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RESEARCH ARTICLE

A theory on constructing blocked two-level designs with general minimum lower order confounding

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Abstract Completely random allocation of the treatment combinations to the experimental units is appropriate only if the experimental units are homogeneous. Such homogeneity may not always be guaranteed when the size of the experiment is relatively large. Suitably partitioning inhomogeneous units into homogeneous groups, known as blocks, is a practical design strategy. How to partition the experimental units for a given design is an important issue. The blocked general minimum lower order confounding is a new criterion for selecting blocked designs. With the help of doubling theory and second order saturated design, we present a theory on constructing optimal blocked designs under the blocked general minimum lower order confounding criterion.

Keywords Aliased effect-number pattern, general minimum lower order confounding, second order saturated design, Yates order
MSC 62K15, 62K05

1 Introduction

Regular two-level designs are often used in factorial experiments due to their simple structure. Such a design usually involves a completely random allocation of the selected treatment combinations to the experimental units. This kind of allocation is appropriate only if the experimental units are homogeneous. However, when the size of the experiment is relatively large, it is difficult or impossible to keep the homogeneity of the experimental units. A practical design strategy is to partition the experimental units into homogeneous groups, known as blocks. It is an important issue to study the optimal way on blocking

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the experimental units.

Three types of optimality criteria have received significant attention in investigating this issue in the passed three decades. The first one extends the idea of the minimum aberration to the blocked case (see [4,10,12,16,17,22,23]). The second one is based on the clear effects criterion [1,3,15,24]. The last one is based on the maximum estimation capacity criterion [7,13].

Zhang et al. [19] proposed the *general minimum lower order confounding* (GMC for short) criterion for ranking 2^{n-m} designs, where the meaning of 2^{n-m} designs will be given in Section 2. The construction of the GMC 2^{n-m} designs were discussed in [6,8,11,18,20]. Zhang and Mukerjee [21] established a blocked GMC (B-GMC) criterion for selecting s-level regular blocked designs. Also for selecting optimal blocked designs, Wei et al. [14] extended the GMC idea with a different consideration from that in [21], and proposed another blocked GMC (B^1-GMC) criterion. The optimal designs with respect to B-GMC and B^1-GMC are different sometimes, but both are very useful in practice. The former is more suitable in the situation that the experimenter has no prior knowledge about the order of the importance of the treatment factors, while the latter is more suitable in the situation that the experimenter has such prior knowledge. For more explanation on the differences of the two criteria, please refer to [14,25].

Wei et al. [14] tabulated the B^1-GMC designs with 16, 32, and 64 runs, which are obtained by computer search. It is a time-consuming task to obtain the B^1-GMC designs in such a way, and necessary to establish a theory to systematically and completely construct B^1-GMC designs. Zhao et al. [25] developed a theory for constructing B^1-GMC $2^{n-m}:2^r$ designs with

$$\frac{5N}{16}+1\leqslant n\leqslant N-1,$$

where $N=2^{n-m}$. The detailed definition of '$2^{n-m}:2^r$ design' will be given in Section 2. The present paper develops a theory for constructing B^1-GMC $2^{n-m}:2^r$ designs with

$$\frac{17N}{64}+1\leqslant n\leqslant \frac{5N}{16}.$$

The rest of this paper is organized as follows. In Section 2, we first introduce the doubling theory which is useful for constructing the B^1-GMC designs and then give the definition of the B^1-GMC criterion. Sections 3 and 4 develop a theory for constructing B^1-GMC $2^{n-m}:2^r$ designs with

$$\frac{9N}{32}+1\leqslant n\leqslant \frac{5N}{16}$$

and

$$\frac{17N}{64}+1\leqslant n\leqslant \frac{9N}{32},$$

respectively. In Appendix, we give the proofs and tabulate some B^1-GMC designs with small run sizes.

2 Preliminaries: doubling and B^1-GMC criterion

Doubling is a simple but powerful method for constructing two-level fractional factorial designs of resolution IV (see [5]). The definition of resolution will be given in the following. Let X be a matrix with entries 1 and -1. Denote $J_0 = (1,1)^{\mathrm{T}}$ and $J_1 = (1,-1)^{\mathrm{T}}$. Then the double of X can be written as

$$D(X) = (J_0\ J_1) \otimes X = \begin{pmatrix} X & X \\ X & -X \end{pmatrix},$$

where $\otimes$ is the Kronecker product. Let $D^u(X)$ denote the matrix obtained by repeatedly doubling X u times. Then $D^u(X)$ can be written as

$$D^u(X) = \underbrace{(J_0\ J_1) \otimes (J_0\ J_1) \otimes \cdots \otimes (J_0\ J_1)}_{u \text{ times}} \otimes X.$$

Especially, for $X = (1)$, we have

$$D^u(1) = (I, 1, 2, 12, 3, 13, 23, 123, \dots, 12\cdots u)_{2^u},$$

where the subscript 2^u stands for the dimension of the columns in $D^u(1)$, and

$$\begin{aligned}
I &= (1, \dots, 1)_{2^u}^{\mathrm{T}}, \\
1 &= (1, -1, 1, -1, \dots, 1, -1, 1, -1)_{2^u}^{\mathrm{T}}, \\
2 &= (1, 1, -1, -1, \dots, 1, 1, -1, -1)_{2^u}^{\mathrm{T}}, \\
3 &= (1, 1, 1, 1, -1, -1, -1, -1, \dots, 1, 1, 1, 1, -1, -1, -1, -1)_{2^u}^{\mathrm{T}}, \\
\dots, \quad u &= (1, 1, \dots, 1, 1, -1, -1, \dots, -1, -1)_{2^u}^{\mathrm{T}},
\end{aligned}$$

and $ij\cdots k$ is the component-wise product of $i, j, \dots, k$, where $i, j, k = 1, 2, \dots, u$. For example,

$$\begin{aligned}
12 &= (1, -1, -1, 1, \dots, 1, -1, -1, 1)_{2^u}^{\mathrm{T}}, \\
23 &= (1, 1, -1, -1, \dots, -1, -1, 1, 1)_{2^u}^{\mathrm{T}}.
\end{aligned}$$

In the following, to avoid confusion, we sometimes use I_{2^u}, 1_{2^u}, 2_{2^u}, $(12)_{2^u}$ instead of I, 1, 2, 12, and so on.

Let

$$H_u = (1, 2, 12, \dots, 12\cdots u)_{2^u}.$$

Then H_u is a $2^u \times (2^u - 1)$ matrix and the columns of H_u are in *Yates order*. Let $H_{r,u}$ denote the matrix which consists of the first $2^r - 1$ columns of H_u with Yates order. For example,

$$H_{1,u} = (1)_{2^u}, \quad H_{2,u} = (1, 2, 12)_{2^u}.$$

In the following, let $q = n - m$. The matrix H_q is just a two-level regular saturated $2^{(N-1)-(N-1-q)}$ design, where $N = 2^q$. A *regular* 2^{n-m} *design* D_t

consists of n columns of H_q with q of them being independent. The remaining m columns are expressed by the q independent columns and determine m independent defining words. The m defining words generate the defining contrast subgroup of the design. Each element in the defining contrast subgroup except for I, the grand mean, is called a defining word. The number of letters in a defining word is called its length. The *resolution* of a 2^{n-m} design D_t is the length of the shortest defining word. Two 2^{n-m} designs are said to be *isomorphic* if the defining contrast subgroup of one of them can be obtained from that of the other by permuting the factor labels.

A *blocked* $2^{n-m}:2^r$ *design* is to arrange a regular 2^{n-m} design D_t into 2^r groups (blocks) of size 2^{n-m-r} by selecting r independent columns from $H_q\backslash D_t$ as the r block factors. Hereafter, $A\backslash B$ denotes the matrix which consists of the columns of A but not those of B. The 2^r-1 columns including the r independent columns and the component-wise products of any v $(2\leqslant v\leqslant r)$ of the r columns constitute a closed submatrix of H_q, denoted as D_b. Hereafter, a closed matrix means that the component-wise product of any two columns of the matrix is still a column of the matrix. Let $D=(D_t:D_b)$ denote a blocked $2^{n-m}:2^r$ design with the columns of D_b corresponding to 2^r-1 block effects of D. Hereafter, $A\cap B$ denotes the matrix which consists of the common columns of A and B, and $A\cap B=\emptyset$ means that A and B have no common column. A design with $D_t\cap D_b\neq\emptyset$ is not a good selection, since it will lead to confounding of some main treatment effects with the block effects which can usually be quite nontrivial. Thus, we consider only the designs $D=(D_t:D_b)$ with $D_t\cap D_b=\emptyset$.

For a $2^{n-m}:2^r$ design $D=(D_t:D_b)$, let ${}_i^{\#}C_j^{(k)}(D)$ denote the number of ith-order treatment effects which are aliased with k jth-order treatment effects, but not with I and the block effects. Under the assumption that all the interactions involving three or more treatment factors are negligible, we consider only the main effects and two-factor interactions. Denote

$${}_1^{\#}C_2(D)=({}_1^{\#}C_2^{(0)}(D),{}_1^{\#}C_2^{(1)}(D),\ldots,{}_1^{\#}C_2^{(K_2)}(D))$$

and

$${}_2^{\#}C_2(D)=({}_2^{\#}C_2^{(0)}(D),{}_2^{\#}C_2^{(1)}(D),\ldots,{}_2^{\#}C_2^{(K_2)}(D)),$$

where $K_2=n(n-1)/2$. Let

$${}^{\#}C(D)=({}_1^{\#}C_2(D),{}_2^{\#}C_2(D)). \tag{1}$$

Pattern (1) is called the *blocked aliased-effect number pattern* (B^1-AENP). Zhao et al. [25] defined the B^1-GMC designs as follows.

Definition 1 Let ${}^{\#}C_l$ be the l-th component of ${}^{\#}C$, and let ${}^{\#}C(D_1)$ and ${}^{\#}C(D_2)$ be the B^1-AENPs of designs D_1 and D_2, respectively. Suppose that ${}^{\#}C_t$ is the first component such that ${}^{\#}C_t(D_1)$ and ${}^{\#}C_t(D_2)$ are different. If ${}^{\#}C_t(D_1)>{}^{\#}C_t(D_2)$, then D_1 is said to have less blocked general lower order confounding (B^1-GLOC for short) than D_2. A blocked design D is said to have blocked general minimum lower order confounding (B^1-GMC for short) if no other blocked design has less B^1-GLOC than D and such a design is called a B^1-GMC design.

3 B^1-GMC $2^{n-m}:2^r$ designs with $\frac{9N}{32}+1 \leqslant n \leqslant 5N/16$

A 2^{n-m} fractional factorial design is called a *second order saturated* (*SOS*) *design* if all its degrees of freedom are completely used to estimate the main effects and two-factor interactions (see [2]). In terms of coding theory, Davydov and Tombak [9] showed that, for $n \geqslant \frac{N}{4}+1$, only when

$$n = \frac{N}{2},\ \frac{5N}{16},\ \frac{9N}{32},\ \frac{17N}{64},\ \frac{33N}{128},\ \ldots,$$

the SOS designs exist, and the SOS designs with $\frac{N}{4}+1 \leqslant n \leqslant N/2$ can be obtained by doubling some given SOS designs. Denote the SOS designs as $S_{(N/2)}, S_{(5N/16)}, S_{(9N/32)}, \ldots$, respectively. For example, let

$$X_1 = (x_1, x_2, x_3, x_4, x_5)_{2^4},$$

where x_i, $i = 1, 2, 3, 4$, are four independent columns of H_4 and $x_5 = x_1x_2x_3x_4$. Then X_1 is the unique 2^{5-1} SOS design up to isomorphism. Thus, when $N \geqslant 32$, the SOS design $S_{(5N/16)}$ can be uniquely obtained by doubling X_1 $\log_2(N/16)$ $(= q-4)$ times, i.e.,

$$S_{(5N/16)} = D^{q-4}(X_1), \quad N = 2^q.$$

Clearly, we have

$$D^q(1) = D^{q-u}(D^u(1)), \quad \forall\, u < q.$$

Suppose that the columns of $X = (x_1, \ldots, x_t)$ are taken from $D^u(1)$. Then $D^q(1)$ can be written as

$$D^q_{RC}(1) = (D^{q-u}(D^u(1)\backslash X), D^{q-u}(x_1), \ldots, D^{q-u}(x_t))$$

up to a permutation upon the columns. We say that $D^q_{RC}(1)$ is in a *re-changed Yates order* (RC-Yates order, for short) related to X.

With an exchange of columns, we can write $D^q(1)\backslash I_N$ as

$$\begin{aligned} &D^q(1)\backslash I_N \\ &= (D^{q-4}(D^4(1)\backslash X_1)\backslash I_N, D^{q-4}(x_1), D^{q-4}(x_2), D^{q-4}(x_3), D^{q-4}(x_4), D^{q-4}(x_5)) \\ &= (D^{q-4}(I_{16})\backslash I_N, D^{q-4}(x_1), D^{q-4}(x_2), D^{q-4}(x_3), D^{q-4}(x_4), D^{q-4}(x_5), Z), \end{aligned} \quad (2)$$

where

$$\begin{aligned} Z = (&D^{q-4}(x_1x_2), D^{q-4}(x_1x_3), D^{q-4}(x_1x_4), D^{q-4}(x_1x_5), D^{q-4}(x_2x_3), \\ &D^{q-4}(x_2x_4), D^{q-4}(x_2x_5), D^{q-4}(x_3x_4), D^{q-4}(x_3x_5), D^{q-4}(x_4x_5)), \end{aligned} \quad (3)$$

and we have

$$S_{(5N/16)} = (D^{q-4}(x_1), D^{q-4}(x_2), D^{q-4}(x_3), D^{q-4}(x_4), D^{q-4}(x_5))$$

in RC-Yates order.

If a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ has B^1-GMC, then D sequentially maximizes (1). Let ${}_1^{\#}C_2^{(k)}(D_t)$ denote the number of main effects aliased with k two-factor interactions in the unblocked 2^{n-m} design D_t. We have

$${}_1^{\#}C_2^{(k)}(D_t) = {}_1^{\#}C_2^{(k)}(D)$$

for any k by noting that $D_t \cap D_b = \emptyset$.

According to (1) and the B^1-GMC criterion, if $D = (D_t : D_b)$ has B^1-GMC, then D must maximize ${}_1^{\#}C_2(D)$, the first part of (1). Note that

$${}_1^{\#}C_2(D) = {}_1^{\#}C_2(D_t),$$

and hence, ${}_1^{\#}C_2(D)$ only depends on D_t. We should consider the part D_t of D as an unblocked design and optimally choose it first. When $n \leqslant 5N/16$, D_t must have resolution at least IV if it maximizes ${}_1^{\#}C_2(D_t)$ (see [19]). Furthermore, when

$$\frac{9N}{32} + 1 \leqslant n \leqslant \frac{5N}{16}, \tag{4}$$

D_t must be an n-projection of $S_{(N/2)}$ or $S_{(5N/16)}$ if it has resolution at least IV (see [5]). Hereafter, the statement 'A is an n-projection of B', denoted as $A \subset B$, implies that A is a submatrix of B with the n columns of A coming from B. For example, X_1 is a 5-projection of H_4. The following lemma shows that D_t must be an n-projection of $S_{(5N/16)}$ and, up to isomorphism,

$$\overline{D}_t = S_{(5N/16)} \backslash D_t \subset D^{q-4}(x_1)$$

if $D = (D_t : D_b)$ has B^1-GMC.

Lemma 1 *When $r \leqslant q-2$, if a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ with (4) has B^1-GMC, then we have*

(a) *D_t must be an n-projection of $S_{(5N/16)}$ but not an n-projection of $S_{(N/2)}$;*

(b) *up to isomorphism, $\overline{D}_t \subset D^{q-4}(x_1)$, where $\overline{D}_t = S_{(5N/16)} \backslash D_t$.*

To give the construction of B^1-GMC designs $D = (D_t : D_b)$ with (4), according to Lemma 1, we can assume that $D_t \subset S_{(5N/16)}$ and $\overline{D}_t \subset D^{q-4}(x_1)$, and should consider the selection of D_b. Hereafter, the term $a \in A$ means that a is a column of the matrix A. For easy presentation, we introduce two more pieces of notation. For given q and $x \in D^u(1) \backslash I_{2^u}$ with $q - u \geqslant r$, let

$$G_{r,u} = H_{r,q-u} \otimes I_{2^u}, \tag{5}$$

and

$$F_{x,r} = (I_{2^{q-u}}, H_{r-1,q-u}) \otimes x = (I_{2^{q-u}} \otimes x, (I_{2^{q-u}} \otimes x)G_{r-1,u}), \tag{6}$$

which consists of the first 2^{r-1} columns of $D^{q-u}(x)$, where $H_{r,q-u}$ consists of the first $2^r - 1$ columns of H_{q-u}, and $(I_{2^{q-u}} \otimes x)G_{r-1,u}$ denotes the matrix obtained by taking the component-wise products of $I_{2^{q-u}} \otimes x$ and the columns of $G_{r-1,u}$.

Lemma 2 *Suppose that*

$$D_t \subset S_{(5N/16)}, \quad \overline{D}_t \subset D^{q-4}(x_1),$$

where $\overline{D}_t = S_{(5N/16)} \backslash D_t$. *Up to isomorphism, there are two classes of the possibilities for the block effects matrix* D_b:

(a)

$$\begin{aligned} D_b = {} & (G_{r-1,4}, F_{x_1,r}), \text{ or} \\ & (G_{r-2,4}, F_{x_1,r-1}, F_{x_2x_3,r-1}, F_{x_4x_5,r-1}), \text{ or} \\ & (G_{r-3,4}, F_{x_1,r-2}, F_{x_2x_3,r-2}, F_{x_2x_4,r-2}, F_{x_2x_5,r-2}, F_{x_3x_4,r-2}, \\ & \ F_{x_3x_5,r-2}, F_{x_4x_5,r-2}), \end{aligned}$$

(b)

$$\begin{aligned} D_b = {} & G_{r,4}, \text{ or} \\ & (G_{r-1,4}, F_{x_ix_j,r}) \text{ (with } \{i,j\} = \{1,2\} \text{ or } \{2,3\}), \text{ or} \\ & (G_{r-2,4}, F_{x_ux_v,r-1}, F_{x_ux_w,r-1}, F_{x_vx_w,r-1}) \\ & \ (\text{with } \{u,v,w\} = \{1,2,3\} \text{ or } \{2,3,4\}). \end{aligned}$$

Suppose $D_t \subset S_{(5N/16)}$ with $\overline{D}_t \subset D^{q-4}(x_1)$. For all the cases of D_b in Lemma 2, let

$$D_{b_0} = D_b \cap (D^{q-4}(I_{16}) \backslash I_N), \tag{7}$$

$$D_{b_1} = D_b \cap (D^{q-4}(x_1x_2), D^{q-4}(x_1x_3), D^{q-4}(x_1x_4), D^{q-4}(x_1x_5)),$$

$$D_{b_2} = D_b \cap S,$$

where

$$\begin{aligned} S = {} & (D^{q-4}(x_2x_3), D^{q-4}(x_2x_4), D^{q-4}(x_2x_5), D^{q-4}(x_3x_4), \\ & D^{q-4}(x_3x_5), D^{q-4}(x_4x_5)). \end{aligned} \tag{8}$$

Denote $D^* = (\overline{D}_t : D_{b_0})$. The following lemma establishes the relation between the B^1-AENPs of $D = (D_t : D_b)$ and $D^* = (\overline{D}_t : D_{b_0})$, which plays an important role in the construction of B^1-GMC designs with (4). Before presenting the lemma, we introduce two more pieces of notation. Let

$$B_2(D_t, \gamma) = \#\{(d_1, d_2) \colon d_1, d_2 \in D_t,\ \gamma \in H_q,\ d_1d_2 = \gamma\}$$

and

$$f(D^*) = \#(D_{b_0}) + \#\{\gamma\colon \gamma \in H_q \backslash D_{b_0},\ B_2(\overline{D}_t, \gamma) > 0\},$$

where # denotes the number of columns of a matrix or the cardinality of a set.

Lemma 3 *Let $D = (D_t : D_b)$ be a $2^{n-m} : 2^r$ design with*

$$D_t \subset S_{(5N/16)}, \quad \overline{D}_t \subset D^{q-4}(x_1), \quad n - m \geqslant 5,$$

and (4). *Then*

(a)

$$B_2(D_t, \gamma) = \begin{cases} 0, & \gamma \in S_{(5N/16)}, \\ \dfrac{N}{16} - l, & \gamma \in D^{q-4}(x_1 x_i),\ i = 2, 3, 4, 5, \\ \dfrac{N}{16}, & \gamma \in D^{q-4}(x_i x_j),\ 2 \leqslant i \neq j \leqslant 5, \\ \dfrac{5N}{32} - l + B_2(\overline{D}_t, \gamma), & \gamma \in D^{q-4}(I_{16}) \backslash I_N; \end{cases}$$

(b)

$${}_1^{\#}C_2^{(k)}(D) = \begin{cases} n, & k = 0, \\ 0, & k \neq 0; \end{cases}$$

(c)

$${}_2^{\#}C_2^{(k)}(D) = \begin{cases} (k+1)\Big(\dfrac{N}{4} - \#(D_{b_1})\Big), & k = \dfrac{N}{16} - l - 1, \\ (k+1)\Big(\dfrac{3N}{8} - \#(D_{b_2})\Big), & k = \dfrac{N}{16} - 1, \\ (k+1)\Big(\dfrac{N}{16} - 1 - f(D^*)\Big), & k = \dfrac{5N}{32} - l - 1, \\ \dfrac{k+1}{k+1-\frac{5N}{32}+l}\, {}_2^{\#}C_2^{(k-\frac{5N}{32}+l)}(D^*), & k > \dfrac{5N}{32} - l - 1, \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$l = \#(\overline{D}_t) = \frac{5N}{16} - n.$$

Lemma 3 implies that a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ with (4) has B^1-GMC if and only if it sequentially maximizes

$$(-\#(D_{b_1}), -\#(D_{b_2}), -f(D^*), {}_2^{\#}C_2(D^*)). \tag{9}$$

With the help of (9), the following theorem constructs B^1-GMC $2^{n-m} : 2^r$ designs with (4).

Theorem 1 *Suppose that $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design with* (4)*. Then D has B*[1]*-GMC if D_t consists of the last n columns of $S_{(5N/16)}$ with RC-Yates order and*

(a) *$D_b = G_{r,4}$ when $r \leqslant q-4$ and*

$$\frac{5N}{16} - 2^{r-1} + 1 \leqslant n \leqslant \frac{5N}{16};$$

(b) *$D_b = (G_{r-1,4}, F_{x_1,r})$ when $r \leqslant q-4$ and*

$$\frac{9N}{32} + 1 \leqslant n \leqslant \frac{5N}{16} - 2^{r-1};$$

(c) *$D_b = (G_{q-4,4}, F_{x_2x_3,q-3})$ when $r = q-3$;*

(d) *$D_b = (G_{q-4,4}, F_{x_2x_3,q-3}, F_{x_2x_4,q-3}, F_{x_3x_4,q-3})$ when $r = q-2$,*

where $G_{\cdot,\cdot}$ and $F_{\cdot,\cdot}$ are as defined in (5) *and* (6)*, respectively.*

The following example illustrates the construction method in Theorem 1. In the example, we take

$$X_1 = (x_1, x_2, x_3, x_4, x_5)_{2^4} = (1, 2, 3, 4, 1234)_{2^4}.$$

Example 1 Consider the construction of $2^{38-31} : 2^r$ B[1]-GMC designs. Then $q = 7$, and $N = 2^7$. Take the last 38 columns of $S_{(5N/16)}$ $(= D^{q-4}(X_1))$ with RC-Yates order as D_t. Next, we consider the block effects matrices D_b for $r = 1, 2, 3, 4, 5$, respectively. For $r = 1, 2$, which correspond to case (b) of Theorem 1, we can get

$$D_b = \begin{cases} F_{x_1,1}, & r = 1, \\ (G_{1,4}, F_{x_1,2}), & r = 2. \end{cases}$$

For $r = 3, 4, 5$, which correspond to cases (a), (c), and (d) of Theorem 1, respectively, we can get

$$D_b = \begin{cases} G_{3,4}, & r = 3, \\ (G_{3,4}, F_{x_2x_3,4}), & r = 4, \\ (G_{3,4}, F_{x_2x_3,4}, F_{x_2x_4,4}, F_{x_3x_4,4}), & r = 5. \end{cases}$$

The constructed designs are all B[1]-GMC designs by Theorem 1.

4 B[1]-GMC $2^{n-m} : 2^r$ designs with $\frac{17N}{64} + 1 \leqslant n \leqslant 9N/32$

This section studies the theory of constructing B[1]-GMC $2^{n-m} : 2^r$ designs with

$$\frac{17N}{64} + 1 \leqslant n \leqslant \frac{9N}{32}. \tag{10}$$

Let

$$X_2 = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)_{2^5}$$

be the 2^{9-4} design determined by

$$x_6 = x_1x_2x_3x_4, \quad x_7 = x_1x_2x_3x_5, \quad x_8 = x_1x_2x_4x_5, \quad x_9 = x_3x_4x_5,$$

where x_i, $i = 1, 2, \ldots, 5$, are five independent columns of H_5. Then X_2 is the unique 2^{9-4} SOS design up to isomorphism. Thus, when $N \geqslant 64$, the SOS design $S_{(9N/32)}$ can be uniquely obtained by doubling X_2 $\log_2(N/32)$ $(= q - 5)$ times, i.e.,

$$S_{(9N/32)} = D^{q-5}(X_2), \quad N = 2^q.$$

With an exchange of columns, we can write $D^q(1)\backslash I_N$ as

$$\begin{aligned} D^q(1)\backslash I_N &= (D^{q-5}(D^5(1)\backslash X_2)\backslash I_N, D^{q-5}(x_1), \ldots, D^{q-5}(x_9)) \\ &= (D^{q-5}(I_{32})\backslash I_N, D^{q-5}(x_1), \ldots, D^{q-5}(x_9), Z'), \end{aligned}$$

where Z' consists of the remaining columns of $D^q(1)\backslash I_N$. Then we have

$$S_{(9N/32)} = (D^{q-5}(x_1), \ldots, D^{q-5}(x_9))$$

in RC-Yates order.

Suppose that $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ blocked design with (10). If D has B^1-GMC, then D must maximize ${}_1^{\#}C_2(D)$, the first part of (1). Noting that

$${}_1^{\#}C_2(D) = {}_1^{\#}C_2(D_t),$$

D_t must maximize ${}_1^{\#}C_2(D_t)$ and have resolution at least IV (see [19]). Furthermore, when (10) is satisfied, D_t must be an n-projection of $S_{(N/2)}$, $S_{(5N/16)}$, or $S_{(9N/32)}$ if it has resolution at least IV (see [5]). The following lemma shows that D_t must be an n-projection of $S_{(9N/32)}$ and, up to isomorphism,

$$\overline{D}_t = S_{(9N/32)}\backslash D_t \subset D^{q-5}(x_1)$$

if $D = (D_t : D_b)$ has B^1-GMC.

Lemma 4 *When $r \leqslant q-2$, if a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ with (10) has B^1-GMC, then we have*

(a) *D_t must be an n-projection of $S_{(9N/32)}$ but not an n-projection of $S_{(N/2)}$ or $S_{(5N/16)}$;*

(b) *up to isomorphism, $\overline{D}_t \subset D^{q-5}(x_1)$, where $\overline{D}_t = S_{(9N/32)}\backslash D_t$.*

To give the construction of B^1-GMC designs $D = (D_t : D_b)$ with (10), by Lemma 4, we can suppose that $D_t \subset S_{(9N/32)}$ and $\overline{D}_t \subset D^{q-5}(x_1)$ and should consider the selection of D_b as follows.

Lemma 5 *Suppose that $D_t \subset S_{(9N/32)}$ and $\overline{D}_t \subset D^{q-5}(x_1)$, where $\overline{D}_t = S_{(9N/32)} \backslash D_t$. Up to isomorphism, there are two classes of the possibilities for the block effects matrix D_b:*

(a)

$$\begin{aligned} D_b = {} & (G_{r-1,5}, F_{x_1,r}), \text{ or} \\ & (G_{r-2,5}, F_{x_1,r-1}, F_{x_2x_3,r-1}, F_{x_4x_6,r-1}), \text{ or} \\ & (G_{r-3,5}, F_{x_1,r-2}, F_{x_2x_3,r-2}, F_{x_2x_4,r-2}, F_{x_2x_6,r-2}, \\ & \ F_{x_3x_4,r-2}, F_{x_3x_6,r-2}, F_{x_4x_6,r-2}), \text{ or} \\ & (G_{r-4,5}, F_{x_1,r-3}, F_{x_2x_3,r-3}, F_{x_2x_4,r-3}, F_{x_2x_5,r-3}, F_{x_2x_6,r-3}, \\ & \ F_{x_2x_7,r-3}, F_{x_2x_8,r-3}, F_{x_2x_9,r-3}, F_{x_3x_4,r-3}, F_{x_3x_5,r-3}, \\ & \ F_{x_3x_6,r-3}, F_{x_3x_7,r-3}, F_{x_3x_8,r-3}, F_{x_3x_9,r-3} F_{x_4x_6,r-3}), \end{aligned}$$

(b)

$$\begin{aligned} D_b = {} & G_{r,5}, \text{ or} \\ & (G_{r-1,5}, F_{x_ix_j,r}) \ (\text{with } \{i,j\} = \{1,2\},\ \{2,3\},\ \text{or } \{3,4\}), \text{ or} \\ & (G_{r-2,5}, F_{x_ux_v,r-1}, F_{x_ux_w,r-1}, F_{x_vx_w,r-1}) \\ & \ (\text{with } \{u,v,w\} = \{1,2,3\},\ \{1,3,4\},\ \{2,3,4\},\ \text{or } \{3,4,5\}), \text{ or} \\ & (G_{r-3,5}, F_{x_ix_3,r-2}, F_{x_ix_4,r-2}, F_{x_ix_5,r-2}, F_{x_ix_9,r-2}, F_{x_3x_4,r-2}, \\ & \ F_{x_3x_5,r-2}, F_{x_3x_9,r-2}) \ (\text{ with } i = 1,\ 2,\ \text{or } 6). \end{aligned}$$

Suppose $D_t \subset S_{(9N/32)}$ with $\overline{D}_t \subset D^{q-5}(x_1)$. Let $A \cup B$ denote the matrix with columns coming from A or B. For example, if

$$A = (a_1, a_2), \quad B = (b_1, b_2),$$

then

$$A \cup B = (a_1, a_2, b_1, b_2).$$

For all the cases of D_b in Lemma 5, denote

$$D_{b_0} = D_b \cap (D^{q-5}(I_{32}) \backslash I_N), \quad D_{b_1} = D_b \cap \left(\bigcup_{i=2}^{9} D^{q-5}(x_1x_i) \right), \tag{11}$$

$$D_{b_2} = D_b \cap \left(\bigcup_{i=3}^{9} D^{q-5}(x_2x_i) \right), \quad D_{b_3} = D_b \cap \left(\bigcup_{3 \leqslant i \neq j \leqslant 9} D^{q-5}(x_ix_j) \right).$$

Let $D^* = (\overline{D}_t : D_{b_0})$. The following lemma establishes the relation between the B^1-AENPs of $D = (D_t : D_b)$ and $D^* = (\overline{D}_t : D_{b_0})$, which plays an important role in the construction of B^1-GMC designs with (10).

Lemma 6 *Let $D = (D_t : D_b)$ be a $2^{n-m} : 2^r$ design with*

$$D_t \subset S_{(9N/32)}, \quad \overline{D}_t \subset D^{q-5}(x_1), \quad n - m \geqslant 6,$$

and (10). *Then*

(a)

$$B_2(D_t,\gamma)=\begin{cases}0, & \gamma\in S_{(9N/32)},\\ \dfrac{N}{32}-l, & \gamma\in D^{q-5}(x_1x_i),\ i=2,3,\ldots,9,\\ \dfrac{N}{32}, & \gamma\in D^{q-5}(x_2x_j),\ j=3,4,\ldots,9,\\ \dfrac{3N}{32}, & \gamma\in D^{q-5}(x_ix_j),\ 3\leqslant i\neq j\leqslant 9,\\ \dfrac{9N}{64}-l+B_2(\overline{D}_t,\gamma), & \gamma\in D^{q-5}(I_{32})\backslash I_N;\end{cases}$$

(b)

$$_1^{\#}C_2^{(k)}(D)=\begin{cases}n, & k=0,\\ 0, & k\neq 0;\end{cases}$$

(c)

$$_2^{\#}C_2^{(k)}(D)=\begin{cases}(k+1)\Big(\dfrac{N}{4}-\#(D_{b_1})\Big), & k=\dfrac{N}{32}-l-1,\\ (k+1)\Big(\dfrac{7N}{32}-\#(D_{b_2})\Big), & k=\dfrac{N}{32}-1,\\ (k+1)\Big(\dfrac{7N}{32}-\#(D_{b_3})\Big), & k=\dfrac{3N}{32}-1,\\ (k+1)\Big(\dfrac{N}{32}-1-f(D^*)\Big), & k=\dfrac{9N}{64}-l-1,\\ \dfrac{k+1}{k+1-\frac{9N}{64}+l}\,{}_2^{\#}C_2^{(k-\frac{9N}{64}+l)}(D^*), & k>\dfrac{9N}{64}-l-1,\\ 0, & \textit{otherwise},\end{cases}$$

where

$$l=\#(\overline{D}_t)=\frac{9N}{32}-n.$$

Lemma 6 implies that a $2^{n-m}:2^r$ design $D=(D_t:D_b)$ with (10) has B^1-GMC if and only if it sequentially maximizes

$$(-\#(D_{b_1}),-\#(D_{b_2}),-\#(D_{b_3}),-f(D^*),{}_2^{\#}C_2(D^*)). \tag{12}$$

With the help of (12), the following theorem constructs the B^1-GMC $2^{n-m}:2^r$ designs with (10).

Theorem 2 *Suppose that* $D=(D_t:D_b)$ *is a* $2^{n-m}:2^r$ *design with* (10). *Then* D *has* B^1*-GMC if* D_t *consists of the last* n *columns of* $S_{(9N/32)}$ *with RC-Yates order and*

(a) $D_b = G_{r,5}$ *when* $r \leqslant q-5$ *and*

$$\frac{9N}{32} - 2^{r-1} + 1 \leqslant n \leqslant \frac{9N}{32};$$

(b) $D_b = (G_{r-1,5}, F_{x_1,r})$ *when* $r \leqslant q-5$ *and*

$$\frac{17N}{64} + 1 \leqslant n \leqslant \frac{9N}{32} - 2^{r-1};$$

(c) $D_b = (G_{q-5,5}, F_{x_3x_4,q-4})$ *when* $r = q-4$;

(d) $D_b = (G_{q-5,5}, F_{x_3x_4,q-4}, F_{x_3x_5,q-4}, F_{x_4x_5,q-4})$ *when* $r = q-3$;

(e)

$$\begin{aligned} D_b = (&G_{q-5,5}, F_{x_3x_4,q-4}, F_{x_3x_5,q-4}, F_{x_3x_6,q-4}, F_{x_3x_9,q-4}, F_{x_4x_6,q-4}, \\ &F_{x_5x_6,q-4}, F_{x_6x_9,q-4}) \end{aligned}$$

when $r = q-2$,
where $G_{\cdot,\cdot}$ *and* $F_{\cdot,\cdot}$ *are as defined in* (5) *and* (6), *respectively.*

The following example shows an application of Theorem 2.

Example 2 Consider the construction of $2^{70-62} : 2^r$ B[1]-GMC designs. Here,

$$\begin{aligned} X_2 &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)_{2^5} \\ &= (1, 2, 3, 4, 5, 6\,(=1234),\ 7\,(=1235),\ 8\,(=1245),\ 9\,(=345))_{32}, \end{aligned}$$

$q = 8$, and $N = 2^8$. Take the last 70 columns of $S_{(9N/32)}$ $(= D^{q-5}(X_2))$ with RC-Yates order as D_t. Next, we consider the block effects matrices D_b for $r = 1, 2, 3, 4, 5, 6$, respectively. For $r = 1, 2$, which correspond to case (b) of Theorem 2, we can get

$$D_b = \begin{cases} F_{x_1,1}, & r = 1, \\ (G_{1,5}, F_{x_1,2}), & r = 2. \end{cases}$$

For $r = 3, 4, 5, 6$, which correspond to cases (a), (c), (d), and (e) of Theorem 2, respectively, we can get

$$D_b = \begin{cases} G_{3,5}, & r = 3, \\ (G_{3,5}, F_{x_3x_4,4}), & r = 4, \\ (G_{3,5}, F_{x_3x_4,4}, F_{x_3x_5,4}, F_{x_4x_5,4}), & r = 5, \\ (G_{3,5}, F_{x_3x_4,4}, F_{x_3x_5,4}, F_{x_3x_6,4}, F_{x_3x_9,4}, F_{x_4x_6,4}, F_{x_5x_6,4}, F_{x_6x_9,4}), & r = 6. \end{cases}$$

The constructed designs are all B[1]-GMC designs by Theorem 2.

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Appendix

Table A1 B^1-GMC $2^{n-m}:2^r$ designs with small run sizes

N	n	D_t	r	independent columns of D_b	Theorem
32	9	X_2	1	$\{34\}_{32}$	2 (c)
32	9	X_2	2	$\{34, 35\}_{32}$	2 (d)
32	9	X_2	3	$\{34, 35, 124\}_{32}$	2 (e)
32	10	$D(X_1)$	1	$1_2 \otimes I_{16}$	1 (a)
32	10	$D(X_1)$	2	$1_2 \otimes I_{16},\ I_2 \otimes (23)_{16}$	1 (c)
32	10	$D(X_1)$	3	$1_2 \otimes I_{16},\ I_2 \otimes (23)_{16},\ I_2 \otimes (24)_{16}$	1 (d)
64	18	$D(X_2)$	1	$1_2 \otimes I_{32}$	2 (a)
64	18	$D(X_2)$	2	$1_2 \otimes I_{32},\ I_2 \otimes (34)_{32}$	2 (c)
64	18	$D(X_2)$	3	$1_2 \otimes I_{32},\ I_2 \otimes (34)_{32},\ I_2 \otimes (35)_{32}$	2 (d)
64	18	$D(X_2)$	4	$1_2 \otimes I_{32},\ I_2 \otimes (34)_{32},\ I_2 \otimes (35)_{32},\ I_2 \otimes (124)_{32}$	2 (e)
64	19	$D^2(X_1)\backslash(I_4 \otimes 1_{16})$	1	$I_4 \otimes 1_{16}$	1 (b)
64	19	$D^2(X_1)\backslash(I_4 \otimes 1_{16})$	2	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16}$	1 (a)
64	19	$D^2(X_1)\backslash(I_4 \otimes 1_{16})$	3	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16},\ I_4 \otimes (23)_{16}$	1 (c)
64	19	$D^2(X_1)\backslash(I_4 \otimes 1_{16})$	4	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16},\ I_4 \otimes (23)_{16},\ I_4 \otimes (24)_{16}$	1 (d)
64	20	$D^2(X_1)$	1	$1_4 \otimes I_{16}$	1 (a)
64	20	$D^2(X_1)$	2	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16}$	1 (a)
64	20	$D^2(X_1)$	3	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16},\ I_4 \otimes (23)_{16}$	1 (c)
64	20	$D^2(X_1)$	4	$1_4 \otimes I_{16},\ 2_4 \otimes I_{16},\ I_4 \otimes (23)_{16},\ I_4 \otimes (24)_{16}$	1 (d)

$X_1 = \{1, 2, 3, 4, 5(= 1234)\}_{16}$
$X_2 = \{1, 2, 3, 4, 5, 6\,(= 1234),\ 7\,(= 1235),\ 8\,(= 1245),\ 9\,(= 345)\}_{32}$

Proof of Lemma 1 As discussed above Lemma 1, when (4) is satisfied, if $D = (D_t : D_b)$ has B^1-GMC, then D_t is an n-projection of $S_{(5N/16)}$ or $S_{(N/2)}$. Note that both $S_{(5N/16)}$ and $S_{(N/2)}$ have resolution at least IV. If $D = (D_t : D_b)$ has B^1-GMC, then it is necessary that D has the maximum number of alias sets that contain the minimum number of two-factor interactions but none of the block effects.

For a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ with $D_t \subset S_{(5N/16)}$, let $\overline{D}_t = S_{(5N/16)}\backslash D_t$. According to [18, Theorem 4], D_t has two nonisomorphic choices when it is the projection of $S_{(5N/16)}$:

(i) $\overline{D}_t \subset D^{q-4}(x_i)$ for some $i = 1, 2, 3, 4, 5$ (since x_i, $i = 1, 2, 3, 4, 5$, play the same role in $S_{(5N/16)}$, without loss of generality, we assume $\overline{D}_t \subset D^{q-4}(x_1)$);

(ii) $\overline{D}_t$ contains columns from at least two of $D^{q-4}(x_i)$, $i = 1, 2, 3, 4, 5$.

We first consider case (i), i.e., $D_t \subset S_{(5N/16)}$ and $\overline{D}_t \subset D^{q-4}(x_1)$.

For any column $\gamma \in D^q(1)\backslash I_N$ as defined in (2), γ corresponds to an alias set of D_t. When $\gamma \in D_t$, the alias set contains a main effect. When $\gamma \in \overline{D}_t$, the alias set contains only interactions involving three or more factors. According to [18, Theorem 4], for any $\gamma \in D^{q-4}(x_1x_i)$, $i = 2, 3, 4, 5$, there are $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. There are $N/4$ alias sets with $\gamma \in D^{q-4}(x_1x_i)$, $i = 2, 3, 4, 5$. For any $\gamma \in D^{q-4}(x_ix_j)$, $2 \leqslant i \neq j \leqslant 5$, there are $N/16$ two-factor interactions of D_t aliased with it. There are altogether $3N/8$ alias sets with $\gamma \in D^{q-4}(x_ix_j)$, $2 \leqslant i \neq j \leqslant 5$. For any $\gamma \in D^{q-4}(I_{16})\backslash I_N$, there are at least $n - \frac{5N}{32}$ two-factor interactions of D_t aliased with it. There are altogether $\frac{N}{16} - 1$ alias sets with $\gamma \in D^{q-4}(I_{16})\backslash I_N$ and these γ form a closed submatrix of H_q.

When we choose columns from $D^q(1)\backslash I_N$ as block effects, we should first choose the columns which are neither aliased with the main effects nor with the two-factor interactions, and then the columns which are aliased with the two-factor interactions of D_t in the most serious degree. Note that

$$\#\left\{ \bigcup_{2 \leqslant i \neq j \leqslant 4} D^{q-4}(x_ix_j) \cup (D^{q-4}(I_{16})\backslash I_N) \right\} = \frac{N}{4} - 1.$$

For any $r \leqslant q - 2$, it is available to choose $2^r - 1$ columns from

$$\bigcup_{2 \leqslant i \neq j \leqslant 4} D^{q-4}(x_ix_j) \cup (D^{q-4}(I_{16})\backslash I_N)$$

to form a closed matrix as D_b. With doing that, for any $\gamma \in D^{q-4}(x_1x_i)$, $i = 2, 3, 4, 5$, there are still $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. There are $N/4$ alias sets with $\gamma \in D^{q-4}(x_1x_i)$, $i = 2, 3, 4, 5$. For any $\gamma \in D^{q-4}(x_ix_5)$, $i = 2, 3, 4$, there are still $N/16$ two-factor interactions of D_t aliased with γ. There are $3N/16$ alias sets with $\gamma \in D^{q-4}(x_ix_5)$, $i = 2, 3, 4$.

Now, we consider case (ii), i.e., $\overline{D}_t$ contains columns from at least two of $D^{q-4}(x_i)$, $i = 1, 2, 3, 4, 5$.

According to [18, Theorem 4 (ii)], D_t has no more than $N/16$ alias sets, each of which contains $n - \frac{N}{4}$ two-factor interactions. Suppose that the columns in D_b do not occupy any alias set containing $n - \frac{N}{4}$ two-factor interactions. The blocked design $D = (D_t : D_b)$ just has no more than $N/16$ alias sets, each of which contains $n - \frac{N}{4}$ two-factor interactions.

If $D_t \subset S_{(N/2)}$, according to [18, Theorem 3], besides the $N/4$ alias sets each of which containing $n - \frac{N}{4}$ two-factor interactions, there are at least $N/8$ two-factor interactions in each of the other alias sets of D_t containing two-factor interactions. Comparing with cases (i) and (ii) above, when D_t is the projection of $S_{(5N/16)}$ with $\overline{D}_t \subset D^{q-4}(x_1)$, D has the maximum number of alias sets that contain the minimum number of two-factor interactions but none of the block effects. This completes the proof of Lemma 1. □

Proof of Lemma 2 (a) Suppose $D_t \subset S_{(5N/16)}$ and $\overline{D}_t \subset D^{q-4}(x_1)$. Consider the selection of D_b. Up to isomorphism, there are two possibilities for D_b:

$$D_b \cap D^{q-4}(x_1) \neq \emptyset, \quad D_b \cap D^{q-4}(x_1) = \emptyset.$$

If $D_b \cap D^{q-4}(x_1) \neq \emptyset$, then D_b has at least one column taken from $D^{q-4}(x_1)$. Without loss of generality, we assume $I_{2^{q-4}} \otimes x_1 \subset D_b$. Then

$$D^{q-4}(x_1x_i) \cap D_b = \emptyset, \quad i = 2, 3, 4, 5.$$

Otherwise, without loss of generality, suppose $\gamma \otimes x_1x_2 \in D^{q-4}(x_1x_2) \cap D_b$, where $\gamma \in D^{q-4}(1)$. Then

$$(I_{2^{q-4}} \otimes x_1)(\gamma \otimes x_1x_2) = \gamma \otimes x_2 \in D_t \cap D_b,$$

which contradicts $D_t \cap D_b = \emptyset$. Thus,

$$D_b \subset (D^{q-4}(I_{16})\backslash I_N, D^{q-4}(x_1), S),$$

where S is as defined in (8) and D_b can be expressed as

$$\begin{aligned} D_b &= (D_b \cap (D^{q-4}(I_{16})\backslash I_N), D_b \cap D^{q-4}(x_1), D_b \cap S) \qquad \text{(A1)} \\ &= (I_{2^{q-4}} \otimes x_1, D_b \cap (D^{q-4}(I_{16})\backslash I_N), \\ &\quad D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1)), D_b \cap S). \qquad \text{(A2)} \end{aligned}$$

The defining relation $I = x_1x_2x_3x_4x_5$ implies

$$(I_{2^{q-4}} \otimes x_1)S = ((I_{2^{q-4}} \otimes x_1)s \colon s \in S) = S.$$

By the structure of $D^{q-4}(I_{16})$, we have

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N, S)) \subset (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S).$$

Note that the interaction of any two columns of D_b is still in D_b. Then

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N, S)) \subset D_b,$$

and therefore,

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N, S)) \subset D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S). \qquad \text{(A3)}$$

Similarly, we have

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S)) \subset D_b,$$
$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S)) \subset (D^{q-4}(I_{16})\backslash I_N, S),$$

which imply

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S)) \subset D_b \cap (D^{q-4}(I_{16})\backslash I_N, S),$$

and hence,

$$D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S) \subset (I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N, S)). \quad \text{(A4)}$$

From (A3) and (A4), we get

$$D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1), S) = (I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N, S)),$$

i.e.,

$$\begin{aligned} &((I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N)), (I_{2^{q-4}} \otimes x_1)(D_b \cap S)) \\ &\quad = (D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1)), D_b \cap S). \end{aligned} \quad \text{(A5)}$$

The structures of D_b and S imply that

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap S) = (I_{2^{q-4}} \otimes x_1)D_b \cap (I_{2^{q-4}} \otimes x_1)S = D_b \cap S,$$

which simplifies (A5) to

$$(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N)) = D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1)) \quad \text{(A6)}$$

by noting that, in (A5),

$$\begin{aligned} &(I_{2^{q-4}} \otimes x_1)(D_b \cap (D^{q-4}(I_{16})\backslash I_N)) \cap (I_{2^{q-4}} \otimes x_1)(D_b \cap S) = \emptyset, \\ &(D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1))) \cap (D_b \cap S) = \emptyset. \end{aligned}$$

Note that the four terms in the right-hand side of (A2) are mutually exclusive. Thus, we have

$$\begin{aligned} 2^r - 1 &= \#(D_b) \\ &= 1 + \#(D_b \cap (D^{q-4}(I_{16})\backslash I_N)) \\ &\quad + \#(D_b \cap (D^{q-4}(x_1)\backslash(I_{2^{q-4}} \otimes x_1))) + \#(D_b \cap S) \\ &= 1 + 2\#(D_b \cap (D^{q-4}(I_{16})\backslash I_N)) + \#(D_b \cap S), \end{aligned} \quad \text{(A7)}$$

where the third equation follows from (A6). Suppose that b_1 and b_2 are two columns of $D_b \cap (D^{q-4}(I_{16})\backslash I_N)$. The interaction of any two columns of D_b (or $D^{q-4}(I_{16})\backslash I_N$) is still in D_b (or $D^{q-4}(I_{16})\backslash I_N$), i.e., $b_1 b_2 \in D_b \cap (D^{q-4}(I_{16})\backslash I_N)$. Then $D_b \cap (D^{q-4}(I_{16})\backslash I_N)$ is isomorphic to $G_{r_1-1,4}$ for some r_1, where $G_{\cdot,\cdot}$ is as defined in (5). Up to isomorphism, suppose

$$D_b \cap (D^{q-4}(I_{16})\backslash I_N) = G_{r_1-1,4}. \quad \text{(A8)}$$

Then, by (6) and (A6), we have

$$D_b \cap D^{q-4}(x_1) = (I_{2^{q-4}} \otimes x_1, (I_{2^{q-4}} \otimes x_1)G_{r_1-1,4}) = F_{x_1,r_1}. \quad \text{(A9)}$$

With the help of (A9), the equations in (a) can be established as follows.

(i) If $D_b \cap S = \emptyset$, then

$$\#(D_b \cap (D^{q-4}(I_{16}) \backslash I_N)) = 2^{r-1} - 1$$

and $r_1 = r$ by (A7) and (A8). Then, from (A1), (A8), and (A9), we obtain $D_b = (G_{r-1,4}, F_{x_1,r})$. The first equation of (a) follows.

(ii) When $D_b \cap S \neq \emptyset$, without loss of generality, suppose $D_b \cap D^{q-4}(x_2x_3) \neq \emptyset$ and $I_{2^{q-4}} \otimes x_2x_3 \in D_b$. Then

$$I_{2^{q-4}} \otimes x_4x_5 = (I_{2^{q-4}} \otimes x_1)(I_{2^{q-4}} \otimes x_2x_3) \in D_b.$$

Similar to (A9), we have

$$D_b \cap D^{q-4}(x_2x_3) = F_{x_2x_3,r_1} \text{ and } D_b \cap D^{q-4}(x_4x_5) = F_{x_4x_5,r_1}.$$

Furthermore, if

$$D_b \cap D^{q-4}(x_2x_4) = D_b \cap D^{q-4}(x_2x_5) = D_b \cap D^{q-4}(x_3x_4) = D_b \cap D^{q-4}(x_3x_5) = \emptyset,$$

then

$$D_b \cap S = (F_{x_2x_3,r_1}, F_{x_4x_5,r_1})$$

and

$$2^r - 1 = 1 + 2(2^{r_1-1} - 1) + 2 \cdot 2^{r_1-1}$$

by (A7). Thus, $r_1 = r - 1$ and the second equation of (a) follows from (A1), (A8), and (A9).

(iii) Based on (ii), when

$$D_b \cap D^{q-4}(x_2x_3) \neq \emptyset, \quad D_b \cap D^{q-4}(x_2x_4) \neq \emptyset,$$

without loss of generality, suppose

$$I_{2^{q-4}} \otimes x_2x_3 \in D_b, \quad I_{2^{q-4}} \otimes x_2x_4 \in D_b.$$

Then

$$I_{2^{q-4}} \otimes x_3x_4 = (I_{2^{q-4}} \otimes x_2x_3)(I_{2^{q-4}} \otimes x_2x_4) \in D_b,$$
$$I_{2^{q-4}} \otimes x_3x_5 = (I_{2^{q-4}} \otimes x_1)(I_{2^{q-4}} \otimes x_2x_4) \in D_b,$$
$$I_{2^{q-4}} \otimes x_2x_5 = (I_{2^{q-4}} \otimes x_1)(I_{2^{q-4}} \otimes x_3x_4) \in D_b.$$

Similar to (A9), we also have

$$D_b \cap D^{q-4}(x_2x_4) = F_{x_2x_4,r_1}, \quad D_b \cap D^{q-4}(x_3x_4) = F_{x_3x_4,r_1},$$
$$D_b \cap D^{q-4}(x_3x_5) = F_{x_3x_5,r_1}, \quad D_b \cap D^{q-4}(x_2x_5) = F_{x_2x_5,r_1}.$$

Then

$$D_b \cap S = (F_{x_2x_3,r_1}, F_{x_4x_5,r_1}, F_{x_2x_4,r_1}, F_{x_3x_5,r_1}, F_{x_3x_4,r_1}, F_{x_2x_5,r_1}),$$

and

$$2^r - 1 = 1 + 2(2^{r_1-1} - 1) + 6 \cdot 2^{r_1-1}$$

by (A7). Thus, $r_1 = r - 2$ and the third equation of (a) follows from (A1), (A8), and (A9).

(b) From (2), if $D_b \cap D^{q-4}(x_1) = \emptyset$, then

$$D_b \subset (D^{q-4}(I_{16}) \backslash I_N, Z),$$

where Z is as defined in (3). The following proof is similar to that of (a). We only provide the outline for saving space. Consider the following three mutually exclusive cases:

(i) $D_b \cap Z = \emptyset$;

(ii) $D_b \cap Z \neq \emptyset$ and $D_b \cap Z \subset D^{q-4}(x_i x_j)$ with $\{i, j\} = \{1, 2\}$ or $\{2, 3\}$;

(iii) $D_b \cap Z \neq \emptyset$, $D_b \cap D^{q-4}(x_u x_v) \neq \emptyset$, $D_b \cap D^{q-4}(x_u x_w) \neq \emptyset$, $D_b \cap D^{q-4}(x_v x_w) \neq \emptyset$, and $D_b \cap Z \subset (D^{q-4}(x_u x_v), D^{q-4}(x_u x_w), D^{q-4}(x_v x_w))$ with $\{u, v, w\} = \{1, 2, 3\}$ or $\{2, 3, 4\}$.

Except the cases which will result in $D_t \cap D_b \neq \emptyset$, any other case is isomorphic to one of the three cases above.

For (i), it is easy to see that $D_b \subset D^{q-4}(I_{16}) \backslash I_N$. Up to isomorphism, we can assume that $D_b = G_{r,4}$. The first equation of (b) follows.

For (ii), without loss of generality, suppose $I_{2^{q-4}} \otimes x_i x_j \in D_b$. Similar to (A8) and (A9), we can assume

$$D_b \cap (D^{q-4}(I_{16}) \backslash I_N) = G_{r_1-1,4},$$

and then obtain

$$D_b \cap D^{q-4}(x_i x_j) = (I_{2^{q-4}} \otimes x_i x_j, (I_{2^{q-4}} \otimes x_i x_j) G_{r_1-1,4}) = F_{x_i x_j, r_1},$$

with $r_1 = r$. The second equation of (b) follows.

For (iii), without loss of generality, suppose

$$I_{2^{q-4}} \otimes x_u x_v,\ I_{2^{q-4}} \otimes x_u x_w,\ I_{2^{q-4}} \otimes x_v x_w \in D_b.$$

Similar to case (ii), we can assume

$$D_b \cap (D^{q-4}(I_{16}) \backslash I_N) = G_{r_1-1,4},$$

and then obtain

$$\begin{aligned} D_b \cap D^{q-4}(x_u x_v) &= (I_{2^{q-4}} \otimes x_u x_v, (I_{2^{q-4}} \otimes x_u x_v) G_{r_1-1,4}) = F_{x_u x_v, r_1}, \\ D_b \cap D^{q-4}(x_u x_w) &= (I_{2^{q-4}} \otimes x_u x_w, (I_{2^{q-4}} \otimes x_u x_w) G_{r_1-1,4}) = F_{x_u x_w, r_1}, \\ D_b \cap D^{q-4}(x_v x_w) &= (I_{2^{q-4}} \otimes x_v x_w, (I_{2^{q-4}} \otimes x_v x_w) G_{r_1-1,4}) = F_{x_v x_w, r_1}, \end{aligned}$$

with $r_1 = r - 1$. Then the third equation of (b) follows. This completes the proof of Lemma 2. □

Proof of Lemma 3 (a) It can be obtained from the proof of [18, Theorem 4].

(b) It follows by noting that ${}_1^{\#}C_2^{(k)}(D) = {}_1^{\#}C_2^{(k)}(D_t)$ and D_t has resolution at least IV.

(c) By the definition of ${}_2^{\#}C_2^{(k)}(D)$, we have

$$ {}_2^{\#}C_2^{(k)}(D) = (k+1)\#\{\gamma\colon \gamma \in H_q\backslash D_b, B_2(D_t,\gamma) = k+1\} =: \Delta_0 + \Delta_1 + \Delta_2, $$

where

$$ \Delta_0 = (k+1)\#\{\gamma\colon \gamma \in (D^{q-4}(I_{16})\backslash I_N)\backslash D_{b_0},\, B_2(D_t,\gamma) = k+1\}, $$

$$ \Delta_1 = (k+1)\#\Big\{\gamma\colon \gamma \in \Big(\bigcup_{i=2}^{5} D^{q-4}(x_1x_i)\Big)\backslash D_{b_1},\, B_2(D_t,\gamma) = k+1\Big\}, $$

$$ \Delta_2 = (k+1)\#\{\gamma\colon \gamma \in S\backslash D_{b_2},\, B_2(D_t,\gamma) = k+1\}. $$

By (a), we have the following results:

1° for $k+1 < \frac{N}{16} - l$, $\Delta_0 = \Delta_1 = \Delta_2 = 0$;

2° for $k+1 = \frac{N}{16} - l$, $\Delta_0 = \Delta_2 = 0$, and

$$ \begin{aligned} \Delta_1 &= (k+1)\#\Big\{\gamma\colon \gamma \in \Big(\bigcup_{i=2}^{5} D^{q-4}(x_1x_i)\Big)\backslash D_{b_1},\, B_2(D_t,\gamma) = \frac{N}{16} - l\Big\} \\ &= (k+1)\Big(\frac{N}{4} - \#(D_{b_1})\Big); \end{aligned} $$

3° for $\frac{N}{16} - l < k+1 < N/16$, $\Delta_0 = \Delta_1 = \Delta_2 = 0$;

4° for $k+1 = N/16$, $\Delta_0 = \Delta_1 = 0$, and

$$ \Delta_2 = (k+1)\#\Big\{\gamma\colon \gamma \in S\backslash D_{b_2},\, B_2(D_t,\gamma) = \frac{N}{16}\Big\} = (k+1)\Big(\frac{3N}{8} - \#(D_{b_2})\Big); $$

5° for $N/16 < k+1 < \frac{5N}{32} - l$, $\Delta_0 = \Delta_1 = \Delta_2 = 0$;

6° for $k+1 = \frac{5N}{32} - l$, $\Delta_1 = \Delta_2 = 0$, and

$$ \begin{aligned} \Delta_0 &= (k+1)\#\Big\{\gamma\colon \gamma \in (D^{q-4}(I_{16})\backslash I_N)\backslash D_{b_0},\, B_2(D_t,\gamma) = \frac{5N}{32} - l\Big\} \\ &= (k+1)\#\{\gamma\colon \gamma \in (D^{q-4}(I_{16})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) = 0\} \\ &= (k+1)\Big(\frac{N}{16} - 1 - \#(D_{b_0})\Big) \\ &\quad - (k+1)\#\{\gamma\colon \gamma \in (D^{q-4}(I_{16})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) > 0\} \\ &= (k+1)\Big(\frac{N}{16} - 1 - \#(D_{b_0})\Big) - (k+1)\#\{\gamma\colon \gamma \in H_q\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) > 0\} \\ &= (k+1)\Big(\frac{N}{16} - 1 - f(D^*)\Big), \end{aligned} $$

where the second equality is from (a), and the forth equality is from the fact that $\gamma_1\gamma_2 \in D^{q-4}(I_{16})\backslash I_N$ for any two different columns γ_1 and γ_2 of $\overline{D}_t$;

7° for $k+1 > \frac{5N}{32} - l$, $\Delta_1 = \Delta_2 = 0$, and

$$\begin{aligned}\Delta_0 &= (k+1)\#\Big\{\gamma\colon \gamma \in (D^{q-4}(I_{16})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) = (k+1) - \frac{5N}{32} + l\Big\}\\ &= (k+1)\#\Big\{\gamma\colon \gamma \in H_q\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) = (k+1) - \frac{5N}{32} + l\Big\}\\ &= \frac{k+1}{k+1-\frac{5N}{32}+l}\, {}_2^{\#}C_2^{(k-\frac{5N}{32}+l)}(D^*),\end{aligned}$$

where the first equality is from (a) and the second equality is from the same fact as in 6°.

Then (c) follows from 1°–7°. □

Before proving Theorem 1, let us first see a lemma in [25].

Lemma A1 *Let $\widetilde{S} = (\widetilde{S}_t : S_b)$ with $\widetilde{S}_t$ consisting of the first or the last s columns of $S_{(N/2)}$ with Yates order and $S_b = H_{a,q}$. Then $\widetilde{S}$ maximizes*

$$(-f(S), {}_2^{\#}C_2(S))$$

among all $S = (S_t : S_b)$ with S_t being an s-projection of $S_{(N/2)}$, where

$$S_{(N/2)} = H_q\backslash H_{q-1,q}.$$

By Lemma A1, we can obtain the following result.

Lemma A2 *Let $S_b = G_{a,u}$ and $\widetilde{S}_t$ consist of the first s columns of $D^{q-u}(x)$ with RC-Yates order, where $x \in D^u(1)\backslash I_{2^u}$ for some $u \leqslant q-a$. Then $\widetilde{S} = (\widetilde{S}_t : S_b)$ maximizes*

$$(-f(S), {}_2^{\#}C_2(S)) \tag{A10}$$

among all $S = (S_t : S_b)$ with S_t being an s-projection of $D^{q-u}(x)$.

Proof Recall the definition of doubling in Section 2, for any $x \in D^u(1)\backslash I_{2^u}$, we have

$$G_{q-u,u} = H_{q-u,q-u} \otimes I_{2^u} = H_{q-u} \otimes I_{2^u},$$
$$D^{q-u}(x) = (I_{2^{q-u}} \otimes x, (I_{2^{q-u}} \otimes x)G_{q-u,u}).$$

Let F be the matrix consisting of the first 2^{q-u} columns of $S_{(N/2)}$, i.e.,

$$F = (q_{2^q}, q_{2^q}H_{q-u,q}),$$

where

$$q_{2^q} = (1,\dots,1,-1\dots,-1)^{\mathrm{T}}_{2^q}$$

is the q-th independent column of H_q. Note that $G_{q-u,u}$ is isomorphic to $H_{q-u,q}$. Then $D^{q-u}(x)$ and F are isomorphic. Thus, the two designs $(G_{a,u} : D^{q-u}(x))$

and $(H_{a,q} : F)$ are isomorphic, since $G_{a,u} \subset G_{q-u,u}$ and $H_{a,q} \subset H_{q-u,q}$. Therefore, the two designs $\widetilde{S}$ in Lemmas A1 and A2 are also isomorphic. Then the result follows from Lemma A1. □

With the help of Lemma A2, we now prove Theorem 1.

Proof of Theorem 1 Let $D^* = (\overline{D}_t : D_{b_0})$, where

$$\overline{D}_t = S_{(5N/16)} \backslash D_t (\subset D^{q-4}(x_1))$$

and D_{b_0} is as defined in (7). By (9), if $D = (D_t : D_b)$ is a B^1-GMC $2^{n-m} : 2^r$ design with (4), then D should sequentially minimize $\#(D_{b_1})$ and $\#(D_{b_2})$ first.

(a) When $r \leqslant q-4$ and

$$\frac{5N}{16} - 2^{r-1} + 1 \leqslant n \leqslant \frac{5N}{16},$$

by Lemma 2, we can select $D_b = G_{r,4}$ or $D_b = (G_{r-1,4}, F_{x_1,r})$ since only the two cases have $\#(D_{b_1}) = 0$ and $\#(D_{b_2}) = 0$. Note that $\overline{D}_t = S_{(5N/16)} \backslash D_t$ consists of $l = \frac{5N}{16} - n \ (< 2^{r-1})$ columns of $D^{q-4}(x_1)$. Then $F_{x_1,r}$, which has 2^{r-1} columns, cannot be accommodated into $\overline{D}_t$. Therefore, it is impossible to have the case of $D_b = (G_{r-1,4}, F_{x_1,r})$. Thus, $D_b = G_{r,4}$. Let $\widetilde{D} = (\widetilde{D}_t : \widetilde{D}_b)$ with $\widetilde{D}_t$ consisting of the last n columns of $S_{(5N/16)}$ and $\widetilde{D}_b = G_{r,4}$. Let

$$\widetilde{D}_t^* = S_{(5N/16)} \backslash \widetilde{D}_t \ (\subset D^{q-4}(x_1)), \quad \widetilde{D}_b^* = \widetilde{D}_b \cap (D^{q-4}(I_{16}) \backslash I_N) = G_{r,4},$$

and let

$$\widetilde{D}^* = (\widetilde{D}_t^* : \widetilde{D}_b^*).$$

By Lemma A2, $\widetilde{D}^*$ maximizes (A10) among all the designs $D^* = (D_t^* : D_b^*)$ with $D_t^* \subset D^{q-4}(x_1)$ and $D_b^* = G_{r,4}$. Then (a) follows immediately from (9).

(b) Similar to (a), when $r \leqslant q-4$ and

$$\frac{9N}{32} + 1 \leqslant n \leqslant \frac{5N}{16} - 2^{r-1},$$

by Lemma 2, we can also select $D_b = G_{r,4}$ or $D_b = (G_{r-1,4}, F_{x_1,r})$.

Let $\widetilde{D} = (\widetilde{D}_t : \widetilde{D}_b)$ with $\widetilde{D}_t$ consists of the last n columns of $S_{(5N/16)}$ and $\widetilde{D}_b = G_{r,4}$. Let $\widetilde{D}^* = (\widetilde{D}_t^* : \widetilde{D}_b^*)$ with

$$\widetilde{D}_t^* = S_{(5N/16)} \backslash \widetilde{D}_t, \quad \widetilde{D}_b^* = \widetilde{D}_b \cap (D^{q-4}(I_{16}) \backslash I_N) = G_{r,4}.$$

Then $\widetilde{D}_t^*$ consists of the first l columns of $D^{q-4}(x_1)$. By Lemma A2, $\widetilde{D}^*$ maximizes (A10) among all the designs $D^* = (D_t^* : D_b^*)$ with $D_t^* \subset D^{q-4}(x_1)$ and $D_b^* = G_{r,4}$. Therefore, $\widetilde{D}$ has B^1-GMC among all the designs $D = (D_t : D_b)$ with $D_t \subset S_{(5N/16)}$, $\overline{D}_t \subset D^{q-4}(x_1)$, and $D_b = G_{r,4}$.

Let $\widetilde{E} = (\widetilde{D}_t : \widetilde{E}_b)$ with $\widetilde{E}_b = (G_{r-1,4}, F_{x_1,r})$. Let

$$\widetilde{E}_b^* = \widetilde{E}_b \cap (D^{q-4}(I_{16}) \backslash I_N) = G_{r-1,4}$$

and $\widetilde{E}^* = (\widetilde{D}_t^* : \widetilde{E}_b^*)$. By Lemma A2, $\widetilde{E}^*$ maximizes (A10) among all the designs $E^* = (D_t^* : E_b^*)$ with $D_t^* \subset D^{q-4}(x_1)$ and $E_b^* = G_{r-1,4}$. Therefore, $\widetilde{E}$ has B^1-GMC among all the designs $E = (D_t : E_b)$ with $D_t \subset S_{(5N/16)}$, $\overline{D}_t \subset D^{q-4}(x_1)$, and $E_b = (G_{r-1,4}, F_{x_1,r})$.

Now, since ${}_1^\#C_2^{(k)}(\widetilde{D}) = {}_1^\#C_2^{(k)}(\widetilde{E})$ for each $k \geqslant 0$, to show the validity of (b), it suffices to show that ${}_2^\#C_2^{(k)}(\widetilde{E}) \geqslant {}_2^\#C_2^{(k)}(\widetilde{D})$ for each $k \geqslant 0$ and there exists a k_0 such that

$$ {}_2^\#C_2^{(k_0)}(\widetilde{E}) > {}_2^\#C_2^{(k_0)}(\widetilde{D}). \tag{A11}$$

By the definition of ${}_2^\#C_2^{(k)}(\widetilde{E})$ and ${}_2^\#C_2^{(k)}(\widetilde{D})$, we have

$$\begin{aligned} {}_2^\#C_2^{(k)}(\widetilde{E}) - {}_2^\#C_2^{(k)}(\widetilde{D}) = (k+1)\#\{\gamma\colon \gamma \in H_q \backslash \widetilde{E}_b,\, B_2(\widetilde{D}_t, \gamma) = k+1\} \\ - (k+1)\#\{\gamma\colon \gamma \in H_q \backslash \widetilde{D}_b,\, B_2(\widetilde{D}_t, \gamma) = k+1\}. \end{aligned}$$

Note that $B_2(\widetilde{D}_t, \gamma) = 0$ for any $\gamma \in F_{x_1,r}$. Then

$$\begin{aligned} {}_2^\#C_2^{(k)}(\widetilde{E}) - {}_2^\#C_2^{(k)}(\widetilde{D}) &= (k+1)\#\{\gamma\colon \gamma \in H_q \backslash G_{r-1,4},\, B_2(\widetilde{D}_t, \gamma) = k+1\} \\ &\quad - (k+1)\#\{\gamma\colon \gamma \in H_q \backslash G_{r,4},\, B_2(\widetilde{D}_t, \gamma) = k+1\} \\ &= (k+1)\#\{\gamma\colon \gamma \in G_{r,4} \backslash G_{r-1,4},\, B_2(\widetilde{D}_t, \gamma) = k+1\}. \end{aligned}$$

So, ${}_2^\#C_2^{(k)}(\widetilde{D}) \geqslant {}_2^\#C_2^{(k)}(\widetilde{E})$ for each $k \geqslant 0$. Next, we show that there exists a k_0 such that (A11) holds. It suffices to show that there exists a $\gamma \in G_{r,4} \backslash G_{r-1,4}$ such that $B_2(\widetilde{D}_t, \gamma) > 0$. Recall that x_1, x_2, x_3, x_4 are four independent columns of $D^4(1) \backslash I_{2^4}$. Suppose

$$D^{q-4}(1) = (I, 1, 2, 12, \ldots, 12\cdots(q-4))_{2^{q-4}}.$$

Note that $r \leqslant q-4$. Then $r_{2^{q-4}} \in D^{q-4}(1)$ and

$$G_{r,4} = (1 \otimes I_{2^4}, 2 \otimes I_{2^4}, 12 \otimes I_{2^4}, \ldots, r \otimes I_{2^4}, 1r \otimes I_{2^4}, \ldots, 12\cdots r \otimes I_{2^4})_{2^q}$$

consists of the first $2^r - 1$ columns of $(D^{q-4}(1) \backslash I_{2^{q-4}}) \otimes I_{2^4}$. Recall that $D^{q-4}(x_2) \subset \widetilde{D}_t$. Then $I_{2^{q-4}} \otimes x_2$ and $r_{2^{q-4}} \otimes x_2$ are two columns of $\widetilde{D}_t$, and their interaction

$$\gamma = (I_{2^{q-4}} \otimes x_2)(r_{2^{q-4}} \otimes x_2) = r_{2^{q-4}} \otimes I_{2^4} \in G_{r,4} \backslash G_{r-1,4}.$$

This completes the proof of (b).

(c) Suppose $r = q-3$. In this case, D_b has $q-3$ independent columns, but $G_{r,4} = H_{r,q-4} \otimes I_{2^4}$ has at most $q-4$ independent columns, thus $D_b \neq G_{r,4}$. Note that when (4) is satisfied, we have

$$l = \frac{5N}{16} - n < \frac{N}{32}.$$

Thus, we delete less than $N/32$ columns from $D^{q-4}(x_1)$. As a result,

$$D_b \neq (G_{r-1,4}, F_{x_1,r}), \quad D_b \neq (G_{r-2,4}, F_{x_1,r-1}, F_{x_2x_3,r-1}, F_{x_4x_5,r-1})$$

since neither $F_{x_1,r}$ nor $F_{x_1,r-1}$ can be accommodated into $\overline{D}_t$. Then by Lemma 2, up to isomorphism, we can select

$$\begin{aligned} D_b = {} & (G_{q-4,4}, F_{x_1x_2,q-3}), \text{ or} \\ & (G_{q-4,4}, F_{x_2x_3,q-3}), \text{ or} \\ & (G_{q-5,4}, F_{x_1x_2,q-4}, F_{x_1x_3,q-4}, F_{x_2x_3,q-4}), \text{ or} \\ & (G_{q-5,4}, F_{x_2x_3,q-4}, F_{x_2x_4,q-4}, F_{x_3x_4,q-4}), \text{ or} \\ & (G_{q-6,4}, F_{x_1,q-5}, F_{x_2x_3,q-5}, F_{x_2x_4,q-5}, F_{x_2x_5,q-5}, \\ & F_{x_3x_4,q-5}, F_{x_3x_5,q-5}, F_{x_4x_5,q-5}) \end{aligned} \tag{A12}$$

as the block effects matrix, where for these matrices,

$$(\#(D_{b_1}), \#(D_{b_2})) = \Big(\frac{N}{16}, 0\Big),\ \Big(0, \frac{N}{16}\Big),\ \Big(\frac{N}{16}, \frac{N}{32}\Big),\ \Big(0, \frac{3N}{32}\Big),\ \Big(0, \frac{3N}{32}\Big),$$

respectively. We should choose (A12) as D_b since it sequentially minimizes $\#(D_{b_1})$ and $\#(D_{b_2})$. Then (c) follows from a similar argument to (a).

(d) Suppose $r = q - 2$. Note that $l < N/32$. Then, similar to (c),

$$\begin{aligned} D_b = {} & (G_{q-4,4}, F_{x_1x_2,q-3}, F_{x_1x_3,q-3}, F_{x_2x_3,q-3}) \quad \text{or} \\ & (G_{q-4,4}, F_{x_2x_3,q-3}, F_{x_2x_4,q-3}, F_{x_3x_4,q-3}) \end{aligned} \tag{A13}$$

can be selected as the block effects matrix by Lemma 2. We should choose (A13) as the block effects matrix since it sequentially minimizes $\#(D_{b_1})$ and $\#(D_{b_2})$. Then (d) follows from a similar argument to (a).

This completes the proof of Theorem 1. □

Proof of Lemma 4 As discussed above Lemma 4, when (10) is satisfied, if $D = (D_t : D_b)$ has B^1-GMC, then D_t is an n-projection of $S_{(9N/32)}$, $S_{(5N/16)}$, or $S_{(N/2)}$.

For a $2^{n-m} : 2^r$ design $D = (D_t : D_b)$ with $D_t \subset S_{(9N/32)}$, let $\overline{D}_t = S_{(9N/32)} \backslash D_t$. There are three kinds of choices for D_t (see [18]):

(i) $\overline{D}_t \subset D^{q-5}(x_i)$, $i = 1$ or 2;

(ii) $\overline{D}_t \subset D^{q-5}(x_i)$, $i = 3, 4, \ldots, 9$;

(iii) $\overline{D}_t$ contains columns from at least two of $D^{q-5}(x_i)$, $i = 1, 2, \ldots, 9$.

Note that, for (i), since x_1 and x_2 play the same role in $S_{(9N/32)}$, without loss of generality, we assume that $\overline{D}_t \subset D^{q-5}(x_1)$. While for (ii), from the defining contrast subgroup of X_2, we find that $x_3, x_4, \ldots, x_9$ play the same role in $S_{(9N/32)}$. So, without loss of generality, we assume that $\overline{D}_t \subset D^{q-5}(x_3)$.

We first consider case (i), i.e., $D_t \subset S_{(9N/32)}$ and $\overline{D}_t \subset D^{q-5}(x_1)$. Let

$$l = \#(\overline{D}_t) = \frac{9N}{32} - n.$$

For any column $\gamma \in D^q(1)\backslash I_N$, γ corresponds to an alias set of D_t. When $\gamma \in D_t$, each alias set contains a main effect. When $\gamma \in \overline{D}_t$, each alias set contains only interactions involving three or more factors. According to [18, Theorem 6], for any $\gamma \in D^{q-5}(x_1x_j)$, $j = 2, 3, \dots, 9$, there are $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. There are $8N/32$ alias sets with $\gamma \in D^{q-5}(x_1x_j)$, $j = 2, 3, \dots, 9$. For any $\gamma \in D^{q-5}(x_2x_j)$, $j = 3, 4, \dots, 9$, there are $N/32$ two-factor interactions of D_t aliased with γ. There are $7N/32$ alias sets with $\gamma \in D^{q-5}(x_2x_j)$, $j = 3, 4, \dots, 9$. For any $\gamma \in D^{q-5}(x_ix_j)$, $i, j = 3, 4, \dots, 9, i \neq j$, there are $3N/32$ two-factor interactions of D_t aliased with γ. There are altogether $7N/32$ alias sets with $\gamma \in D^{q-5}(x_ix_j)$, $3 \leqslant i \neq j \leqslant 9$. For any $\gamma \in D^{q-5}(I_{32})\backslash I_N$, there are at least $n - \frac{9N}{64}$ two-factor interactions of D_t aliased with γ. There are altogether $\frac{N}{32} - 1$ alias sets with $\gamma \in D^{q-5}(I_{32})\backslash I_N$ and these γ form a closed submatrix of H_q.

When we choose columns from $D^q(1)\backslash I_N$ as block effects, we should first choose the columns which are neither aliased with the main effects nor with the two-factor interactions, then the columns which are aliased with the two-factor interactions of D_t in the most serious degree. Note that

$$\#\left\{\bigcup_{3\leqslant i\neq j\leqslant 9} D^{q-5}(x_ix_j) \cup (D^{q-5}(I_{32})\backslash I_N)\right\} = \frac{N}{4} - 1.$$

For any $r \leqslant q-2$, it is available to choose $2^r - 1$ columns from

$$\bigcup_{3\leqslant i\neq j\leqslant 9} D^{q-5}(x_ix_j) \cup (D^{q-5}(I_{32})\backslash I_N)$$

to form a closed matrix as D_b. With doing that, for any $\gamma \in D^{q-5}(x_1x_i)$, $i = 2, 3, \dots, 9$, there are still $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. There are $8N/32$ alias sets with $\gamma \in D^{q-5}(x_1x_i)$, $i = 2, 3, \dots, 9$. For any $\gamma \in D^{q-5}(x_2x_i)$, $i = 3, 4, \dots, 9$, there are still $N/32$ two-factor interactions of D_t aliased with γ. There are $7N/32$ alias sets with $\gamma \in D^{q-5}(x_2x_i)$, $i = 3, 4, \dots, 9$.

Consider case (ii), i.e., $D_t \subset S_{(9N/32)}$ and $\overline{D}_t \subset D^{q-5}(x_3)$. For any $\gamma \in (D^{q-5}(x_1x_3), D^{q-5}(x_2x_3))$, there are $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. There are $2N/32$ alias sets with $\gamma \in (D^{q-5}(x_1x_3), D^{q-5}(x_2x_3))$. For any other $\gamma \in D^{q-5}(x_ix_j)$, there are at least $N/32$ two-factor interactions of D_t aliased with γ.

We now turns to consider case (iii), i.e., $\overline{D}_t$ contains columns from at least two of $D^{q-5}(x_i)$, $i = 1, 2, \dots, 9$. Let

$$\overline{D}_{ti} = \overline{D}_t \cap D^{q-5}(x_i), \quad l_i = \#(\overline{D}_{ti}), \quad i = 1, 2, \dots, 9.$$

Clearly, we have

$$l_i + l_j \leqslant l, \quad 1 \leqslant i \neq j \leqslant 9.$$

For any $\gamma \in D^{q-5}(x_i x_j)$, there are at least $\frac{N}{32} - l_i - l_j$ ($\geqslant \frac{N}{32} - l = n - \frac{N}{4}$) two-factor interactions of D_t aliased with γ. There is at most one pair of (i, j) such that $\gamma \in D^{q-5}(x_i x_j)$ and $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ. Therefore, there are at most $N/32$ alias sets each containing exactly $n - \frac{N}{4}$ two-factor interactions of D_t. For any other $\gamma \in D^{q-5}(x_i x_j)$, there are more than $n - \frac{N}{4}$ two-factor interactions of D_t aliased with γ.

When $D_t \subset S_{(5N/16)}$, according to the proof of Lemma 1, besides the $N/4$ alias sets each of which containing $n - \frac{N}{4}$ two-factor interactions, there are at least $N/16$ two-factor interactions in each of the other alias sets of D_t containing two-factor interactions. When $D_t \subset S_{(N/2)}$, according to [18, Theorem 3], besides the $N/4$ alias sets each of which containing $n-\frac{N}{4}$ two-factor interactions, there are at least $N/8$ two-factor interactions in each of the other alias sets of D_t containing two-factor interactions. Comparing with the three cases of $D_t \subset S_{(9N/32)}$, if $D = (D_t : D_b)$ has B^1-GMC, then D_t must be the projection of $S_{(9N/32)}$ with $\overline{D}_t \subset D^{q-5}(x_1)$ but not the projection of $S_{(N/2)}$ or $S_{(5N/16)}$. This completes the proof of Lemma 4. □

Proof of Lemma 5 Up to isomorphism, there are two possibilities for D_b: $D_b \cap D^{q-5}(x_1) \neq \emptyset$ and $D_b \cap D^{q-5}(x_1) = \emptyset$, which result in (a) and (b), respectively, with a similar deduction method to the proof of Lemma 2. We omit the proof for saving space. □

Proof of Lemma 6 (a) It is the result of [18, Theorem 6].

(b) It follows by noting that ${}_1^{\#}C_2^{(k)}(D) = {}_1^{\#}C_2^{(k)}(D_t)$ and D_t has resolution at least IV.

(c) The definition of ${}_2^{\#}C_2^{(k)}(D)$ means that

$${}_2^{\#}C_2^{(k)}(D) = (k+1)\#\{\gamma\colon \gamma \in H_q \backslash D_b,\, B_2(D_t, \gamma) = k+1\} =: \Delta_0' + \Delta_1' + \Delta_2' + \Delta_3',$$

where

$$\Delta_0' = (k+1)\#\{\gamma\colon \gamma \in (D^{q-5}(I_{32}) \backslash I_N) \backslash D_{b_0},\, B_2(D_t, \gamma) = k+1\},$$

$$\Delta_1' = (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{i=2}^{9} D^{q-5}(x_1 x_i) \backslash D_{b_1},\, B_2(D_t, \gamma) = k+1\Big\},$$

$$\Delta_2' = (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{i=3}^{9} D^{q-5}(x_2 x_i) \backslash D_{b_2},\, B_2(D_t, \gamma) = k+1\Big\},$$

$$\Delta_3' = (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{3 \leqslant i \neq j \leqslant 9} D^{q-5}(x_i x_j) \backslash D_{b_3},\, B_2(D_t, \gamma) = k+1\Big\}.$$

From (a), we can get the following results:

1° for $k+1 < \frac{N}{32} - l$, $\Delta'_0 = \Delta'_1 = \Delta'_2 = \Delta'_3 = 0$;
2° for $k+1 = \frac{N}{32} - l$, $\Delta'_0 = \Delta'_2 = \Delta'_3 = 0$, and

$$\begin{aligned}\Delta'_1 &= (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{i=2}^{9} D^{q-5}(x_1x_i)\backslash D_{b_1},\, B_2(D_t,\gamma) = \frac{N}{32} - l\Big\}\\ &= (k+1)\Big(\frac{N}{4} - \#(D_{b_1})\Big);\end{aligned}$$

3° for $\frac{N}{32} - l < k+1 < N/32$, $\Delta'_0 = \Delta'_1 = \Delta'_2 = \Delta'_3 = 0$;
4° for $k+1 = N/32$, $\Delta'_0 = \Delta'_1 = \Delta'_3 = 0$, and

$$\begin{aligned}\Delta'_2 &= (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{j=3}^{9} D^{q-5}(x_2x_j)\backslash D_{b_2},\, B_2(D_t,\gamma) = \frac{N}{32}\Big\}\\ &= (k+1)\Big(\frac{7N}{32} - \#(D_{b_2})\Big);\end{aligned}$$

5° for $N/32 < k+1 < 3N/32$, $\Delta'_0 = \Delta'_1 = \Delta'_2 = \Delta'_3 = 0$;
6° for $k+1 = 3N/32$, $\Delta'_0 = \Delta'_1 = \Delta'_2 = 0$, and

$$\begin{aligned}\Delta'_3 &= (k+1)\#\Big\{\gamma\colon \gamma \in \bigcup_{3\leqslant i\neq j\leqslant 9} D^{q-5}(x_ix_j)\backslash D_{b_3},\, B_2(D_t,\gamma) = \frac{3N}{32}\Big\}\\ &= (k+1)\Big(\frac{7N}{32} - \#(D_{b_3})\Big);\end{aligned}$$

7° for $3N/32 < k+1 < \frac{9N}{64} - l$, $\Delta'_0 = \Delta'_1 = \Delta'_2 = \Delta'_3 = 0$;
8° for $k+1 = \frac{9N}{64} - l$, $\Delta'_1 = \Delta'_2 = \Delta'_3 = 0$, and

$$\begin{aligned}\Delta'_0 &= (k+1)\#\Big\{\gamma\colon \gamma \in (D^{q-5}(I_{32})\backslash I_N)\backslash D_{b_0},\, B_2(D_t,\gamma) = \frac{9N}{64} - l\Big\}\\ &= (k+1)\#\{\gamma\colon \gamma \in (D^{q-5}(I_{32})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) = 0)\}\\ &= (k+1)(\#(D^{q-5}(I_{32})\backslash I_N\backslash D_{b_0})\\ &\quad - \#\{\gamma\colon \gamma \in (D^{q-5}(I_{32})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) > 0\})\\ &= (k+1)\Big(\frac{N}{32} - 1 - \#(D_{b_0}) - \#\{\gamma\colon \gamma \in H_q\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) > 0\}\Big)\\ &= (k+1)\Big(\frac{N}{32} - 1 - f(D^*)\Big),\end{aligned}$$

where the second equality is from (a), and the forth equality is from the fact that $\gamma_1\gamma_2 \in D^{q-5}(I_{32})\backslash I_N$ for any two different columns γ_1 and γ_2 of $\overline{D}_t$;
9° for $k+1 > \frac{9N}{64} - l$, $\Delta'_1 = \Delta'_2 = \Delta'_3 = 0$, and

$$\Delta'_0 = (k+1)\#\Big\{\gamma\colon \gamma \in (D^{q-5}(I_{32})\backslash I_N)\backslash D_{b_0},\, B_2(\overline{D}_t,\gamma) = k+1 - \frac{9N}{64} + l\Big\}$$

$$= (k+1)\#\Big\{\gamma\colon \gamma \in H_q \backslash D_{b_0},\ B_2(\overline{D}_t,\gamma) = k+1-\frac{9N}{64}+l\Big\}$$
$$= \frac{k+1}{k+1-\frac{9N}{64}+l}\,{}_2^{\#}C_2^{(k-\frac{9N}{64}+l)}(D^*),$$

where the first equality is from (a) and the second equality is from the same fact as in 8°.

Then (c) follows from 1°–9°. □

Proof of Theorem 2 Let $D^* = (\overline{D}_t : D_{b_0})$, where

$$\overline{D}_t = S_{(9N/32)} \backslash D_t\ (\subset D^{q-5}(x_1))$$

and D_{b_0} is as defined in (11). By (12), if $D = (D_t : D_b)$ is a $2^{n-m} : 2^r$ design with (10), then D should sequentially minimize $(\#(D_{b_1}), \#(D_{b_2}), \#(D_{b_3}))$ first.

(a) When $r \leqslant q-5$ and

$$\frac{9N}{32} - 2^{r-1} + 1 \leqslant n \leqslant \frac{9N}{32},$$

by Lemma 5, we can select $D_b = G_{r,5}$ or $D_b = (G_{r-1,5}, F_{x_1,r})$ since only the two cases have

$$\#(D_{b_1}) = \#(D_{b_2}) = \#(D_{b_3}) = 0. \tag{A14}$$

Note that $\overline{D}_t = S_{(9N/32)} \backslash D_t$ consists of $l = \frac{9N}{32} - n$ $(< 2^{r-1})$ columns of $D^{q-5}(x_1)$. Then $F_{x_1,r}$, which has 2^{r-1} columns, cannot be accommodated into $\overline{D}_t$. Therefore, it is impossible to have the case of $D_b = (G_{r-1,5}, F_{x_1,r})$. Thus, $D_b = G_{r,5}$. Then (a) follows from a similar argument to the proof of Theorem 1 (a).

(b) By Lemma 5, when $r \leqslant q-5$ and

$$\frac{17N}{64} + 1 \leqslant n \leqslant \frac{9N}{32} - 2^{r-1},$$

similar to (a), we can also select $D_b = G_{r,5}$ or $D_b = (G_{r-1,5}, F_{x_1,r})$. Then, we have (A14), and maximizing (12) is equivalent to maximizing $(-f(D^*), {}_2^{\#}C_2(D^*))$. The remaining of the proof is similar to that of Theorem 1 (b). We omit it for saving space.

(c)–(e) The proofs are similar to that of Theorem 1 (c). □

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