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Mixed 2- and 2^r-level fractional factorial split-plot designs with clear effects

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ABSTRACT

Fractional factorial split-plot (FFSP) designs are often used when the levels of some factors are difficult to change or control. If not all experimental factors have the same number of levels, mixed-level designs are natural choices. This paper provides the necessary and sufficient conditions for mixed 2- and 2^r-level FFSP designs of resolution III or IV to contain clear main effects or two-factor interaction components. Particularly, the sufficient conditions are proved through constructing the corresponding designs. The new results here are more general and include Zhao and Chen (2012a,b)'s results as special cases for $r = 2$.

1. Introduction

Fractional factorial (FF) designs, due to the run size economy, are commonly used for factorial experiments in many fields, such as agriculture, medicine, chemistry, and high-tech industry. A 2^{n-k} design denotes a regular fraction with 2^{n-k} runs and n two-level factors, which has $n-k$ independent columns and is determined by k independent defining words. Such designs have a simple aliasing structure in that any two effects are either orthogonal to or fully aliased with each other. If the levels of some factors in an experiment are difficult to change or control, it may be impractical or even impossible to conduct the runs in a completely random order, which makes one consider a special design called the fractional factorial split-plot (FFSP) design. In general, an FFSP experiment has two types of factors: hard-to-change factors named whole-plot (WP) factors and the rest factors named sub-plot (SP) factors. One can randomly choose one of the level-settings of WP factors and then run all of the level-combinations of the SP factors in a random order with the WP factors fixed. The procedure above is repeated for other WP factor-level combinations until the experiment is complete.

In terms of ranking designs, minimum aberration and clear effects are two most commonly used criteria. Minimum aberration criterion was first introduced by Fries and Hunter (1980) for distinguishing FF designs and was studied extensively by Chen (1992), Chen and Hedayat (1996), Tang and Wu (1996) and others. Huang et al. (1998) and Bingham and Sitter (1999a,b) extended this criterion to ranking FFSP designs. The notion of clear effect was proposed by Wu and Chen (1992). A main effect or a two-factor interaction (2FI) of a 2^{n-k} design is said to be clear if it is not aliased with any other main effect or 2FI. A clear effect can be estimated unbiasedly under the weak assumption that effects involving

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three or more factors can be ignored. In any resolution IV 2^{n-k} design, all the main effects are clear. Then, a resolution IV 2^{n-k} design with the most clear two-factor interactions is preferred, see Wu and Hamada (2009, p. 217). For a resolution III 2^{n-k} design, we can assume the magnitude of the main effect is much larger than that of the 2FI according to effect hierarchy principle (Wu and Hamada, 2009, p. 172). When some background knowledge suggests that certain effects are potentially important, we will choose the designs with more clear effects. Yang et al. (2006) gave necessary and sufficient conditions for the existence of two-level FFSP designs containing various clear effects. Zi et al. (2006) derived the upper and lower bounds on the maximum numbers of clear effects for FFSP designs. For comprehensive discussions on the theory of clear effects, we refer the readers to Tang et al. (2002), Wu and Wu (2002), Ai and Zhang (2004), and Chen et al. (2006). Cheng and Tsai (2009) proposed a general and unified approach to the selection of regular FF designs with split-plot designs as a special case.

In practice, mixed-level FF designs are commonly used in the experiments when the numbers of levels of the factors are not all equal to each other. Zhao et al. (2008), Zhao and Zhang (2008) and Zhao and Zhao (2015) studied the mixed-level designs containing clear effects. Zhao and Chen (2012a,b) investigated the mixed-level split-plot designs with a four-level and some two-level factors and gave the conditions for such designs to have various clear effects.

High-level factors are often encountered in real life. One such example can be found in Example 1.6 of Montgomery (2013), which considers an eight-level factor in an experiment of designing a web page. If frequently changing the levels of the factor is not allowed, we will need an FFSP design with one eight-level factor in WP part. Usually, for designs containing clear effects, there are two main research topics. The first one considers the conditions for a design to have clear effects, and the second one is about the construction of the design with the most clear effects. This paper focuses on the first one and considers the $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1$ regular mixed-level FFSP design that contains n_1 2-level WP factors, n_2 2-level SP factors and one 2^r-level WP or SP factor. Section 2 gives such notation and definitions. Section 3 provides the conditions for $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_s$ designs to contain clear main effects and two-factor interaction components, and Section 4 discusses when $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_w$ designs contain clear main effects and two-factor interaction components, where the subscript s or w means the 2^r-level factor is an SP or WP factor. Section 5 contains some concluding remarks.

2. Notation and definitions

We first consider the construction of $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP designs with $p = p_1 + p_2$ independent columns, where $p_1 = n_1 - k_1, p_2 = n_2 - k_2$. Such a design contains n_1 WP factors, n_2 SP factors, and 2^p runs. Let

$$H = H(a_1, \ldots, a_{p_1}, b_1, \ldots, b_{p_2})$$

be the saturated design generated by the independent columns $a_1, \ldots, a_{p_1}, b_1, \ldots, b_{p_2}$. A $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design corresponds to two subsets of columns of H. Let $H_a = H(a_1, \ldots, a_{p_1})$ be the closed subset of $2^{p_1} - 1$ columns of H generated by $a_1, \ldots, a_{p_1}$. Hereafter, a closed subset of H means that the interaction of any two columns of it is still a column of it. We can choose n_1 columns with p_1 independent ones in $H_a(\subset H)$ as WP factors, and n_2 columns with another p_2 independent ones from $H \backslash H_a$ as SP factors. Denote the selected n_1 and n_2 columns by $B_1 = \{c_1, \ldots, c_{n_1}\}$ and $B_2 = \{c_{n_1+1}, \ldots, c_{n_1+n_2}\}$, respectively. Then (B_1, B_2) corresponds to a $2^{(n_1+n_2)-(k_1+k_2)}$ FFSP design, where $k_1 = n_1 - p_1$ and $k_2 = n_2 - p_2$.

Now, we consider the construction of a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_s$ FFSP design. A 2^r-level column can be obtained from $2^r - 1$ two-level columns, which compose a closed subset of H, using the method of replacement introduced by Addelman (1962) and developed by Wu (1989). For illustration of the replacement method, we consider replacing two-level columns by an eight-level column. Because an eight-level column has seven degrees of freedom, we need seven two-level columns each having one degree of freedom. Three independent two-level columns a, b, c and all their possible interaction columns ab, ac, bc, abc just compose the closed subset with seven two-level columns. Then the replacement can be done according to the rule in Table 1. Suppose (B_1, B_2) corresponds to a $2^{(n'_1+n'_2)-(k'_1+k'_2)}$ FFSP design D. If there are $t(= 2^r - 1)$ two-level columns of B_2 which compose a closed subset, then replacing them with a 2^r-level column we can get a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_s$ design with $n_1 = n'_1$, $k_1 = k'_1$, $n_2 = n'_2 - t$, and $k_2 = k'_2 - t + r$. Similarly, if there are $t(= 2^r - 1)$ two-level columns of B_1 which compose a closed subset, then replacing them with a 2^r-level column we can get a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_w$ design with $n_1 = n'_1 - t$, $k_1 = k'_1 - t + r$, $n_2 = n'_2$, and $k_2 = k'_2$. Denote the 2^r-level column as E.

We call the $2^r - 1$ columns replaced with the 2^r-level column as the components of the 2^r-level factor E. For convenience, both the main effects of the two-level factors and the components of the 2^r-level factor are called the main effect components. For the same reason, both the two-factor interaction of two two-level factors and the two-factor interaction components (2FICs) of a two-level factor and a 2^r-level factor are called 2FICs. We divide the 2FICs into three types: WP2FIC, SP2FIC and WS2FIC, where a WP2FIC (or SP2FIC) means a 2FIC in which both factors are WP (or SP) factors, and similarly a WS2FIC means a 2FIC in which one factor is a WP factor and the other is an SP factor.

Wu and Zhang (1993) proposed an extension of the minimum aberration criterion for designs with two-level and four-level factors. We extend their idea here. For a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1$ design, there are two types of defining words: type 0, which involves only the two-level factors, and type 1, which involves one component of the 2^r-level factor and some of the two-level factors. We introduce two important definitions as follows.

Table 1
Rule for replacing seven two-level columns by an eight-level column.

a	b	c	ab	ac	bc	abc	8-level column
0	0	0	0	0	0	0	0
0	0	1	0	1	1	1	1
0	1	0	1	0	1	1	2
0	1	1	1	1	0	0	3
1	0	0	1	1	0	1	4
1	0	1	1	0	1	0	5
1	1	0	0	1	1	0	6
1	1	1	0	0	0	1	7

Definition 1. Let A_{i0} and A_{i1} be the numbers of type 0 and type 1 words of length i in the defining contrasts subgroup of a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1$ design D, respectively. The resolution of D is defined to be the smallest i such that $A_{i0}(D)+A_{i1}(D)$ is positive.

Definition 2. A main effect component of a factor is said to be clear if it is not aliased with any main effect component of the other factors or any 2FIC. A 2FIC is said to be clear if it is not aliased with any main effect component or any other 2FIC. A main effect or two-factor interaction is said to be clear if all its components are clear.

Hereafter, let $2_R^{(n_1+n_2)-(k_1+k_2)}(2^r)^1$ denote a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1$ design with resolution R.
We introduce a lemma which will be used in the proofs of theorems. Its proof is obvious and thus omitted here.

Lemma 1. *Let* $Q_1 = \{a_1\} \cup H(a_2, \ldots, a_{p_1})$ *and* $Q_2 = H(a_2, \ldots, a_{p_1}, b_1, \ldots, b_{p_2})\backslash H(a_2, \ldots, a_{p_1})$*. Then* $Q = (Q_1, Q_2)$ *is a* $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}$ *design D with* $n_1 = 2^{p_1-1}$ *and* $n_2 = 2^{p-1} - 2^{p_1-1}$*, and the WP main effect* a_1 *and WP2FIC* a_1c $(c \in Q_1\backslash\{a_1\})$ *are clear in D.*

3. $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear effects

This section discusses the conditions for the existence of $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing various clear effects. A $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D can be denoted by

$$C = \{c_1, \ldots, c_{n_1}; c_{n_1+1}, \ldots, c_{n_1+n_2}; d_1, \ldots, d_t\},$$

where $B_1 = \{c_1, \ldots, c_{n_1}\}(\subset H_a)$ consists of the WP factors, $B_2 = \{c_{n_1+1}, \ldots, c_{n_1+n_2}\}(\subset H\backslash H_a)$ consists of the two-level SP factors, and $E = \{d_1, \ldots, d_t\}(\subset H\backslash H_a)$ consists of $t(= 2^r - 1)$ two-level columns replaced by a 2^r-level factor. Here, we denote $p_1 = n_1 - k_1, p_2 = (n_2 + t) - (k_2 + t - r)$ and $p = p_1 + p_2$ for simplicity.

In this section, without loss of generality, we say that a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D is determined by C with $H_{b_r} = H(b_1, \ldots, b_r) = \{d_1, \ldots, d_t\} \subset H\backslash H_a$ being substituted by a 2^r-level factor denoted as E. Let us first give the necessary and sufficient conditions for $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs to contain clear WP main effects or WP2FICs.

Theorem 1. *There exist* $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ *designs containing clear WP main effects or WP2FICs if and only if* $n_1 \le 2^{p_1-1}$ *and* $n_2 \le 2^{p-1} - 2^{p_1-1} - t$*.*

Proof. Suppose that a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D is determined by C and a two-level WP main effect, say c_1, is clear in D. Then we have

$$c_1c_i \in H_a\backslash B_1, i = 2, \ldots, n_1;$$
$$c_1c_j \in (H\backslash H_a)\backslash(B_2 \cup H_{b_r}), j = n_1 + 1, \ldots, n_1 + n_2; \text{ and}$$
$$c_1d_l \in (H\backslash H_a)\backslash(B_2 \cup H_{b_r}), l = 1, \ldots, t.$$

The above columns are different from each other, which implies that

$$n_1 - 1 \le 2^{p_1} - 1 - n_1, \text{ i.e., } n_1 \le 2^{p_1-1}, \text{ and}$$

$$n_2 + t \le 2^p - 1 - (2^{p_1} - 1) - n_2 - t, \text{ i.e., } n_2 \le 2^{p-1} - 2^{p_1-1} - t.$$

One can easily obtain the same result by similar arguments when D has a clear WP2FIC.

When $n_1 = 2^{p_1-1}$ and $n_2' = 2^{p-1} - 2^{p_1-1}$, we can get a $2_{\text{III}}^{(n_1+n_2')-(k_1+k_2')}$ design D with the clear WP main effect a_1 and WP2FIC a_1c $(c \in Q_1\backslash\{a_1\})$ according to Lemma 1. Replacing H_{b_r} with a 2^r-level factor E in Q_2, we can obtain a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design with the same types of clear effects as D, where $n_2 = 2^{p-1} - 2^{p_1-1} - t$. When $n_1 < 2^{p_1-1}$ and/or $n_2 < 2^{p-1} - 2^{p_1-1} - t$, we can get the designs with clear WP main effects or WP2FICs by deleting any $2^{p_1-1} - n_1$ columns from $Q_1\backslash\{a_1\}$ and/or any $2^{p-1} - 2^{p_1-1} - t - n_2$ columns from $Q_2\backslash H_{b_r}$. This completes the proof. □

The proof of Theorem 1, in fact, provides us a method to construct such $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear WP main effects or WP2FICs. The following theorem shows when a $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design can have the clear 2^r-level SP main effect and illustrates the corresponding construction.

Theorem 2. *There exist $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing the clear 2^r-level SP main effect if and only if $n_1 \le 2^{p_1}-1$ and $n_2 \le 2^{p-r}-n_1-1$.*

Proof. Suppose that a $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D is determined by C. Obviously, $n_1 \le 2^{p_1}-1$, since $\{c_1,\dots,c_{n_1}\} \subset H_a$. If the 2^r-level factor E is clear, then we have $d_ic_j \in H\backslash C$, for $i=1,\dots,t, j=1,\dots,n_1+n_2$. Note that the columns d_ic_j are different from each other (otherwise d_i is not clear) and $t=2^r-1$, we can conclude that

$$t(n_1+n_2) \le 2^p-1-n_1-n_2-t, \text{ i.e., } n_2 \le 2^{p-r}-n_1-1.$$

When $n_1 \le 2^{p_1}-1$ and $n_2 = 2^{p-r}-n_1-1$, let $M_1 \subset H_a$ such that $\#\{M_1\}=n_1$ and M_1 has p_1 independent columns, and

$$M_2 = H_{b_r} \cup \left(\{b_1\} \otimes (H(a_1,\dots,a_{p_1},b_{r+1},\dots,b_{p_2})\backslash M_1)\right).$$

Hereafter, $\#\{\cdot\}$ denotes the cardinality of the set, and $T_1 \otimes T_2 = \{t_1t_2 : t_1 \in T_1, t_2 \in T_2\}$ for two sets T_1 and T_2. Then, $n_1 \le 2^{p_1}-1$ and $\#\{M_2\} = 2^{p-r}-1-n_1+t$. Replacing H_{b_r} with a 2^r-level factor E in M_2, we can get a $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design with $n_1 \le 2^{p_1}-1$ and $n_2 = 2^{p-r}-1-n_1$. Obviously, the 2^r-level factor E is clear. For $n_1 \le 2^{p_1}-1$ and $n_2 < 2^{p-r}-n_1-1$, we can delete any $2^{p-r}-n_1-1-n_2$ columns from $M_2\backslash H_{b_r}$ to get the required designs. This completes the proof. □

Next, we will discuss the conditions that the $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design contains clear WS2FICs, two-level SP main effects or SP2FICs, respectively.

Theorem 3.

(a) *For $p_2 \ge r$, there exist $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear WS2FICs if and only if $n_1 \le 2^{p_1}-1$ and $n_2 \le 2^{p-1}-n_1-t$.*

(b) *For $p_2 \ge r+1$, there exist $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear two-level SP main effects or SP2FICs if and only if $n_1 \le 2^{p_1}-1$ and $n_2 \le 2^{p-1}-n_1-t$.*

(c) *For $p_2 = r$, there exist $2_{\rm III}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear two-level SP main effects or SP2FICs if and only if $n_1 \le 2^{p_1}-2$ and $n_2 \le 2^{p-1}-n_1-t$.*

The proof of Theorem 3 is lengthy and we give it in Appendix. The following two examples help us to better understand the structure of the designs constructed in the proof.

Example 1. Let $p_1=2, p_2=3, r=2$ and $B_1=H_2=\{1,2,12\}$. There are altogether 14 disjoint column pairs in H_5, which join $\{1,3\}$ to form distinct words with length four. Among them, there are two pairs, $\{2,123\}$ and $\{12,23\}$, with one column from H_2 and the other from $H_5\backslash H_2$. Apart from these, the remaining twelve pairs have both columns from $H_5\backslash H_2$. They are $\{4,134\}, \{34,14\}, \{24,1234\}, \{124,234\}, \{5,135\}, \{15,35\}, \{25,1235\}, \{125,235\}, \{45,1345\}, \{145,345\}, \{245,12345\}, \{1245,2345\}$. By choosing one column from each of these pairs (say the first element) as the element of B_2 and adding $\{3\}$ into B_2, we have

$$B_2 = \{3,4,34,24,124,5,15,25,125,45,145,245,1245\}.$$

Replacing $\{3,4,34\}$ in B_2 with a four-level factor, we get a $2_{\rm III}^{(3+10)-(1+9)}4_s^1$ design with a clear WS2FIC $\{1,3\}$.

We can also get a $2_{\rm III}^{(2+11)-(0+10)}4_s^1$ design with a clear WS2FIC $\{1,3\}$ by deleting $\{12\}$ from B_1 and adding $\{23\}$ into B_2. Similarly, we can get the designs with the same p_1 and p_2. □

Example 2. Suppose $p_1=2, p_2=3, r=2$ and $B_1=\{1,2,12\}, B_2=\{3,4,5,45,14,15,145,24,25,245,124,125,1245\}$. After replacing $\{4,5,45\}$ with a four-level factor, we have a $2_{\rm III}^{(3+10)-(1+9)}4_s^1$ design with clear SP main effect $\{3\}$ and clear SP2FIC $\{3,5\}$. Similarly, we can get a $2_{\rm III}^{(2+11)-(0+10)}4_s^1$ design with same clear effects by deleting $\{12\}$ from B_1 and adding $\{1235\}$ into B_2. □

As we know, designs with resolution IV are vital in practice because the main effects are all clear. In the following two theorems, we provide the conditions for a $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design to contain clear all kinds of 2FICs.

Theorem 4.

(a) *If there exist $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear WS2FICs or SP2FICs involving some 2^r-level factor component, then $n_1 \le 2^{p_1-1}$ and $n_2 \le 2^{p-2}-n_1-2^{r-1}+1$.*

(b) *If there exist $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear WS2FICs or SP2FICs involving only two-level factors, then $n_1 \le 2^{p_1-1}$ and $n_2 \le 2^{p-2}-n_1-t+1$.*

Proof. For both cases (a) and (b), the WP part of a $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design is a $2_{\mathrm{IV}}^{n_1-k_1}$ design and there does not exist $2_{\mathrm{IV}}^{n_1-k_1}$ design when $n_1 > 2^{p_1-1}$.

For (a), we now show $n_2 \leq 2^{p-2} - n_1 - 2^{r-1} + 1$. Suppose that a $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design is determined by C and SP2FIC $c_{n_1+1}d_1$ is clear. Then we have

$$c_{n_1+1}c_i \in H\backslash C,\ \ i = 1, \ldots, n_1+n_2, i \neq n_1+1;$$
$$c_{n_1+1}d_j \in H\backslash C,\ \ j = 1, \ldots, t;$$
$$d_1c_l \in H\backslash C,\ \ l = 1, \ldots, n_1, n_1+2, \ldots, n_1+n_2;\ \text{ and}$$
$$c_{n_1+1}d_1c_k \in H\backslash C,\ \ k = 1, \ldots, n_1, n_1+2, \ldots, n_1+n_2.$$

Because the above columns are different from each other, we get

$$3(n_1+n_2-1)+t \leq 2^p - 1 - n_1 - n_2 - t,$$

which implies $n_2 \leq 2^{p-2} - n_1 - 2^{r-1} + 1$. For the conditions for a $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design to contain the same type of clear WS2FICs, the proof is similar.

For (b), we now show $n_2 \leq 2^{p-2} - n_1 - t + 1$. Similar to the above, suppose that a $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design is determined by C and SP2FIC $c_{n_1+1}c_{n_1+2}$ is clear. Then we have

$$c_{n_1+1}c_{i_1} \in H\backslash C,\ \ i_1 = 1, \ldots, n_1+n_2, i_1 \neq n_1+1;$$
$$c_{n_1+2}c_{i_2} \in H\backslash C,\ \ i_2 = 1, \ldots, n_1+n_2, i_2 \neq n_1+1, n_1+2;$$
$$c_{n_1+1}d_{i_3} \in H\backslash C,\ \ i_3 = 1, \ldots, t;$$
$$c_{n_1+2}d_{i_4} \in H\backslash C,\ \ i_4 = 1, \ldots, t;$$
$$c_{n_1+1}c_{n_1+2}c_{i_5} \in H\backslash C,\ \ i_5 = 1, \ldots, n_1+n_2, i_5 \neq n_1+1, n_1+2;\ \text{ and}$$
$$c_{n_1+1}c_{n_1+2}d_{i_6} \in H\backslash C,\ \ i_6 = 1, \ldots, t.$$

Since the above columns are different from each other, we get

$$(n_1+n_2-1)+2(n_1+n_2-2)+3t \leq 2^p - 1 - n_1 - n_2 - t,$$

which implies that $n_2 \leq 2^{p-2} - n_1 - t + 1$. This completes the proof. □

Theorem 5. *If there exist $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear WP2FICs, then $n_1 \leq 2^{p_1-2}+1$ and $n_2 \leq 2^{p-2}-2^{p_1-2}-t$.*

Proof. Suppose that a $2_{\mathrm{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design is determined by C and WP2FIC c_1c_2 is clear. Then we have

$$c_1c_{i_1} \in H_a\backslash B_1,\ \ i_1 = 2, \ldots, n_1;$$
$$c_2c_{i_2} \in H_a\backslash B_1,\ \ i_2 = 3, \ldots, n_1;\ \text{ and}$$
$$c_1c_2c_{i_3} \in H_a\backslash B_1,\ \ i_3 = 3, \ldots, n_1.$$

Because the above columns are different from each other, we can get

$$(n_1-1)+2(n_1-2) \leq 2^{p_1} - 1 - n_1,\ \text{ i.e., }\ n_1 \leq 2^{p_1-2}+1.$$

Note that

$$c_1c_{i_4} \in H\backslash H_a\backslash B_2,\ \ i_4 = n_1+1, \ldots, n_1+n_2;$$
$$c_2c_{i_5} \in H\backslash H_a\backslash B_2,\ \ i_5 = n_1+1, \ldots, n_1+n_2;$$
$$c_1c_2c_{i_6} \in H\backslash H_a\backslash B_2,\ \ i_6 = n_1+1, \ldots, n_1+n_2;$$
$$c_1d_{i_7} \in H\backslash H_a\backslash B_2,\ \ i_7 = 1, \ldots, t;$$
$$c_2d_{i_8} \in H\backslash H_a\backslash B_2,\ \ i_8 = 1, \ldots, t;\ \text{ and}$$
$$c_1c_2d_{i_9} \in H\backslash H_a\backslash B_2,\ \ i_9 = 1, \ldots, t.$$

Since the above columns are different from each other, we have

$$3(n_2+t) \leq 2^p - 1 - (2^{p_1}-1) - n_2 - t,\ \text{ i.e., } n_2 \leq 2^{p-2} - 2^{p_1-2} - t.$$

This completes the proof. □

4. $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs containing clear effects

In this section, we will discuss the conditions for $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs to contain clear effects. A $2^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design D can be denoted by

$$C' = \{c_1, \ldots, c_{n_1}; d_1, \ldots, d_t; c_{n_1+1}, \ldots, c_{n_1+n_2}\},$$

where $c_1, \ldots, c_{n_1}$ are the two-level WP factors, $d_1, \ldots, d_t$ are the components of the 2^r-level WP factor, and $c_{n_1+1}, \ldots, c_{n_1+n_2}$ are the two-level SP factors. Let $B_1' = \{c_1, \ldots, c_{n_1}, d_1, \ldots, d_t\}$ and $B_2' = \{c_{n_1+1}, \ldots, c_{n_1+n_2}\}$. For simplicity, we denote $p_1 = (n_1 + t) - (k_1 + (t - r))$, $p_2 = n_2 - k_2$, $p = p_1 + p_2$ and $t = 2^r - 1$.

Without loss of generality, we say that a $2^{(n_1+n_2)-(k_1+k_2)}(2^r)^1_w$ design D is determined by C' with $H_{a_r} = H(a_1, \ldots, a_r) = \{d_1, \ldots, d_t\}$ being replaced by a 2^r-level factor denoted as F. First, Theorems 6 and 7 give necessary and sufficient conditions for the existence of a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design with clear WP main effects or WP2FICs. The proof of the existence of designs is also by construction.

Theorem 6. *There exist $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ designs containing clear two-level WP main effects or WP2FICs if and only if $n_1 \le 2^{p_1-1} - t$ and $n_2 \le 2^{p-1} - 2^{p_1-1}$.*

Proof. Suppose that a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design D is determined by C' and a two-level WP main effect c_1 is clear. Then we have

$c_1c_i \in H_a \backslash B_1'$, $i = 2, \ldots, n_1$;
$c_1d_j \in H_a \backslash B_1'$, $j = 1, \ldots, t$; and
$c_1c_l \in H \backslash H_a \backslash B_2'$, $l = n_1 + 1, \ldots, n_1 + n_2$.

Since the columns above are different from each other, we have

$$n_1 + t - 1 \le 2^{p_1} - 1 - n_1 - t, \text{ i.e., } n_1 \le 2^{p_1-1} - t, \text{ and}$$

$$n_2 \le (2^p - 1) - (2^{p_1} - 1) - n_2, \text{ i.e., } n_2 \le 2^{p-1} - 2^{p_1-1}.$$

The proof of the necessity conditions for designs to contain clear WP2FICs is similar and omitted here.

We can easily get a $2^{(n_1'+n_2)-(k_1'+k_2)}_{\text{III}}$ design with the clear WP main effect a_1 and WP2FIC a_1c ($c \in Q_1 \backslash \{a_1\}$) according to Lemma 1, where $n_1' = 2^{p_1-1}$ and $n_2 = 2^{p-1} - 2^{p_1-1}$. Replacing $H(a_2, \ldots, a_{r+1})$ with a 2^r-level factor F, we can get a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design with $n_1 = 2^{p_1-1} - t$. From the construction, it is obvious that the two-level WP main effect a_1 is clear. For any $c \in Q_1 \backslash \{a_1\}$, the WP2FIC a_1c is clear. When $n_1 < 2^{p_1-1} - t$ and/or $n_2 < 2^{p-1} - 2^{p_1-1}$, we only need to delete any $2^{p_1-1} - t - n_1$ columns from $Q_1 \backslash (\{a_1\} \cup F)$ and/or any $2^{p-1} - 2^{p_1-1} - n_2$ columns from Q_2 to obtain the required designs. The proof is completed. □

Theorem 7. *There exist $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ designs containing the clear 2^r-level WP main effect if and only if $n_1 \le 2^{p_1-r} - 1$ and $n_2 \le 2^{p-r} - 2^{p_1-r}$.*

Proof. Suppose that the design is determined by C' and the 2^r-level WP main effect $F = H(a_1, \ldots, a_r) = \{d_1, \ldots, d_t\}$ is clear. Then we have

$d_ic_j \in H_a \backslash B_1'$, $i = 1, \ldots, t, j = 1, \ldots, n_1$; and
$d_ic_k \in H \backslash H_a \backslash B_2'$, $i = 1, \ldots, t, k = n_1 + 1, \ldots, n_1 + n_2$.

Note that the above columns are different from each other and $t = 2^r - 1$, we can conclude that

$$tn_1 \le 2^{p_1} - 1 - n_1 - t, \text{ i.e., } n_1 \le 2^{p_1-r} - 1, \text{ and}$$

$$tn_2 \le 2^p - 2^{p_1} - n_2, \text{ i.e., } n_2 \le 2^{p-r} - 2^{p_1-r}.$$

Let $F = H(a_1, \ldots, a_r)$ be the 2^r-level factor, $W_1 \subset H(a_{r+1}, \ldots, a_{p_1})$ and

$$W_2 \subset H(a_{r+1}, \ldots, a_{p_1}, b_1, \ldots, b_{p_2}) \backslash H(a_{r+1}, \ldots, a_{p_1}),$$

such that $\#\{W_1\} = n_1 \le 2^{p_1-r} - 1$ and $\#\{W_2\} = n_2 \le 2^{p-r} - 2^{p_1-r}$. Then we can get a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design with F being clear. This completes the proof. □

Remark 1. The proofs of Theorems 1 and 6 share some common features shown in Lemma 1. For Theorems 2 and 7, it looks as if they are similar, but the structures of the designs constructed in the proofs are totally different. Theorem 7 constructs a design containing the clear 2^r-level WP main effect while Theorem 2 constructs a design containing the clear 2^r-level SP main effect. If we apply the method in the proof of Theorem 7 to that of Theorem 2, some factors in $H(a_1, \ldots, a_{p_1}, b_{r+1}, \ldots, b_{p_2}) \backslash M_1$ may not belong to the SP part. So we put b_1 to each element of $H(a_1, \ldots, a_{p_1}, b_{r+1}, \ldots, b_{p_2}) \backslash M_1$ in the proof of Theorem 2 to make sure they are in the SP part.

According to Theorems 6 and 7, we can get the results with clear WP factors. The following theorem considers the condition that a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design can have clear SP main effects.

Theorem 8. *For $p_2 \ge 2$, there exist $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ designs containing clear SP main effects if and only if $n_1 \le 2^{p_1} - t - 1$ and $n_2 \le 2^{p-1} - n_1 - t$.*

Proof. First, since the WP factors belong to H_a, we have

$$n_1 + t \le 2^{p_1} - 1, \text{ i.e., } n_1 \le 2^{p_1} - t - 1.$$

Suppose the design is determined by C' and SP main effect c_{n_1+1} is clear. Then we have

$$c_{n_1+1}c_i \in H\backslash C', \quad i = 1, \ldots, n_1, n_1+2, \ldots, n_1+n_2; \text{ and}$$
$$c_{n_1+1}d_j \in H\backslash C', \quad j = 1, \ldots, t.$$

Since the above columns are different from each other, we can get

$$n_1 + n_2 - 1 + t \le 2^p - 1 - n_1 - n_2 - t, \text{ i.e., } n_2 \le 2^{p-1} - n_1 - t.$$

Let $W_1 = H(a_1, \ldots, a_{p_1})$ and

$$W_2 = \{b_1\} \cup \left(\{b_1\} \otimes (H(a_1, \ldots, a_{p_1}, b_2, \ldots, b_{p_2})\backslash W_1)\right).$$

Then $W = (W_1, W_2)$ is a $2^{(n_1'+n_2)-(k_1'+k_2)}_{\text{III}}$ design with $n_1' = 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1'$. Replacing $H(a_1, \ldots, a_r)$ with a 2^r-level factor F, we can get a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ design with $n_1 = 2^{p_1} - t - 1$ and $n_2 = 2^{p-1} - 2^{p_1} + 1 = 2^{p-1} - n_1 - t$. From the construction of the design, it is obvious that the two-level SP main effect b_1 is clear. When $n_1 < 2^{p_1} - t - 1$ and $n_2 = 2^{p-1} - n_1 - t$, we only need to delete any $2^{p_1} - t - 1 - n_1$ columns c_i from $W_1\backslash F$ and add c_ib_1 into W_2 to get the required designs. Furthermore, when $n_2 < 2^{p-1} - n_1 - t$, we need to delete any $2^{p-1} - n_1 - t - n_2$ columns from $W_2\backslash\{b_1\}$ to get the required designs. This completes the proof. □

The following theorem considers the case of $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ designs containing clear WS2FICs or SP2FICs. Its proof is also lengthy and thus given in Appendix.

Theorem 9. *For $p_2 \ge 2$, there exist $2^{(n_1+n_2)-(k_1+k_2)}_{\text{III}}(2^r)^1_w$ designs containing clear WS2FICs or SP2FICs if and only if $n_1 \le 2^{p_1} - t - 1$ and $n_2 \le 2^{p-1} - n_1 - t$.*

The following example illustrates the construction method in the proof of Theorem 9.

Example 3. Consider the construction of a $2^{(4+9)-(3+7)}_{\text{III}}(4)^1_w$ design with a clear WS2FIC. First, we have $p_1 = 3, p_2 = 2, r = 2, n_1 = 4$ and $n_2 = 9$. Let $B_1' = H_3 = \{1, 2, 12, 3, 13, 23, 123\}$, $c_1 = \{1\}$ and $c_{n_1+1} = \{4\}$. There are altogether 14 disjoint column pairs in H_5, which join $\{1, 4\}$ to form distinct words with length four. Then, among the pairs which join $\{1, 4\}$ to form length four words, there are eight ones with both columns from $H_5\backslash H_3$, which are $\{5, 145\}$, $\{15, 45\}$, $\{25, 1245\}$, $\{125, 245\}$, $\{35, 1345\}$, $\{135, 345\}$, $\{235, 12345\}$, $\{1235, 2345\}$. Choosing one column from each of these pairs as the element of B_2' and adding $\{4\}$ into B_2', we get B_2', say

$$B_2' = \{4, 5, 45, 25, 125, 35, 135, 235, 1235\}.$$

By replacing $\{1, 2, 12\}$ with a four-level factor, we can get a $2^{(4+9)-(3+7)}_{\text{III}}(4)^1_w$ design with a clear WS2FIC $\{1, 4\}$. In the same manner, we can get the designs with the same p_1 and p_2. For instance, we can obtain a $2^{(3+10)-(2+8)}_{\text{III}}(4)^1_w$ design with a clear WS2FIC $\{1, 4\}$ by deleting $\{123\}$ from B_1 and adding $\{234\}$ into B_2. □

Since the main effects are all clear in a design of resolution IV, we now discuss the conditions for a $2^{(n_1+n_2)-(k_1+k_2)}_{\text{IV}}(2^r)^1_w$ design to contain various clear 2FICs in the following two theorems.

Theorem 10. *There exist $2^{(n_1+n_2)-(k_1+k_2)}_{\text{IV}}(2^r)^1_w$ designs with clear WP2FICs or WS2FICs if and only if $n_1 \le 2^{p_1-r} - 1$ and $n_2 \le 2^{p-r} - 2^{p_1-r}$.*

Proof. Let

$$H(a_{r+1}, \ldots, a_{p_1}) = \{\alpha_1, \ldots, \alpha_{2^{p_1-r}-1}\},$$

$$H(a_{r+1}, \ldots, a_{p_1}, b_1, \ldots, b_{p_2})\backslash H(a_{r+1}, \ldots, a_{p_1}) = \{\alpha_{2^{p_1-r}}, \ldots, \alpha_{2^{p-r}-1}\},$$

and $S_i = \{\alpha_i\} \cup (\{\alpha_i\} \otimes H(a_1, \ldots, a_r))$ for $i = 1, \ldots, 2^{p-r} - 1$. Then

$$H = H(a_1, \ldots, a_{p_1}, b_1, \ldots, b_{p_2}) = H(a_1, \ldots, a_r) \cup \left(\cup_{i=1}^{2^{p-r}-1} S_i\right).$$

Clearly, the components of the 2^r-level WP factor belong to $H(a_1, \ldots, a_r)$, the two-level WP factors belong to $S_i, i = 1, \ldots, 2^{p_1-r} - 1$, and the SP factors belong to $S_i, i = 2^{p_1-r}, \ldots, 2^{p-r} - 1$. Note that each S_i contains at most one column of the design, otherwise, the design would have resolution III. Thus, we have $n_1 \le 2^{p_1-r} - 1$ and $n_2 \le 2^{p-r} - 2^{p_1-r}$.

Let

$$E_a = \{a_1, \ldots, a_r\} \cup (\{a_r\} \otimes H(a_{r+1}, \ldots, a_{p_1})),$$

$$W_1 = E_a, W_2 = (E_a \backslash \{a_1, \ldots, a_{r-1}\}) \otimes H(b_1, \ldots, b_{p_2}).$$

Then $W = (W_1, W_2)$ is a $2_{\rm IV}^{(n_1'+n_2)-(k_1'+k_2)}$ design with $n_1' = 2^{p_1-r} - 1 + r$ and $n_2 = 2^{p-r} - 2^{p_1-r}$. Adding the columns of $H(a_1, \ldots, a_r)\backslash(a_1, \ldots, a_r)$ into W, we get a new design W^*. Replacing $H(a_1, \ldots, a_r)$ with a 2^r-level factor F, we get a $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design with $n_1 = 2^{p_1-r} - 1$ and $n_2 = 2^{p-r} - 2^{p_1-r}$. For any $c_1 \in W_1\backslash\{a_1, \ldots, a_r\}$ and $c_2 \in W_2$, all of the WP2FICs a_1c_1 and WS2FICs a_1c_2 are clear. For $n_1 < 2^{p_1-r} - 1$ and/or $n_2 < 2^{p-r} - 2^{p_1-r}$, we can delete any $2^{p_1-r} - 1 - n_1$ columns from $W_1\backslash F$ and/or any $2^{p-r} - 2^{p_1-r} - n_2$ columns from W_2 to get the required designs. This completes the proof. □

Theorem 10 provides a necessary and sufficient condition for a $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design to have clear WP2FICs or WS2FICs. Similarly, the conditions for a $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design to have clear SP2FICs are given in the following theorem.

Theorem 11.

(a) *For $p_2 \geq 2$, there exist $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs containing clear SP2FICs if and only if $n_1 \leq 2^{p_1-r}-1$, $n_2 \leq 2^{p-r}-2^{p_1-r}$ and the equalities cannot hold at the same time.*

(b) *For $p_2 = 1$, there exist $2_{\rm IV}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs containing clear SP2FICs if and only if $n_1 \leq 2^{p_1-r} - 2$ and $n_2 \leq 2^{p-1} - 2^{p_1-r}$.*

The following two examples are used to illustrate the construction of Theorem 11(a).

Example 4. For $p_1 = 4, p_2 = 2$ and $r = 2$, let $E_a = \{1, 2\} \cup \{2 \otimes H(3, 4)\}$ and $E_{ab} = \{2, 23, 24, 234\} \otimes H(5, 6)$. Define $B_1' = E_a \cup \{12\} = \{1, 2, 12, 23, 24, 234\}$ and $B_2' = (E_{ab} \setminus \{25, 256\}) \cup \{15\} = \{15, 235, 245, 2345, 26, 236, 246, 2346, 2356, 2456, 23456\}$. Then, after replacing $\{1, 2, 12\}$ with a four-level factor, (B_1', B_2') corresponds to a $2_{\rm IV}^{(3+11)-(1+9)}4_w^1$ design with clear SP2FIC $\{15, 26\}$. Similarly, we can construct the designs with same p_1 and p_2. □

Example 5. Let us see another example with larger p_1 and p_2 values. For $p_1 = 8$, $p_2 = 7$ and $r = 3$, we denote the 15 independent columns as $1, 2, 3, 4, 5, 6, 7, 8, 9, t_0, t_1, t_2, t_3, t_4$ and t_5. Let $E_a = \{1, 2, 3\} \cup \{3 \otimes H(4, 5, 6, 7, 8)\}$, and $E_{ab} = (E_a \setminus \{1, 2\}) \otimes H(9, t_0, t_1, t_2, t_3, t_4, t_5)$. We get $W_1 = E_a$ and $W_2 = (E_{ab} \setminus \{39, 39t_0\}) \cup \{19\}$. Then adding $H(1, 2, 3) \setminus \{1, 2, 3\}$ into W_1 and replacing $H(1, 2, 3)$ with an eight-level factor, we can get a design $2_{\rm IV}^{(31+4063)-(26+4056)}8_w^1$ with the clear SP2FIC $\{19, 3t_0\}$. If $n_1 = 31$ and $n_2 < 4063$, say $n_2 = 300$, we just need to delete any 3763 (= 4063 − 300) columns from $W_2 \setminus \{19, 3t_0\}$ to get the required design. □

5. Concluding remarks

We have discussed the existence and construction of mixed-level FFSP designs with clear main effects or 2FICs. The necessary and sufficient conditions for $2_R^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ and $2_R^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs with R = III or IV to have clear main effects or 2FICs are obtained. Meanwhile, the structures of these designs are revealed and some useful construction methods are developed. Zhao and Chen (2012a,b) investigated the FFSP designs with some two-level factors and one four-level factor which is in the whole-plot and sub-plot, respectively, and gave the conditions for such designs to have various clear effects. It is obvious that their results are special cases of the results here for $r = 2$. In practice, one may need mixed-level FFSP designs with more than one high-level factor. The methods in this paper can be extended to such designs, but the extension will be more complex since more cases have to be considered. For example, the cases are different depending on whether the 2^r-level factors are in only WP part, only SP part or in both WP and SP parts. We will leave this for further study.

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Appendix. Proofs of theorems

A.1. Proof of Theorem 3

For cases (a) and (b), since the WP factors belong to H_a, we have $n_1 \leq 2^{p_1} - 1$. Suppose that a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D has clear WS2FICs (or SP2FICs). Regarding D as a $2_{\text{III}}^{(n_1+n_2+t)-(k_1+k_2+(t-r))}$ FF design D', the clear WS2FICs (or SP2FICs) of D are clear two-factor interaction of D'. According to Chen and Hedayat (1998), we have $n_1 + n_2 + t \leq 2^{p-1}$, i.e., $n_2 \leq 2^{p-1} - n_1 - t$. Using a similar method to Theorem 2, it is easy to get that $n_1 + n_2 + t \leq 2^{p-1}$, i.e., $n_2 \leq 2^{p-1} - n_1 - t$, if a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design has clear two-level SP main effect.

For case (c), by similar arguments to cases (a) and (b), we can get that $n_2 \leq 2^{p-1} - n_1 - t$, so we only need to prove $n_1 \leq 2^{p_1} - 2$. If $n_1 = 2^{p_1} - 1$, then every element in H_a belongs to B_1, i.e., every element in H_a is a WP factor. Recall that each column of H_{b_r} is a component of the 2^r-level SP factor. Note that when $p_2 = r$, $H = H_a \cup H_{b_r} \cup (H_a \otimes H_{b_r})$, which implies that all of the two-level SP factors come from $H_a \otimes H_{b_r}$, i.e., each two-level SP main effect (or SP2FIC) is aliased with the interaction of a WP factor and a component of the 2^r-level factor. Obviously, the two-level SP main effect (or SP2FIC) is not clear. Thus, we get $n_1 \leq 2^{p_1} - 2$.

Now, it comes to prove the "if" parts. We need to construct $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ designs containing clear two-level SP main effects, WS2FICs or SP2FICs.

For (a), let $E = H_{b_r} = H(b_1, \ldots, b_r)$ and $M_1 = H_a = \{c_1, \ldots, c_{2^{p_1}-1}\}$. Without loss of generality, let $c_1 = a_1$ and $d_1 = b_1$. Then $c_1 d_1 = a_1 b_1 \in H \backslash H_a$. There are altogether $2^{p-1} - 2$ different pairs of columns in H which join $\{c_1, d_1\}$ to form $2^{p-1} - 2$ distinct words of length four (Chen and Hedayat, 1998). Among them, $2^{p_1} - 2$ pairs have the form of $\{c_i, c_1 d_1 c_i\}$, where $c_i \in H_a \backslash \{c_1\}$ and $c_1 d_1 c_i \in H \backslash H_a$, $i = 2, \ldots, 2^{p_1} - 1$. The remaining $2^{p-1} - 2^{p_1}$ pairs have the form of $\{f_{s_i}, f_{t_i}\}$, $i = 1, \ldots, 2^{p-1} - 2^{p_1}$, where both of them are from $H \backslash H_a$. Note that the pairs with the form of $\{d_i, c_1 d_1 d_i\}$, $i = 2, \ldots, t$, are from the latter $2^{p-1} - 2^{p_1}$ pairs. From each pair $\{f_{s_i}, f_{t_i}\}$, we choose one column as an element of M_2, such that $H_{b_r} \subset M_2$. Also adding d_1 into M_2, we have $\#\{M_2\} = 2^{p-1} - 2^{p_1} + 1$. Then replacing H_{b_r} with a 2^r-level factor, we obtain a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D with $n_1 = 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1 - t$. Obviously, the WS2FIC $c_1 d_1$ is clear. When $n_1 < 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1 - t$, we only need to delete the column of $c_i (i \neq 1)$ from M_1 and add $c_1 d_1 c_i$ into M_2 to get the required design. Furthermore, when $n_2 < 2^{p-1} - n_1 - t$, we can delete some columns from $M_2 \backslash H_{b_r}$ to get the required design.

For (b), let $E = H(b_2, \ldots, b_{r+1})$, M_1 be any n_1-subset of H_a and

$$M_2 = \{b_1\} \cup \left(\{b_1\} \otimes (H_a \backslash M_1)\right) \cup H(b_2, \ldots, b_{p_2}) \cup (H_a \otimes H(b_2, \ldots, b_{p_2}))\,.$$

Then $\#\{M_1\} = n_1 \leq 2^{p_1} - 1$ and $\#\{M_2\} = 2^{p-1} - n_1$. Replacing $H(b_2, \ldots, b_{p_2})$ with a 2^r-level factor in (M_1, M_2), we obtain a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D with $n_1 \leq 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1 - t$. Obviously, the two-level SP main effect b_1 is clear. For $n_2 < 2^{p-1} - n_1 - t$, we only need to delete some columns from $M_2 \backslash (E \cup \{b_1\})$ to get the required design.

Let $M_1 = H_a = \{c_1, \ldots, c_{2^{p_1}-1}\}$, $E = H(b_1, \ldots, b_r) = \{d_1, \ldots, d_t\}$ and $c_{n_1+1} = b_{r+1}$. Without loss of generality, let $d_1 = b_1$. Then $c_{n_1+1} d_1 = b_1 b_{r+1} \in H \backslash H_a$. Similar to the above proof, there are $2^{p-1} - 2$ disjoint pairs in H which join $\{c_{n_1+1}, d_1\}$ to form $2^{p-1} - 2$ distinct words of length four. Among them, $2^{p_1} - 1$ pairs have the form of $\{c_i, c_i c_{n_1+1} d_1\}$, where $c_i \in H_a$ and $c_i c_{n_1+1} d_1 \in H \backslash H_a$, $i = 1, \ldots, 2^{p_1} - 1$. The remaining $2^{p-1} - 2^{p_1} - 1$ pairs have the form of $\{f_{s_i}, f_{t_i}\}$, $i = 1, \ldots, 2^{p-1} - 2^{p_1} - 1$, with both columns coming from $H \backslash H_a$. Note that the pairs with the form of $\{d_i, c_{n_1+1} d_1 d_i\} (i = 2, \ldots, t)$ belong to the latter $2^{p-1} - 2^{p_1} - 1$ pairs. From each of $\{f_{s_i}, f_{t_i}\}$, $i = 1, \ldots, 2^{p-1} - 2^{p_1} - 1$, choose one column as an element of M_2, such that $\{d_2, \ldots, d_t\} \subset M_2$. Adding c_{n_1+1} and d_1 into M_2, we have $\#\{M_2\} = 2^{p-1} - 2^{p_1} + 1$. Replacing $E = H(b_1, \ldots, b_r)$ with a 2^r-level factor, we get the $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D with $n_1 = 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1 - t$. From the construction of D, the SP2FIC $c_{n_1+1} d_1$ is clear. When $n_1 < 2^{p_1} - 1$ and $n_2 = 2^{p-1} - n_1 - t$, we only need to delete some columns c_i from M_1 and add $c_{n_1+1} d_1 c_i$ into M_2 to get the required design. We can delete some columns from $M_2 \backslash (E \cup \{c_{n_1+1}\})$ to get the design when $n_2 < 2^{p-1} - n_1 - t$.

For (c), let $H_{b_r} = H(b_1, \ldots, b_r)$, $M_1 = H_a \backslash \{a_1\} = \{c_1, \ldots, c_{2^{p_1}-2}\}$, $d_1 = b_1$ and $c_{n_1+1} = a_1 b_2$. Then we can get $c_{n_1+1} d_1 = a_1 b_1 b_2 \in H \backslash H_a$. There are altogether $2^{p-1} - 2$ disjoint pairs of columns in H which join $\{c_{n_1+1}, d_1\}$ to form $2^{p-1} - 2$ distinct words of length four. Among them, $2^{p_1} - 1$ pairs have the form of $\{c_i, c_{n_1+1} d_1 c_i\}$, $i = 1, \ldots, 2^{p_1} - 1$, where $c_i \in H_a$ and $c_{n_1+1} d_1 c_i \in H \backslash H_a$. The remaining $2^{p-1} - 2^{p_1} - 1$ pairs with both columns in $H \backslash H_a$ are denoted as $\{f_{s_i}, f_{t_i}\}$, $i = 1, \ldots, 2^{p-1} - 2^{p_1} - 1$. Note that $\{b_1 b_2, b_2 c_{n_1+1}\}$ belongs to the former $2^{p_1} - 1$ pairs since $b_2 c_{n_1+1} = a_1 \in H_a$, and the columns of $H(b_1, \ldots, b_r) \backslash \{b_1 b_2, b_1\}$ belong to the latter $2^{p-1} - 2^{p_1} - 1$ pairs. We choose one column from each pair $\{f_{s_i}, f_{t_i}\}$ as an element of M_2 such that $H_{b_r} \backslash \{b_1, b_1 b_2\} \subset M_2$. Then adding the elements of $\{c_{n_1+1}, b_1, b_1 b_2\}$ into M_2, we have $\#\{M_2\} = 2 + 2^{p-1} - 2^{p_1}$. Replacing H_{b_r} with a 2^r-level factor, we can obtain a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_s^1$ design D with $n_1 = 2^{p_1} - 2$ and $n_2 = 2^{p-1} - n_1 - t$. From the above construction of D, the SP2FIC $c_{n_1+1} d_1$ is clear. When $n_1 < 2^{p_1} - 2$ and $n_2 = 2^{p-1} - n_1 - t$, we only need to delete some columns c_i from M_1 and add $c_{n_1+1} d_1 c_i$ into M_2 to get the required design. Furthermore, when $n_2 < 2^{p-1} - n_1 - t$, we can delete some columns from $M_2 \backslash (\{c_{n_1+1}\} \cup H_{b_r})$ to get the design with clear SP2FIC $c_{n_1+1} d_1$.

Deleting c_{n_1+1} and adding $c_{n_1+1} d_1$ into the design constructed above, the SP main effect $c_{n_1+1} d_1$ is still clear in the new design. The proof is completed. □

A.2. Proof of Theorem 9

Suppose that a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design is determined by C' and the WS2FIC $c_1c_{n_1+1}$ is clear. Since the WP factors belong to H_a, we have

$$n_1 + t \le 2^{p_1} - 1,\ \text{i.e., } n_1 \le 2^{p_1} - t - 1.$$

Then we have

$$c_1c_{n_1+1} \in H\backslash C';$$
$$c_1c_{n_1+1}c_i \in H\backslash C',\ i = 2, \ldots, n_1, n_1+2, \ldots, n_1+n_2;\ \text{and}$$
$$c_1c_{n_1+1}d_j \in H\backslash C',\ j = 1, \ldots, t.$$

Since the above columns are different from each other, we can get

$$1 + n_1 + n_2 - 2 + t \le 2^p - 1 - n_1 - n_2 - t,\ \text{i.e., } n_2 \le 2^{p-1} - n_1 - t.$$

The conditions for a design to contain clear SP2FICs can be proved similarly.

Without loss of generality, suppose $W_1 = H_a = H(a_1, \ldots, a_{p_1})$, $c_1 = a_1$ and $c_{n_1+1} = b_1$. Then $c_1c_{n_1+1} = a_1b_1 \in H\backslash H_a$. There are altogether $2^{p-1} - 2$ disjoint pairs of columns in H which join $\{c_1, c_{n_1+1}\}$ to form $2^{p-1} - 2$ distinct words of length four (Chen and Hedayat, 1998). Among them, $2^{p_1} - 2$ pairs have the form of $\{c_i, c_1c_{n_1+1}c_i\}$ with $c_i \in H_a\backslash\{a_1\}$ and $c_1c_{n_1+1}c_i \in H\backslash H_a, i = 2, \ldots, n_1$. The remaining $2^{p-1} - 2^{p_1}$ pairs with both columns in $H\backslash H_a$ are denoted by $\{f_{s_i}, f_{t_i}\}, i = 1, \ldots, 2^{p-1} - 2^{p_1}$. Choose either one from each pair of the latter $2^{p-1} - 2^{p_1}$ ones as an element of W_2. Also by adding c_{n_1+1} into W_2, we have $\#\{W_2\} = 2^{p-1} - 2^{p_1} + 1$. Then replacing $H(a_1, \ldots, a_r)$ with a 2^r-level factor F, we can obtain a $2_{\text{III}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design with $n_1 = 2^{p_1} - 1 - t$ and $n_2 = 2^{p-1} - n_1 - t$. It is obvious that the WS2FIC $c_1c_{n_1+1}$ is clear. For $n_1 < 2^{p_1} - 1 - t$ and $n_2 = 2^{p-1} - n_1 - t$, we only need to delete some $c_i (i \neq 1)$ from $W_1\backslash F$ and add $c_1c_{n_1+1}c_i$ into W_2 to get the required design. Furthermore, when $n_2 < 2^{p-1} - n_1 - t$, we can delete some columns from $M_2\backslash\{c_{n_1+1}\}$ to get the design. The construction of designs with clear SP2FICs is similar. We omit the proof for saving space. The proof is completed. □

A.3. Proof of Theorem 11

(a) Without loss of generality, let $F = H(a_1, \ldots, a_r)$ be the 2^r-level factor. With a discussion similar to that in the proof of Theorem 10, we can get $n_1 \le 2^{p_1-r} - 1$ and $n_2 \le 2^{p-r} - 2^{p_1-r}$. Now we prove that the equalities cannot hold at the same time. If $n_1 = 2^{p_1-r} - 1$ and $n_2 = 2^{p-r} - 2^{p_1-r}$, then there is exactly one column of each of S_i $(i = 1, \ldots, 2^{p-r} - 1)$ belonging to the design. Suppose these columns are $e_i \in S_i$ $(i = 1, \ldots, 2^{p-r} - 1)$. For any SP2FIC e_ie_j $(2^{p_1-r} \le i < j \le 2^{p-r} - 1)$. If $e_ie_j \in S_k$ for some k $(1 \le k \le 2^{p-r} - 1)$, then $e_ie_je_k \in H(a_1, \ldots, a_r)$, which implies that e_ie_j is not clear. Hence, the equalities cannot hold at the same time.

For $p_2 \ge 2$, $n_1 = 2^{p_1-r} - 1$ and $n_2 = 2^{p-r} - 2^{p_1-r} - 1$, let

$$E_a = \{a_1, \ldots, a_r\} \cup (\{a_r\} \otimes H(a_{r+1}, \ldots, a_{p_1})),$$
$$E_{ab} = (E_a\backslash\{a_1, \ldots, a_{r-1}\}) \otimes H(b_1, \ldots, b_{p_2}),$$
$$W_1 = E_a \text{ and } W_2 = (E_{ab}\backslash\{a_rb_1, a_rb_1b_2\}) \cup \{a_1b_1\}.$$

Then, $W = (W_1, W_2)$ is a $2_{\text{IV}}^{(n_1'+n_2)-(k_1'+k_2)}$ design with $n_1' = 2^{p_1-r} - 1 + r$ and $n_2 = 2^{p-r} - 2^{p_1-r} - 1$. Adding the columns $H(a_1, \ldots, a_r)\backslash(a_1, \ldots, a_r)$ into the design W and replacing $H(a_1, \ldots, a_r)$ with a 2^r-level factor F, we can get a $2_{\text{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design with $n_1 = 2^{p_1-r} - 1$ and $n_2 = 2^{p-r} - 2^{p_1-r} - 1$. Then we can check that the SP2FIC $\{a_1b_1, a_rb_2\}$ is clear in the design. For $n_1 \le 2^{p_1-r} - 1$ and/or $n_2 < 2^{p-r} - 2^{p_1-r} - 1$, we delete some columns from $W_1\backslash F$ and/or $W_2\backslash\{a_1b_1, a_rb_2\}$ to get the required designs.

Now we consider the construction of $2_{\text{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs with $n_1 \le 2^{p_1-r} - 2$ and $n_2 \le 2^{p-1} - 2^{p_1-r}$. Let $W_1' = E_a\backslash\{a_ra_{r+1}\}$ and $W_2' = (E_{ab}\backslash\{a_rb_1\}) \cup \{a_1b_1\}$. Then, the design $W' = (W_1', W_2')$ is a $2_{\text{IV}}^{(n_1'+n_2)-(k_1'+k_2)}$ design with $n_1' = 2^{p_1-r}+r-2$ and $n_2 = 2^{p-r}-2^{p_1-r}$. Adding the columns $H(a_1, \ldots, a_r)\backslash\{a_1, \ldots, a_r\}$ into W' and replacing $H(a_1, \ldots, a_r)$ with a 2^r-level factor F, we get a $2_{\text{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ design with $n_1 = 2^{p_1-r} - 2$ and $n_2 = 2^{p-r} - 2^{p_1-r}$. The SP2FIC $\{a_1b_1, a_ra_{r+1}b_1\}$ is clear in the design. For $n_1 < 2^{p_1-r} - 2$ and/or $n_2 < 2^{p-r} - 2^{p_1-r}$, we only need to delete some columns from $W_1'\backslash H(a_1, \ldots, a_r)$ and/or $W_2'\backslash\{a_1b_1, a_ra_{r+1}b_1\}$ to get the required designs.

(b) By the proof of (a), we need only to prove $n_1 \le 2^{p_1-r} - 2$. If $n_1 = 2^{p_1-r} - 1$, then there is exactly one column of each $S_i (i = 1, \ldots, 2^{p_1-r} - 1)$ belonging to the design. When $p_2 = 1$, for any SP2FIC $e_ie_j (2^{p_1-r} \le i < j \le 2^{p-r} - 1)$, we have $e_ie_j \in S_k$ for some $k (1 \le k \le 2^{p_1-r} - 1)$, which leads to $e_ie_je_k \in H(a_1, \ldots, a_r)$. Thus, e_ie_j cannot be clear. So, we get $n_1 \le 2^{p_1-r} - 2$. The construction of $2_{\text{IV}}^{(n_1+n_2)-(k_1+k_2)}(2^r)_w^1$ designs containing clear SP2FICs in (a) also applies here. This completes the proof. □

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