3	−1	−3	−1	−1	−1	−1	1	1
31	−1	3	1	−1	1	−1	−1	1	1	1	63	3	3	1	3	1	1	−1	−1	−1	−1
32	−1	3	3	−3	1	−1	1	−1	−1	−1	64	3	3	3	1	1	1	1	1	1	1

Table 4 COD(64; $(64)^2 8^2 2^2$) in Example 4.2

Run	1	2	3	4	5	6	Run	1	2	3	4	5	6
1	−63	−57	−7	−5	−1	−1	33	−59	23	3	5	−1	1
2	−31	−51	1	−3	1	1	34	−27	29	−5	3	1	−1
3	1	−45	−1	3	1	1	35	5	11	−3	−1	−1	−1
4	33	−39	7	5	−1	−1	36	37	1	5	−7	1	1
5	−55	−49	−1	7	−1	1	37	−51	31	5	−7	−1	−1
6	−23	−59	7	1	1	−1	38	−19	21	−3	−1	1	1
7	9	−37	−7	−1	1	−1	39	13	3	−5	3	−1	1
8	41	−47	1	−7	−1	1	40	45	9	3	5	1	−1
9	−47	−41	5	−3	1	−1	41	−43	15	−7	−5	1	1
10	−15	−35	−3	−5	−1	1	42	−11	5	1	−3	−1	−1
11	17	−53	3	1	−1	−1	43	21	27	7	5	1	1
12	49	−63	−5	7	1	1	44	53	17	−1	3	−1	−1
13	−39	−33	3	1	1	1	45	−35	7	−1	7	1	−1
14	−7	−43	−5	7	−1	−1	46	−3	13	7	1	−1	1
15	25	−61	5	−3	−1	1	47	29	19	1	−7	1	−1
16	57	−55	−3	−5	1	−1	48	61	25	−7	−1	−1	1
17	−61	−17	7	1	1	−1	49	−57	63	−3	−1	1	1
18	−29	−27	−1	7	−1	1	50	−25	53	5	−7	−1	−1
19	3	−5	1	−7	−1	1	51	7	35	3	5	1	−1
20	35	−15	−7	−1	1	−1	52	39	41	−5	3	−1	1
21	−53	−25	1	−3	1	1	53	−49	55	−5	3	1	−1
22	−21	−19	−7	−5	−1	−1	54	−17	61	3	5	−1	1
23	11	−13	7	5	−1	−1	55	15	43	5	−7	1	1
24	43	−7	−1	3	1	1	56	47	33	−3	−1	−1	−1
25	−45	−1	−5	7	−1	−1	57	−41	39	7	1	−1	1
26	−13	−11	3	1	1	1	58	−9	45	−1	7	1	−1
27	19	−29	−3	−5	1	−1	59	23	51	−7	−1	−1	1
28	51	−23	5	−3	−1	1	60	55	57	1	−7	1	−1
29	−37	−9	−3	−5	−1	1	61	−33	47	1	−3	−1	−1
30	−5	−3	5	−3	1	−1	62	−1	37	−7	−5	1	1
31	27	−21	−5	7	1	1	63	31	59	−1	3	−1	−1
32	59	−31	3	1	−1	−1	64	63	49	7	5	1	1

5 Construction of orthogonal LHDs with multi-dimensional stratifications

In this section, we will propose a method to construct orthogonal LHDs with multi-dimensional stratifications using regular factorial designs with prime power levels.

For a regular saturated factorial design of strength two with prime power levels, Ai et al. [1] provided a general scheme to partition the columns into groups each being a full factorial design. But for a regular factorial design of strength larger than two, there is no general partition scheme and thus the grouping process has to be carried out by the computer. In particular, the OA($q^3; q^{q+1}, 3$), constructed through the Bush's method (see [9]), can be partitioned into $\lfloor (q+1)/3 \rfloor$ groups each being a $q^3 \times 3$ full factorial design with q levels. By using a grouped design, we can construct orthogonal LHDs with multi-dimensional stratifications. Moreover, if there exists a small orthogonal LHD, then a larger orthogonal LHD with multi-dimensional stratifications can be obtained.

For a prime power q, assume that the regular factorial design of strength $e \geqslant 2$,

$$C = \mathrm{OA}(q^m; q^{km}, e) \tag{5.1}$$

can be partitioned into k groups $C = (C_1, C_2, \ldots, C_k)$ with each C_i $(i = 1, \ldots, k)$ being a full factorial design $\mathrm{OA}(q^m; q^m, m)$. For an even m, define

$$X_3 = I_k \otimes H_m^q, \tag{5.2}$$

where $\otimes$ denotes the Kronecker product and H_m^q has the same form as in (3.4). Then we can obtain the following results.

Theorem 5.1. *For the C in* (5.1) *and X_3 in* (5.2), *define $D = CX_3$.*

(i) *If $e \geqslant 3$, then D is a second-order orthogonal $L(q^m, 2k)$ which achieves stratifications on $q^{u_1} \times \cdots \times q^{u_g}$ grids in g-dimensional projections, where $g \leqslant e$ and $\sum_{l=1}^{g} u_l = e$. The two columns in the same rotation part* (*i.e., the* $(2i-1)$*-th and* $(2i)$*-th columns of D, where $i = 1, 2, \ldots, k$*) *achieve stratifications on $q^{u_1} \times q^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = m$.*

(ii) *If $e = 2$, then D is an orthogonal $L(q^m, 2k)$. The two columns in the same rotation part* (*i.e., the* $(2i-1)$*-th and* $(2i)$*-th columns of D, where $i = 1, 2, \ldots, k$*) *achieve stratifications on $q^{u_1} \times q^{u_2}$ grids in a two-dimensional projection, where $u_1 + u_2 = m$.*

Remark 5.2. For the case of odd m in C with strength two, we can construct orthogonal $L(q^m, k)$, which only achieves stratifications on $q \times q$ grids when projected onto two dimensions if we take $H_m^q = (1, q, \ldots, q^{m-2}, q^{m-1})$. On the other hand, for the case of odd m in C with strength $e \geqslant 3$, a second-order orthogonal $L(q^m, k)$ can be obtained which achieves stratifications on $q^{u_1} \times \cdots \times q^{u_g}$ grids in g-dimensional projections, where $g \leqslant e$ and $\sum_{l=1}^{g} u_l = e$.

Let C be of the form in (5.1) and $E = (\gamma_1, \gamma_2, \ldots, \gamma_d)$ be an orthogonal $L(q, d)$. For $l = 1, 2, \ldots, d$, the matrix $C^{(l)}$ is obtained by replacing the q levels of C with $\gamma_{1l}, \ldots, \gamma_{ql}$ in γ_l. Construct a $q^m \times 2kd$ matrix

$$D = (C^{(1)} X_3, C^{(2)} X_3, \ldots, C^{(d)} X_3). \tag{5.3}$$

According to [1, Theorem 2.2], we have the following results.

Corollary 5.3. (i) *If C is an* OA *of strength three and E is a second-order orthogonal* LHD, *then the D constructed in* (5.3) *is a second-order orthogonal $L(q^m, 2kd)$.*

(ii) *If C is an* OA *of strength two and E is an orthogonal* LHD, *then the D constructed in* (5.3) *is an orthogonal $L(q^m, 2kd)$.*

According to Corollary 5.3, we can construct larger (second-order) orthogonal LHDs with multi-dimensional stratifications. An example is presented to illustrate the details of the construction.

Example 5.4. Consider the construction of a second-order orthogonal LHD of 9^4 runs. For the case of $m \geqslant 4$, it is hard to partition an OA of strength three into groups of q^m-run full factorial designs for m factors. Therefore, the needed regular factorial designs have to be obtained by computer search. In this example, we use $C = \mathrm{OA}(9^4; 9^{36}, 3)$, which can be partitioned into 9 groups each being a $9^4 \times 4$ full factorial design with 9 levels (see [1]), to construct a second-order orthogonal LHD of 9^4 runs with multi-dimensional stratifications.

Let $X_3 = I_9 \otimes H_4^9$. From Theorem 5.1, we know that $D = CX_3$ is a second-order orthogonal $L(9^4, 18)$. If E is taken to be the second-order orthogonal $L(9, 4)$ (see [17]) of the form

$$E = 2\begin{pmatrix} 1 & 2 & 3 & 4 & 0 & -1 & -2 & -3 & -4 \\ 2 & -1 & 4 & -3 & 0 & -2 & 1 & -4 & 3 \\ 3 & -4 & -1 & 2 & 0 & -3 & 4 & 1 & -2 \\ 4 & 3 & -2 & -1 & 0 & -4 & -3 & 2 & 1 \end{pmatrix}^{\mathrm{T}},$$

according to Corollary 5.3, the D constructed in (5.3) is a second-order orthogonal $L(9^4, 72)$. In particular, the $(2i-1)$-th and $(2i)$-th columns ($i = 1, 2, \ldots, 36$) of D achieve stratifications on 9×9^3, $9^2 \times 9^2$ and $9^3 \times 9$ grids in a two-dimensional projection.

Some new (second-order) orthogonal LHDs with multi-dimensional stratifications are listed in Table 5. In this table, the first column displays the newly constructed orthogonal or second-order orthogonal LHDs.

Table 5 New (second-order) orthogonal LHDs with multi-dimensional stratifications

LHD (D)	Grouped OA (C)	H_m^q in X_3	Small LHD (E)	Methods
$L(9^4, 2050)^*$	$\mathrm{OA}(9^4; 9^{820}, 2)$	$(9, 4)$	$L(9, 5)$	Th 5.1(ii) and Cor 5.3(ii)
$L(5^4, 156)^*$	$\mathrm{OA}(5^4; 5^{156}, 2)$	$(5, 4)$	$L(5, 2)$	Th 5.1(ii) and Cor 5.3(ii)
$L(7^4, 600)^*$	$\mathrm{OA}(7^4; 7^{400}, 2)$	$(7, 4)$	$L(7, 3)$	Th 5.1(ii) and Cor 5.3(ii)
$L((11)^4, 5124)^*$	$\mathrm{OA}((11)^4; (11)^{1464}, 2)$	$(11, 4)$	$L(11, 7)$	Th 5.1(ii) and Cor 5.3(ii)
$L(3^4, 4)$	$\mathrm{OA}(3^4; 3^8, 3)$	$(3, 4)$		Th 5.1(i)
$L(3^5, 4)$	$\mathrm{OA}(3^5; 3^{20}, 3)$	$(3, 5)$		Th 5.1(i)
$L(9^3, 12)$	$\mathrm{OA}(9^3; 9^9, 3)$	$(9, 3)$	$L(9, 4)$	Th 5.1(i) and Cor 5.3(i)
$L(9^4, 72)$	$\mathrm{OA}(9^4; 9^{36}, 3)$	$(9, 4)$	$L(9, 4)$	Th 5.1(i) and Cor 5.3(i)
$L(5^3, 4)$	$\mathrm{OA}(5^3; 5^6, 3)$	$(5, 3)$	$L(5, 2)$	Th 5.1(i) and Cor 5.3(i)
$L(5^4, 24)$	$\mathrm{OA}(5^4; 5^{24}, 3)$	$(5, 4)$	$L(5, 2)$	Th 5.1(i) and Cor 5.3(i)
$L(5^5, 22)$	$\mathrm{OA}(5^5; 5^{55}, 3)$	$(5, 5)$	$L(5, 2)$	Th 5.1(i) and Cor 5.3(i)
$L((25)^3, 32)$	$\mathrm{OA}((25)^3; (25)^{24}, 3)$	$(25, 3)$	$L(25, 4)$	Th 5.1(i) and Cor 5.3(i)
$L(7^3, 6)$	$\mathrm{OA}(7^3; 7^6, 3)$	$(7, 3)$	$L(7, 3)$	Th 5.1(i) and Cor 5.3(i)
$L(7^4, 66)$	$\mathrm{OA}(7^4; 7^{44}, 3)$	$(7, 4)$	$L(7, 3)$	Th 5.1(i) and Cor 5.3(i)
$L(7^5, 42)$	$\mathrm{OA}(7^5; 7^{70}, 3)$	$(7, 5)$	$L(7, 3)$	Th 5.1(i) and Cor 5.3(i)
$L((11)^3, 12)$	$\mathrm{OA}((11)^3; (11)^{12}, 3)$	$(11, 3)$	$L(11, 3)$	Th 5.1(i) and Cor 5.3(i)
$L((11)^4, 174)$	$\mathrm{OA}((11)^4; (11)^{116}, 3)$	$(11, 4)$	$L(11, 3)$	Th 5.1(i) and Cor 5.3(i)

Note. In the first column, designs with $*$ are orthogonal LHDs, and designs without $*$ are second-order orthogonal LHDs. In the third column, we list "(q, m)" to represent H_m^q, where H_m^q takes the form of (3.4) for even m and $(1, q, \ldots, q^{m-2}, q^{m-1})^{\mathrm{T}}$ for odd m.

All of these designs are new and possess the multi-dimensional stratifications as described above. The second column displays the needed OAs of strength two or three, abbreviated as "Grouped OA", which can be partitioned into groups each being a full factorial design. In particular, the OAs of strength two are all regular saturated factorial designs. The OAs of the form $\mathrm{OA}(q^3; q^{q+1}, 3)$ can be constructed by Bush's method and the others are obtained by computer search. The third column lists the parameters (q, m) to represent the H_m^q for X_3 in (5.2), where H_m^q takes the form of (3.4) for even m and $(1, q, \ldots, q^{m-2}, q^{m-1})^{\mathrm{T}}$ for odd m. Small orthogonal LHDs are listed in the forth column, which can be obtained from [5,11,15,17]. The used methods are listed in the last column, where we use "Th" and "Cor" to denote "Theorem" and "Corollary", respectively.

6 Concluding remarks

In this paper, we propose some methods to construct column-orthogonal designs (CODs) and orthogonal Latin hypercube designs (LHDs) with good stratifications by rotating symmetric and asymmetric orthogonal arrays (OAs). These methods are easy to implement and the resulting designs have many attractive properties. The newly constructed CODs not only guarantee that the estimate of each linear effect is uncorrelated with all other linear effects and the second-order effects if an OA with strength larger than two is rotated, but also preserve stratifications on multiple dimensions. Therefore, such designs can be used for computer experiments especially for the case where some factors need to be studied in more details than others (see [3]). Orthogonal and second-order orthogonal LHDs can be obtained by rotating a class of grouped regular prime power-level factorial designs. In addition, the regular factorial designs used for constructing orthogonal LHDs (not for second-order orthogonal LHDs) are all saturated.

Before ending this section, it should be mentioned that there are plenty of alternative projection designs from a given OA when constructing CODs. To make it clear, we take the saturated $\mathrm{OA}(16; 2^{15}, 2)$ (in Example 3.3) for an illustration. Without loss of generality, we suppose that a COD with 16-level factors and multi-dimensional stratifications is required. Following our construction, we need to partition the columns of the saturated $\mathrm{OA}(16; 2^{15}, 2)$ to get three four-factor projection designs of strength four as the rotation parts. We now focus on the number of such partitions. There are altogether 2,627,625 ($=\binom{15}{4}\times\binom{11}{4}\times\binom{7}{4}/6$) partition candidates. Here, we use the generalized word length pattern (GWLP), $\{B_1(D), B_2(D), \ldots, B_m(D)\}$, proposed by [22] to obtain the strength of each projection design. The value $B_j(D)$ measures the overall aliasing between all j-factor interactions and the general mean in design D, where $j = 1, 2, \ldots, m$ and m denotes the number of factors. If $B_t(D) = 0$ for $t = 1, 2, \ldots, e$ and $B_{e+1}(D) \neq 0$, then design D has strength e. With the above method, we use a MATLAB program to search the partitions which result in three full 2^4 factorial designs, and finally 594,160 such partitions are found. That is to say, 594,160 CODs with the same orthogonality and the property of multi-dimensional stratifications can be constructed. The ratio is 594,160/2,627,625 = 0.2261. One can arbitrarily choose one of these CODs for practical use. Note that the uniformity and column-orthogonality do not necessarily agree with each other, i.e., the low correlations among the effects of a design cannot guarantee the uniformity of its points, and vice versa (see [19]). Although the newly constructed CODs guarantee the nice column-orthogonality and property of stratifications, they may not be optimal under some uniformity criteria. Therefore, if we want to further evaluate these resulting CODs, an additional uniformity criterion can be adopted alternatively such as one of the discrepancy measures (see [6]) and distance criteria (see [10]). In this sense, the COD with the best uniformity is preferred. Once the levels of CODs are given (say $\mathrm{COD}(n; (q^{t_1})^2(q^{t_2})^2\cdots(q^{t_v})^2q^r)$), it is not hard to find the projection designs (or rotation parts) by a searching program.

In the following, an exhaustive partition algorithm is provided for illustrating how to divide an $\mathrm{OA}(n; q^m, e)$ with $e \geqslant 2$ (denoted by A) into projection designs with different strengths, i.e., $A = (A_1, \ldots, A_v, A^*)$ with A_i being an $\mathrm{OA}(n; q^{f_i}, t_i)$ for $i = 1, 2, \ldots, v$ and A^* being an $\mathrm{OA}(n; q^r, t_{v+1})$.

Algorithm 6.1 (Partition algorithm for symmetric OAs). **Step 1.** Give an $\mathrm{OA}(n; q^m, e)$, f_i and t_i with $q^{t_i} \,|\, n$, where $q^{t_i} \,|\, n$ means "q^{t_i} divides n". Set $f_0 = 0$ and $i = 1$.

Step 2. Choose f_i columns from the remaining $m-\sum_{j=0}^{i-1} f_j$ columns of the $\mathrm{OA}(n; q^m, e)$ and generate

$$N_i = \binom{m-\sum_{j=0}^{i-1} f_j}{f_i}$$

possible projection designs $A_i^1, \ldots, A_i^{N_i}$.

Step 3. For $j = 1$ to N_i, check whether A_i^j has strength t_i by GWLP. If yes, retain this A_i^j. If no, drop this A_i^j.

Step 4. If $i < v$, for each retained A_i^j in the above step, set $i = i + 1$ and go to Step 2. If $i = v$, record the current partitions $(A_1, \ldots, A_v, A^*)$ for all retained A_v^j with A^* consisting of the remaining $m - \sum_{j=0}^{v} f_j$ columns of the $\mathrm{OA}(n; q^m, e)$.

Note that the partition algorithm for asymmetric OAs is similar and thus omitted. Algorithm 6.1 along with the above illustration example shows that, for a given OA and the parameter settings of the needed COD, the number of partition candidates is usually very large, and it will be a time-costing task if we want to enumerate all these candidates.

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Appendix A Proofs of theorems

The following lemmas will be helpful in proving the theorems in this paper.

Lemma A.1 (See [19]). *Suppose A is an $n \times m$ matrix with*

$$1_n^{\mathrm{T}} A = 0_{1\times m} \quad \textit{and} \quad A^{\mathrm{T}} A = cI_m,$$

where 1_n denotes an $n \times 1$ vector with all entries being unity and c is a constant. Let $D = AX$, where X is a matrix with m rows. Then

(i) *if X is a column-orthogonal matrix, then D is also a column-orthogonal matrix*; *and*

(ii) *if A is a second-order orthogonal design, then the estimates of all linear main effects of D are uncorrelated with the estimates of all quadratic effects and bilinear interactions. Furthermore, if X is a column-orthogonal matrix, then D is a second-order orthogonal matrix.*

Lemma A.2 (See [12]). *Let $A=(a_1,a_2,\ldots,a_m)$ be an* $\mathrm{OA}(s^m;s^m,m)$, *and*

$$d_k=(d_{k_1},d_{k_2},\ldots,d_{k_k})^{\mathrm{T}}$$

be a vector of $(s^{k-1},s^{k-2},\ldots,s,1)^{\mathrm{T}}$ up to sign changes, where k is a positive integer with $k\leqslant m$. Then

(i) *if $\sum_{r=1}^m a_rd_{m_r}$ is collapsed into s^u levels, it becomes $\sum_{r=1}^u \mathrm{sgn}(d_{m_r})a_rs^{u-r}$, where $\mathrm{sgn}(x)$ means the sign of x, $1\leqslant u\leqslant m-1$. In particular, if $\sum_{r=1}^m a_rs^{m-r}$ is collapsed into s^u levels, it becomes $\sum_{r=1}^u a_rs^{u-r}$;*

(ii) *for $g\leqslant m$ and $u_1+\cdots+u_g=m$,*

$$\left(\sum_{r=1}^{u_1}a_rd_{u_{1_r}},\sum_{r=1}^{u_2}a_{u_1+r}d_{u_{2_r}},\ldots,\sum_{r=1}^{u_g}a_{\sum_{j=1}^{g-1}u_j+r}d_{u_{g_r}}\right)$$

is an $\mathrm{OA}(s^m;g,(s^{u_1})^1\cdots(s^{u_g})^1,g)$. *In particular,*

$$\left(\sum_{r=1}^{u_1}a_rs^{u_1-r},\sum_{r=1}^{u_2}a_{u_1+r}s^{u_2-r},\ldots,\sum_{r=1}^{u_g}a_{\sum_{j=1}^{g-1}u_j+r}s^{u_g-r}\right)$$

is an $\mathrm{OA}(s^m;g,(s^{u_1})^1\cdots(s^{u_g})^1,g)$.

Remark A.3. Following the proof of Lemma A.2, the level collapsed in this paper is also done by

$$2\lfloor(i+q^t)/(2q^{t-u})\rfloor-q^u+1,$$

if q^t levels are collapsed into q^u levels.

Lemma A.4 (See [2]). *Let A be a full factorial design with m factors each of s levels and u be an m-dimensional vector. Then the vector Au is equally spaced if and only if u is a permutation of*

$$\{\pm\lambda,\pm\lambda s,\pm\lambda s^2,\ldots,\pm\lambda s^{m-1}\}$$

up to sign specification, where λ is any nonzero constant.

Proof of Theorem 3.1. (i) Since X_1 is column-orthogonal and $A=(a_1,a_2,\ldots,a_m)$ is an $\mathrm{OA}(n;q^m,e)$ of strength $e\geqslant 3$,

$$D=AX_1=(d_1,d_2,\ldots,d_{2v+r})$$

is a second-order orthogonal COD according to Lemma A.1. Let

$$D=(D_1,D_2,\ldots,D_v,D^*),$$

where $D_i=(d_{2i-1},d_{2i})$, $i=1,2,\ldots,v$, and D^* contains the remaining r columns. Moreover, let $H^q_{t_i}=(h^{t_i}_1,h^{t_i}_2)$, where $h^{t_i}_k=(h^{t_i}_{k_1},h^{t_i}_{k_2},\ldots,h^{t_i}_{k_{f_i}})'$, $i=1,2,\ldots,v,k=1,2$.

For the case of even $t_i,i=1,2,\ldots,v$, the two columns in D, d_{2i-1} and d_{2i}, can be expressed as

$$d_{2i-1}=\sum_{l=1}^{t_i}\mathrm{sgn}(h^{t_i}_{1_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{l-1}\quad\text{and}\quad d_{2i}=\sum_{l=1}^{t_i}\mathrm{sgn}(h^{t_i}_{2_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{t_i-l}.$$

According to Lemma A.2, when columns d_{2i-1} and d_{2i} are collapsed into q^{u_1} levels and q^{u_2} levels, respectively, they become

$$\sum_{l=t_i-u_1+1}^{t_i}\mathrm{sgn}(h^{t_i}_{1_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{l-t_i+u_1-1}\quad\text{and}\quad\sum_{l=1}^{u_2}\mathrm{sgn}(h^{t_i}_{2_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}.$$

Now we only need to consider the stratifications of D_i obtained by rotating $A_i=\mathrm{OA}(n;q^{f_i},t_i)$. For $u_1+u_2=t_i$, when q^{t_i}-level columns d_{2i-1} and d_{2i} are collapsed into q^{u_1} and q^{u_2} levels respectively, they can be expressed as

$$\sum_{l=u_2+1}^{t_i}\mathrm{sgn}(h^{t_i}_{1_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{l-u_2-1}\quad\text{and}\quad\sum_{l=1}^{u_2}\mathrm{sgn}(h^{t_i}_{2_l})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}.$$

For the case of odd t_i, the two columns in D, d_{2i-1} and d_{2i}, can be expressed as

$$d_{2i-1}=\sum_{l=1}^{t_i}\mathrm{sgn}(h_{1_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{l-1}$$

and

$$d_{2i}=\sum_{l=1}^{\frac{t_i-1}{2}}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{t_i-l}+\sum_{l=\frac{t_i+3}{2}}^{t_i}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{t_i-l}+a_{\sum_{j=1}^{i-1}f_j+t_i+1}q^{\frac{t_i-1}{2}}.$$

For $u_1+u_2=t_i$, when the q^{t_i}-level column d_{2i-1} is collapsed into q^{u_1} levels, it can be expressed as

$$2\left\lfloor\frac{d_{2i-1}+q^{t_i}}{2q^{t_i-u_1}}\right\rfloor-q^{u_1}+1=\sum_{l=u_2+1}^{t_i}\mathrm{sgn}(h_{1_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{l-u_2-1},$$

and when the q^{t_i}-level column d_{2i} is collapsed into q^{u_2} levels, it can be expressed as

$$2\left\lfloor\frac{d_{2i}+q^{t_i}}{2q^{t_i-u_2}}\right\rfloor-q^{u_2}+1=\begin{cases}\displaystyle\sum_{l=1}^{u_2}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}, & u_2\leqslant\dfrac{t_i-1}{2},\\ \displaystyle\sum_{l=1}^{u_2-1}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}+a_{\sum_{j=1}^{i-1}f_j+t_i+1}, & u_2=\dfrac{t_i+1}{2},\\ \displaystyle\sum_{l=1}^{\frac{t_i-1}{2}}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}+a_{\sum_{j=1}^{i-1}f_j+t_i+1}q^{u_2-\frac{t_i+1}{2}} & \\ \displaystyle\quad+\sum_{l=\frac{t_i+3}{2}}^{u_2}\mathrm{sgn}(h_{2_l}^{t_i})a_{\sum_{j=1}^{i-1}f_j+l}q^{u_2-l}, & u_2>\dfrac{t_i+1}{2}.\end{cases}$$

Meanwhile, since

$$A_i=(a_{\sum_{j=1}^{i-1}f_j+1},a_{\sum_{j=1}^{i-1}f_j+2},\ldots,a_{\sum_{j=1}^{i-1}f_j+f_i})$$

is an $\mathrm{OA}(n;q^{f_i},t_i)$, following Lemma A.2 and the above discussion, $D_i=(d_{2i-1},d_{2i})$ can be collapsed into an $\mathrm{OA}(n;2,q^{u_1}q^{u_2},2)$ for any given t_i. Hence, the two q^{t_i}-level columns achieve stratifications on $q^{u_1}\times q^{u_2}$ ($u_1+u_2=t_i$) grids in a two-dimensional projection. For the case of $\sum_{l=1}^{g}u_l=e$ where $2\leqslant g\leqslant e$, if any g columns come from different groups $D_{i1},D_{i2},\ldots,D_{ig}$, it is easy to verify that these g columns can be collapsed into an $\mathrm{OA}(n;g,q^{u_1}q^{u_2}\cdots q^{u_g},g)$ because any e columns $a_{i1},a_{i2},\ldots,a_{ie}$ of A form an $\mathrm{OA}(n;q^e,\ e)$. Thus D achieves stratifications on $q^{u_1}\times q^{u_2}\times\cdots\times q^{u_g}$ ($\sum_{l=1}^{g}u_l=e$) grids in g-dimensional projections.

(ii) The condition of (ii) is a special case of (i) for $e=2$; therefore we omit the details of proof. This completes the proof. □

Proof of Theorem 4.1. (i) As B is an $\mathrm{OA}(n;m,(q_1)^{m_1}(q_2)^{m_2},e)$ of $e\geqslant3$ and X_2 is a column-orthogonal matrix, we know that $D=BX_2$ is a second-order orthogonal $\mathrm{COD}(n;(s_1)^2\cdots(s_{v_1+v_2})^2(q_1)^{r_1}(q_2)^{r_2})$, where $s_j=(q_1)^{t_j^1}$ for $1\leqslant j\leqslant v_1$ and $s_{v_1+j}=(q_2)^{t_j^2}$ for $1\leqslant j\leqslant v_2$. We just need to prove the multi-dimensional stratifications of D.

Here, we only prove the property of stratifications of D when any g columns come from different groups $(D_{i1},D_{i2},\ldots,\ D_{ig})$, where $2\leqslant g\leqslant e$. Note that any e columns $(a_{i1},a_{i2},\ldots,a_{ie})$ of

$$B=\mathrm{OA}(n;m,(q_1)^{m_1}(q_2)^{m_2},e)$$

form an $\mathrm{OA}(n;e,(q_1)^{k_1}(q_2)^{k_2},e)$ with $k_1+k_2=e$. According to Lemma A.2, these g columns can be collapsed into an $\mathrm{OA}(n;g,\ (q_i)^{u_1}\cdots(q_{i'})^{u_g},g)$ where $i,i'\in\{1,2\}$ and $\sum_{l=1}^{g}u_l=e$. Thus D achieves stratifications on $q_i^{u_1}\times\cdots\times q_{i'}^{u_g}$ ($i,i'\in\{1,2\},\ \sum_{l=1}^{g}u_l=e$) grids in g-dimensional projections. The other case is similar to that of Theorem 3.1(i) and we omit its proof here.

(ii) The proof of (ii) is similar to that of (i) and is thus omitted. This completes the proof of Theorem 4.1. □

Proof of Theorem 5.1. (i) For the resulting design D, its second-order orthogonality and property of stratifications can be easily obtained from Theorem 3.1. Next, what we only need to prove is that D is an LHD. To show this, we note that C can be partitioned into k groups $C = (C_1, C_2, \ldots, C_k)$ with C_i being a full factorial design $\mathrm{OA}(q^m; q^m, m)$ for $i = 1, 2, \ldots, k$. Following Lemma A.4, each column of D has q^m equally spaced levels, so D is an LHD.

(ii) Part (ii) is a special case of Part (i), so the proof is omitted. This completes the proof of Theorem 5.1. □