

COLUMN-ORTHOGONAL STRONG ORTHOGONAL ARRAYS AND SLICED STRONG ORTHOGONAL ARRAYS

Haiyan Liu and Min-Qian Liu

Nankai University

Abstract: A strong orthogonal array of strength t can achieve uniformity on finer grids when projected onto any g dimensions for any g less than t . It can be regarded as a kind of uniform space-filling design. Meanwhile, orthogonality is also desirable for space-filling designs. In this paper, we construct strong orthogonal arrays through ordinary orthogonal arrays, in a different fashion than do He and Tang (2013). The resulting strong orthogonal arrays have comparable columns with that of He and Tang (2013), and can achieve near or exact column-orthogonality in most cases, and even 3-orthogonality when the ordinary orthogonal arrays have strength no less than three. On the other hand, sliced space-filling designs are very useful for computer experiments with both qualitative and quantitative factors, multiple computer experiments, data pooling, and cross-validation procedures. Employing the good space-filling property of strong orthogonal arrays, we further propose a new kind of sliced space-filling design, called the sliced strong orthogonal array, and provide two methods for constructing designs for which the resulting designs and their slices also perform well in terms of both column-orthogonality and 3-orthogonality.

Key words and phrases: Column-orthogonality, sliced space-filling design, strong orthogonal array, 3-orthogonality.

1. Introduction

Latin hypercube designs (LHDs), proposed by McKay, Beckman, and Conover (1979), are popular space-filling designs for computer experiments. An LHD of n runs possesses n equally spaced levels. This is a desirable feature as the design achieves the maximum stratification when projected onto any one dimension. It is well known that orthogonality can be viewed as a stepping stone to space-filling designs (Bingham, Sitter, and Tang (2009)). A large number of papers have made efforts to find column-orthogonal LHDs, see e.g. Ye (1998), Steinberg and Lin (2006), Cioppa and Lucas (2007), Bingham, Sitter, and Tang (2009), Georgiou (2009), Lin, Mukerjee, and Tang (2009), Pang, Liu, and Lin (2009), Sun, Liu, and Lin (2009, 2010), Lin et al. (2010), Georgiou and Stylianou (2011), Sun, Pang, and Liu (2011), Ai, He, and Liu (2012), Yang and Liu (2012), Yin and Liu (2013) and Georgiou and Efthimiou (2014). However, there is no guarantee that the LHDs achieve uniformity when projected onto multi-dimensions.

For computer experiments, it is not necessary that the run size equal the number of levels at which each factor is observed. Recently, He and Tang (2013) proposed a new class of arrays, called the strong orthogonal arrays (SOAs), and constructed SOAs using generalized orthogonal arrays (GOAs). Although the number of levels in an SOA is not always equal to the run size, an SOA of strength t not only has equally spaced levels for each factor but also achieves uniformity on finer grids when projected onto g dimensions for any g less than t . Such a design is a space-filling design which can achieve uniformity when projected onto multi-dimensions. However, He and Tang (2013) did not discuss the correlations among the columns of an SOA. As we know, in a regression model, it is preferable to include orthogonal variables so that the estimates of the regression coefficients be uncorrelated. Steinberg and Lin (2006) also pointed out that, the presence of highly correlated input factors can complicate the subsequent data analysis and make it more difficult to identify the most important input factors. In this paper, we propose methods to construct SOAs through ordinary orthogonal arrays such that the proposed methods have more direct and simpler mathematical forms than those of He and Tang (2013). Besides, the resulting SOAs are evaluated in terms of orthogonality, and most of them achieve near or exact column-orthogonality.

Sliced space-filling designs, proposed by Qian and Wu (2009), are intended for computer experiments with qualitative and quantitative factors. They can also be used for multiple computer experiments, data pooling, and cross-validation procedures. Such a design is a special space-filling design that can be divided into slices each of which is also a space-filling design. Inspired by this, we further propose a special kind of sliced space-filling design, called the sliced SOA, where each slice can be collapsed into an SOA. Methods are provided for constructing sliced SOAs, and in some cases, not only the whole SOA, but also each slice can be column-orthogonal.

The rest of this paper is organized as follows. Section 2 provides some definitions and notation. The construction methods of SOAs are given in Section 3. Section 4 introduces two methods for constructing sliced SOAs. Some further discussion and concluding remarks are given in the last section. All proofs are deferred to the Appendix.

2. Definitions and Notation

Let $D(n, s_1 \cdots s_m)$ denote a design which has n runs and m factors with s_1, \dots, s_m levels, respectively. For convenience, the levels of the j th column are taken to be $-(s_j - 1), -(s_j - 3), \dots, s_j - 1$. A design $D(n, s_1 \cdots s_m)$ is called an orthogonal array of strength t , denoted by $OA(n, m, s_1 \times \cdots \times s_m, t)$, if all possible level-combinations for any t columns occur with the same frequency. When

all the s_j 's are equal to s , the array is symmetric and denoted by $OA(n, m, s, t)$. Here this orthogonality is called combinatorial orthogonality. If the inner product of any two columns of a design $D(n, s_1 \cdots s_m)$ is zero, then this design is called a column-orthogonal design. Further, a column-orthogonal design is called 3-orthogonal (Bingham, Sitter, and Tang (2009)) if the sum of element-wise products of any three columns (whether they are distinct or not) is zero. In a first-order regression model, if the design is a column-orthogonal design, the estimates of linear main effects are uncorrelated with each other. Sun, Pang, and Liu (2011) pointed out that if the design is 3-orthogonal, then the estimates of all linear main effects are uncorrected with the estimates of all second-order effects. This is desirable when fitting a first-order model with second-order effects present.

The correlation between two vectors $a = (a_1, \dots, a_n)^T$ and $b = (b_1, \dots, b_n)^T$ is

$$\rho(a, b) = \frac{\sum_{i=1}^n (a_i - \bar{a})(b_i - \bar{b})}{\sqrt{\sum_{i=1}^n (a_i - \bar{a})^2 \sum_{i=1}^n (b_i - \bar{b})^2}},$$

where $\bar{a} = \sum_{i=1}^n a_i/n$ and $\bar{b} = \sum_{i=1}^n b_i/n$. The correlation matrix of design D with m columns is $\rho(D) = (\rho_{ij}(D))_{m \times m}$, where $\rho_{ij}(D)$ is the correlation between the i th and j th columns of D . Two commonly used measures for evaluating the orthogonality of D are $\rho_M(D) = \max_{i < j} |\rho_{ij}(D)|$ and $\rho^2(D) = 2 \sum_{i < j} \rho_{ij}^2(D)/(m(m - 1))$. If $\bar{a} = \bar{b} = 0$, $\rho(a, b)$ reduces to

$$\tilde{\rho}(a, b) = \frac{\sum_{i=1}^n a_i b_i}{\sqrt{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}} = \frac{a^T b}{\sqrt{a^T a} \sqrt{b^T b}}$$

with $\tilde{\rho}(D)$, $\tilde{\rho}_M(D)$ and $\tilde{\rho}^2(D)$ defined similarly.

The concept of SOA with non-negative levels was proposed by He and Tang (2013). Here we restate it with a slight modification to account for designs with centered levels.

Definition 1. An $n \times m$ matrix with entries from s^t levels $\{-(s^t - 1), -(s^t - 3), \dots, s^t - 1\}$ is called a strong orthogonal array (SOA) of size n , m factors, and strength t , denoted by $SOA(n, m, s^t, t)$, if for any integer g with $1 \leq g \leq t$, any subarray of g columns can be collapsed into an $OA(n, g, s^{u_1} \times \cdots \times s^{u_g}, g)$ for any positive integers u_1, \dots, u_g with $u_1 + \cdots + u_g = t$, where collapsing into s^{u_j} levels is done using $2 \lfloor (i + s^t)/2s^{t-u_j} \rfloor - s^{u_j} + 1$, and $\lfloor x \rfloor$ denotes the largest integer not exceeding x .

From this definition, we know that an SOA of strength t achieves uniformity on any g -dimensional finer grid for g less than t , i.e., it achieves stratification on

$s^{u_1} \times \cdots \times s^{u_g}$ grids in g dimensions. Since an $SOA(n, m, s^t, t)$ can be collapsed into an $OA(n, m, s, t)$, we must have $n = \lambda s^t$, where λ is called the index of the SOA as well as the OA.

We now propose an SOA, termed the sliced SOA, that has something like sliced space-filling designs. For $s^t = ab$, an $SOA(\lambda s^t, m, s^t, t)$ is called a sliced SOA if it can be divided into b slices, each of which is a $\lambda a \times m$ matrix that can be collapsed into an SOA of size λa , m factors, and s^t/b levels. Obviously, a sliced SOA is also a sliced space-filling design.

3. Construction of Strong Orthogonal Arrays Using Orthogonal Arrays

In this section, we provide several methods for constructing SOAs using orthogonal arrays. We consider the construction of SOAs with even strength t in Section 3.1. When t is even, the newly constructed SOAs can easily achieve column-orthogonality, and even 3-orthogonality by sacrificing at most one column. In Section 3.2, SOAs of odd strength are constructed, where the column-orthogonality can be achieved by sacrificing more columns than in the case of even t . If we need more columns, SOAs achieving near column-orthogonality can also be constructed.

3.1. Construction of $SOA(n, m', s^t, t)$'s for even t

For $m = kt + q$, where q is an integer with $0 \leq q < t$, let

$$V_1 = \begin{pmatrix} 1 & s^{t-1} \\ s & s^{t-2} \\ \vdots & \vdots \\ s^{t-2} & s \\ s^{t-1} & 1 \end{pmatrix}, \quad R_{1(-s)} = \begin{pmatrix} V_1 & 0_{t \times 2} & \cdots & 0_{t \times 2} \\ 0_{t \times 2} & V_1 & \cdots & 0_{t \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{t \times 2} & 0_{t \times 2} & \cdots & V_1 \\ 0_{q \times 2} & 0_{q \times 2} & \cdots & 0_{q \times 2} \end{pmatrix},$$

and $R_{1(-s)}^* = (R_{1(-s)}, d)$, where V_1 occurs k times in $R_{1(-s)}$, $0_{t \times v}$ denotes a $t \times v$ matrix with all entries zero, and $d = (1, s, \dots, s^{t-q-1}, 0, \dots, 0, s^{t-q}, \dots, s^{t-1})^T$ is an $m \times 1$ vector. Let R_1 be an $m \times 2k$ matrix which consists of columns of $R_{1(-s)}$ up to sign changes, and R_1^* be an $m \times (2k + 1)$ matrix which consists of columns of $R_{1(-s)}^*$ up to sign changes.

Theorem 1. *Suppose A is an $OA(n, m, s, t)$ with $m = kt + q$ and $0 \leq q < t$, and R_1 and R_1^* are as defined above. Then*

- (i) for $0 \leq q < t/2$, $B = AR_1$ is an $SOA(n, 2k, s^t, t)$;
- (ii) for $t/2 \leq q < t$, $B = AR_1^*$ is an $SOA(n, 2k + 1, s^t, t)$.

Remark 1. From Theorem 1, we can use an orthogonal array and a matrix R_1 or R_1^* to construct an SOA with even strength. He and Tang (2013) pointed out that the existence of an SOA is equivalent to that of a GOA (see their definition of a GOA), and provided a method for constructing an SOA using a GOA. Essentially, the construction of SOAs in Theorem 1 is also achieved with some construction of GOAs. Take $m = kt, R_1 = R_{1(-s)}$, for example, and let

$$u_1 = (1, s, \dots, s^{t-1})^T, P = \text{diag}\{(I_t, N_t), \dots, (I_t, N_t)\}, \text{ and } Q = \text{diag}\{u_1, \dots, u_1\},$$

where I_t is the identity matrix of order t , N_t is the back diagonal identity matrix of order t , (I_t, N_t) occurs k times in P , and u_1 occurs $2k$ times in Q . Then R_1 can be decomposed as $R_1 = PQ$. So the construction process can be done in two steps. With $A = (a_1, \dots, a_{kt})$,

$$C = AP = ((a_1, \dots, a_t), (a_t, \dots, a_1), (a_{t+1}, \dots, a_{2t}), \dots, (a_{kt}, a_{kt-1}, \dots, a_{(k-1)t+1}))$$

is a $GOA(n, 2k, s, t)$. Then CQ is an SOA. This process is similar to that of He and Tang (2013) in the framework of OAs to GOAs to SOAs. Here, Theorem 1 constructs SOAs in one step, since, in this way, for mathematical forms and proofs it is more direct; for orthogonality, we want to discuss this through R . In the following, we give matrices like R_1 straightforwardly, instead of giving matrices such as P and Q .

Lemma 1 (Sun, Pang, and Liu (2011)). *Suppose A is an $n \times m$ matrix with $1_n^T A = 0_{1 \times m}$ and $A^T A = cI_m$, where 1_n denotes an $n \times 1$ vector with all entries one and c is a constant. Let $D = AT$, where T is a matrix with m rows. Then*

- (i) *if T is a column-orthogonal matrix, D is a column-orthogonal matrix;*
- (ii) *$\rho(D) = \tilde{\rho}(T)$, $\rho_M(D) = \tilde{\rho}_M(T)$ and $\rho^2(D) = \tilde{\rho}^2(T)$; and*
- (iii) *if A is a 3-orthogonal design, the estimates of all linear main effects of D are uncorrelated with the estimates of all quadratic effects and bilinear interactions. Furthermore, if T is a column-orthogonal matrix, D is a 3-orthogonal matrix.*

Accordingly, when $0 \leq q < t/2$, if R_1 is a column-orthogonal matrix, then the SOA $B = AR_1$ is a column-orthogonal SOA. When $t/2 \leq q < t$, we know that no matter how one changes the signs of the elements of d , d cannot be orthogonal to the first column of R_1^* . In this case, in order to get a column-orthogonal SOA,

we remove the last column of R_1^* to use R_1 instead. Let

$$V_2 = \begin{pmatrix} 1 & s & \cdots & s^{t/2-1} & s^{t/2} & \cdots & s^{t-1} \\ s^{t-1} & s^{t-2} & \cdots & s^{t/2} & -s^{t/2-1} & \cdots & -1 \end{pmatrix}^T, \text{ and}$$

$$R_2 = \begin{pmatrix} V_2 & 0_{t \times 2} & \cdots & 0_{t \times 2} \\ 0_{t \times 2} & V_2 & \cdots & 0_{t \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{t \times 2} & 0_{t \times 2} & \cdots & V_2 \\ 0_{q \times 2} & 0_{q \times 2} & \cdots & 0_{q \times 2} \end{pmatrix}, \tag{3.1}$$

where V_2 occurs k times in R_2 .

Theorem 2. *Suppose A is an $OA(n, m, s, t)$ with $m = kt + q$ and $0 \leq q < t$, and R_2 is as defined in (3.1). Then $B = AR_2$ is a column-orthogonal $SOA(n, 2k, s^t, t)$. Furthermore, B is a 3-orthogonal SOA if $t \geq 3$.*

Remark 2. For the case of even t , if an $OA(n, m, s, t)$ exists, we can construct an ordinary SOA by Theorem 1. The number of columns m' can also be expressed as $\lfloor 2m/t \rfloor$, which is the same as that of He and Tang (2013). According to Lemma 1, the correlation matrix of the resulting SOA in Theorem 1 is $\tilde{\rho}(R_1)$ or $\tilde{\rho}(R_1^*)$, which can be calculated more easily than directly using the SOA . It follows from Theorem 2 that we can construct a column-orthogonal SOA . When $0 \leq q < t/2$, the SOA has the same number of columns as the one constructed by Theorem 1, and when $t/2 \leq q < t$, the number of columns is only one less than that of the SOA , by Theorem 1.

Example 1. Suppose A is an $OA(8, 7, 2, 2)$,

$$R_2 = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } R_1^* = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Here R_2 is a column-orthogonal matrix, $\tilde{\rho}_{1,7}(R_1^*) = \tilde{\rho}_{7,1}(R_1^*) = 0.2$, $\tilde{\rho}_{2,7}(R_1^*) = \tilde{\rho}_{7,2}(R_1^*) = 0.4$, and other elements of $\tilde{\rho}(R_1^*)$ are all zero. Then $B = AR_1^*$ is an ordinary $SOA(8, 7, 4, 2)$, and $C = AR_2$, formed by the first six columns of B , is a column-orthogonal $SOA(8, 6, 4, 2)$. Besides, $\rho_{1,7}(B) = \rho_{7,1}(B) = 0.2$, $\rho_{2,7}(B) = \rho_{7,2}(B) = 0.4$, and other elements of $\rho(B)$ are all zero. The $OA(8, 7, 2, 2)$ and $SOA(8, 7, 4, 2)$ are listed in Table 1.

Table 1. The $OA(8, 7, 2, 2)$ and $SOA(8, 7, 4, 2)$ in Example 1.

$OA(8, 7, 2, 2)$							$SOA(8, 7, 4, 2)^*$						
-1	-1	-1	-1	-1	-1	-1	-3	-1	-3	-1	-3	-1	-3
1	-1	-1	1	-1	1	1	-1	3	1	-3	1	-3	3
1	1	-1	-1	1	-1	1	3	1	-3	-1	-1	3	3
1	1	1	-1	-1	1	-1	3	1	-1	3	1	-3	-1
-1	1	1	1	-1	-1	1	1	-3	3	1	-3	-1	1
1	-1	1	1	1	-1	-1	-1	3	3	1	-1	3	-1
-1	1	-1	1	1	1	-1	1	-3	1	-3	3	1	-3
-1	-1	1	-1	1	1	1	-3	-1	-1	3	3	1	1

* The first six columns form a column-orthogonal $SOA(8, 6, 4, 2)$

Example 2. Suppose A is an $OA(64, 8, 2, 4)$, and

$$R_2 = \begin{pmatrix} 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 8 & 4 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 & 8 & 4 & -2 & -1 \end{pmatrix}^T.$$

Here R_2 is a column-orthogonal matrix, and $B = AR_2$ is a 3-orthogonal $SOA(64, 4, 16, 4)$.

3.2. Construction of $SOA(n, m', s^t, t)$'s for odd t

For $m - 1 = k(t - 1) + q$, where q is an integer with $0 \leq q < t - 1$, let

$$V_3 = \begin{pmatrix} 1 & s & \dots & s^{(t-3)/2} & s^{(t+1)/2} & \dots & s^{t-1} \\ s^{t-1} & s^{t-2} & \dots & s^{(t+1)/2} & s^{(t-3)/2} & \dots & 1 \end{pmatrix}^T,$$

$$R_{3(-s)} = \begin{pmatrix} s^{(t-1)/2} 1_2^T & s^{(t-1)/2} 1_2^T & \dots & s^{(t-1)/2} 1_2^T \\ V_3 & 0_{(t-1) \times 2} & \dots & 0_{(t-1) \times 2} \\ 0_{(t-1) \times 2} & V_3 & \dots & 0_{(t-1) \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{(t-1) \times 2} & 0_{(t-1) \times 2} & \dots & V_3 \\ 0_{q \times 2} & 0_{q \times 2} & \dots & 0_{q \times 2} \end{pmatrix}, \text{ and } R_{3(-s)}^* = (R_{3(-s)}, d),$$

where V_3 occurs k times in $R_{3(-s)}$, and d is an $m \times 1$ vector with

$$d = \begin{cases} (s^{(t-1)/2}, 1, s, \dots, s^{(t-3)/2}, 0, \dots, 0, s^{(t+1)/2}, \dots, s^{t-1})^T, & \text{for } q = \frac{t-1}{2}; \\ (s^{(t-1)/2}, 1, s, \dots, s^{t-q-2}, 0, \dots, 0, s^{t-q-1}, \dots, \\ s^{(t-3)/2}, s^{(t+1)/2}, \dots, s^{t-1})^T, & \text{for } \frac{t-1}{2} < q < t-1. \end{cases}$$

Let R_3 be an $m \times 2k$ matrix which consists of columns of $R_{3(-s)}$ up to sign changes, and R_3^* be an $m \times (2k + 1)$ matrix which consists of columns of $R_{3(-s)}^*$ up to sign changes.

Theorem 3. *Suppose A is an $OA(n, m, s, t)$ with $m - 1 = k(t - 1) + q$ and $0 \leq q < t - 1$, and R_3 and R_3^* are as defined above. Then*

- (i) *for $0 \leq q < (t - 1)/2$, $B = AR_3$ is an $SOA(n, 2k, s^t, t)$;*
- (ii) *for $(t - 1)/2 \leq q < t - 1$, $B = AR_3^*$ is an $SOA(n, 2k + 1, s^t, t)$.*

According to Lemma 1, in order to obtain a column-orthogonal SOA by Theorem 3, R_3 or R_3^* needs to be a column-orthogonal matrix. However, R_3 and R_3^* cannot be orthogonal. We propose a new matrix R_4 that ensures a column-orthogonal SOA. The number of columns of R_4 is usually less than that of R_3 , the price we pay for the orthogonality.

For $m = k(t + 1) + q$, where q is an integer with $0 \leq q < t + 1$, let

$$\begin{aligned}
 V_4 &= \begin{pmatrix} 1 & s & \cdots & s^{(t-3)/2} & s^{(t-1)/2} & s^{(t+1)/2} & \cdots & s^{t-1} & 0 \\ s^{t-1} & s^{t-2} & \cdots & s^{(t+1)/2} & 0 & -s^{(t-3)/2} & \cdots & -1 & s^{(t-1)/2} \end{pmatrix}^T, \\
 R_4 &= \begin{pmatrix} V_4 & 0_{(t+1) \times 2} & \cdots & 0_{(t+1) \times 2} \\ 0_{(t+1) \times 2} & V_4 & \cdots & 0_{(t+1) \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{(t+1) \times 2} & 0_{(t+1) \times 2} & \cdots & V_4 \\ 0_{q \times 2} & 0_{q \times 2} & \cdots & 0_{q \times 2} \end{pmatrix}, \text{ and } R_4^* = (R_4, d), \tag{3.2}
 \end{aligned}$$

where V_4 occurs k times in R_4 , and $d = (0, \dots, 0, 1, s, \dots, s^{t-1})^T$ (for $q = t$) is an $m \times 1$ vector. It is easy to see that both R_4 and R_4^* are column-orthogonal matrices.

Theorem 4. *Suppose A is an $OA(n, m, s, t)$ with $m = k(t + 1) + q$ and $0 \leq q < t + 1$, and R_4 and R_4^* are as defined in (3.2). Then*

- (i) *for $q < t$, $B = AR_4$ is a column-orthogonal $SOA(n, 2k, s^t, t)$, and B is a 3-orthogonal SOA if $t \geq 3$;*
- (ii) *for $q = t$, $B = AR_4^*$ is a column-orthogonal $SOA(n, 2k + 1, s^t, t)$, and B is a 3-orthogonal SOA if $t \geq 3$.*

Example 3. Suppose A is an $OA(16, 8, 2, 3)$. Let

$$R_3^* = \begin{pmatrix} 2 & -2 & 2 & -2 & 2 & -2 & 2 \\ 1 & -4 & 0 & 0 & 0 & 0 & -1 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}, \text{ and } R_4 = \begin{pmatrix} 1 & 4 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Table 2. The $OA(16, 8, 2, 3)$, $SOA(16, 7, 8, 3)$ and 3-orthogonal $SOA(16, 4, 8, 3)$ in Example 3.

$OA(16, 8, 2, 3)$								$SOA(16, 7, 8, 3)$						$SOA(16, 4, 8, 3)$				
-1	-1	-1	-1	-1	-1	-1	-1	-7	5	-7	-3	-7	5	-5	-7	-5	-7	-5
-1	1	-1	1	-1	1	-1	1	-5	-3	-5	5	-5	-3	1	-3	-1	-3	-1
-1	-1	1	1	-1	-1	1	1	1	7	-5	5	1	7	3	1	-3	1	-3
-1	1	1	-1	-1	1	1	-1	3	-1	-7	-3	3	-1	-7	5	-7	5	-7
-1	-1	-1	-1	1	1	1	1	-7	5	1	-1	3	-1	3	-7	-5	7	5
-1	1	-1	1	1	-1	1	-1	-5	-3	3	7	1	7	-7	-3	-1	3	1
-1	-1	1	1	1	1	-1	-1	1	7	3	7	-5	-3	-5	1	-3	-1	3
-1	1	1	-1	1	-1	-1	1	3	-1	1	-1	-7	5	1	5	-7	-5	7
1	1	1	1	1	1	1	1	7	-5	7	3	7	-5	5	7	5	7	5
1	-1	1	-1	1	-1	1	-1	5	3	5	-5	5	3	-1	3	1	3	1
1	1	-1	-1	1	1	-1	-1	-1	-7	5	-5	-1	-7	-3	-1	3	-1	3
1	-1	-1	1	1	-1	-1	1	-3	1	7	3	-3	1	7	-5	7	-5	7
1	1	1	1	-1	-1	-1	-1	7	-5	-1	1	-3	1	-3	7	5	-7	-5
1	-1	1	-1	-1	1	-1	1	5	3	-3	-7	-1	-7	7	3	1	-3	-1
1	1	-1	-1	-1	-1	1	1	-1	-7	-3	-7	5	3	5	-1	3	1	-3
1	-1	-1	1	-1	1	1	-1	-3	1	-1	1	7	-5	-1	-5	7	5	-7

Then R_4 is a column-orthogonal matrix, and

$$\tilde{\rho}(R_3^*) = \begin{pmatrix} 1 & -0.1905 & 0.1905 & -0.1905 & 0.1905 & -0.1905 & 0.1429 \\ -0.1905 & 1 & -0.1905 & 0.1905 & -0.1905 & 0.1905 & 0 \\ 0.1905 & -0.1905 & 1 & 0.1905 & 0.1905 & -0.1905 & 0.1905 \\ -0.1905 & 0.1905 & 0.1905 & 1 & -0.1905 & 0.1905 & -0.1905 \\ 0.1905 & -0.1905 & 0.1905 & -0.1905 & 1 & -0.1905 & 0.1905 \\ -0.1905 & 0.1905 & -0.1905 & 0.1905 & -0.1905 & 1 & -0.1905 \\ 0.1429 & 0 & 0.1905 & -0.1905 & 0.1905 & -0.1905 & 1 \end{pmatrix}.$$

Based on Theorem 3(ii), $B = AR_3^*$ is an ordinary $SOA(16, 7, 8, 3)$ with a correlation matrix $\rho(B) = \tilde{\rho}(R_3^*)$ and, from Lemma 1(iii), we can know that the estimates of all linear main effects of B are uncorrelated with the estimates of all quadratic effects and bilinear interactions. Based on Theorem 4(i), $C = AR_4$ is a 3-orthogonal $SOA(16, 4, 8, 3)$. Both designs achieve stratifications on 2×4 and 4×2 grids in any two-dimensional projection, and achieve a stratification on a $2 \times 2 \times 2$ grid in any three-dimensional projection. The $OA(16, 8, 2, 3)$, $SOA(16, 7, 8, 3)$, and 3-orthogonal $SOA(16, 4, 8, 3)$ are listed in Table 2.

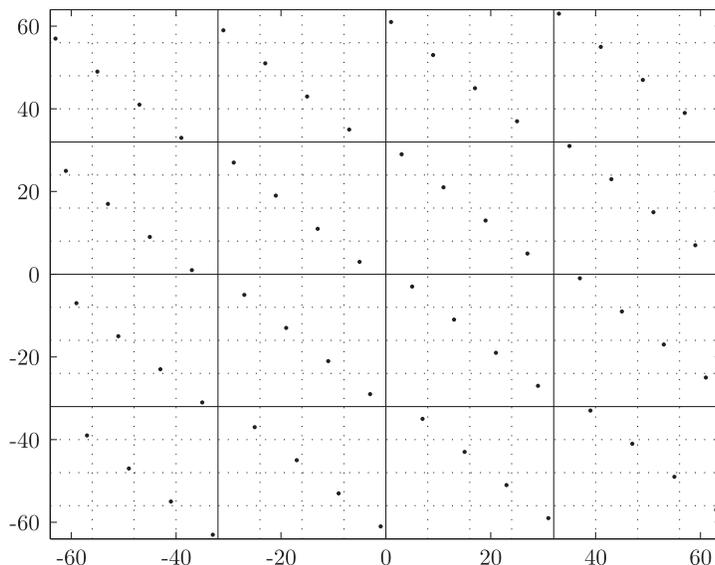


Figure 1. The binary projection of the first two columns of the $SOA(64, 5, 64, 3)$ in Example 4.

Example 4. Suppose A is an $OA(64, 6, 4, 3)$, and

$$R_3^* = \begin{pmatrix} 4 & -4 & 4 & -4 & 4 \\ 1 & -16 & 0 & 0 & -1 \\ 16 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 16 & 0 \\ 0 & 0 & 16 & 1 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}.$$

Let $B = AR_3^*$, then it can be calculated that $\rho_M(B) = \tilde{\rho}_M(R_3^*) = 0.0586$, and $\rho^2(B) = \tilde{\rho}^2(R_3^*) = 0.00305$. We can see that B is a nearly column-orthogonal $SOA(64, 5, 64, 3)$, which is obviously a nearly column-orthogonal LHD with 64 runs and 5 factors. The estimates of all linear main effects of B are uncorrelated with the estimates of all quadratic effects and bilinear interactions. Furthermore, it achieves stratifications on 16×4 and 4×16 grids in any two-dimensional projection, and achieves a stratification on a $4 \times 4 \times 4$ grid in any three-dimensional projection. The binary projection of the first two columns of this design is displayed in Figure 1. Design B is given in Table 3.

Theorem 4 ensures that the constructed SOA is column-orthogonal, but it usually has many fewer columns than the one in Theorem 3. To obtain an SOA with more columns, its orthogonality is discussed below.

Table 3. The nearly column-orthogonal $SOA(64, 5, 64, 3)$ in Example 4.

Run	1	2	3	4	5	Run	1	2	3	4	5
1	-63	57	-63	-39	-57	33	-47	41	-11	11	55
2	-31	59	-29	-5	-25	34	-15	43	-41	41	23
3	1	61	5	29	7	35	17	45	49	-49	-9
4	33	63	39	63	39	36	49	47	19	-19	-41
5	-61	25	3	-3	37	37	-45	9	55	47	-43
6	-29	27	33	-33	5	38	-13	11	21	13	-11
7	3	29	-57	57	-27	39	19	13	-13	-21	21
8	35	31	-27	27	-59	40	51	15	-47	-55	53
9	-59	-7	37	31	-29	41	-43	-23	17	-51	19
10	-27	-5	7	61	-61	42	-11	-21	51	-17	51
11	5	-3	-31	-37	35	43	21	-19	-43	9	-45
12	37	-1	-61	-7	3	44	53	-17	-9	43	-13
13	-57	-39	-25	59	1	45	-41	-55	-45	-23	-15
14	-25	-37	-59	25	33	46	-9	-53	-15	-53	-47
15	7	-35	35	-1	-63	47	23	-51	23	45	49
16	39	-33	1	-35	-31	48	55	-49	53	15	17
17	-55	49	43	-9	15	49	-39	33	31	37	-1
18	-23	51	9	-43	47	50	-7	35	61	7	-33
19	9	53	-17	51	-49	51	25	37	-37	-31	63
20	41	55	-51	17	-17	52	57	39	-7	-61	31
21	-53	7	-23	-45	-19	53	-37	1	-35	1	29
22	-21	19	-53	-15	-51	54	-5	3	-1	35	61
23	11	21	45	23	45	55	27	5	25	-59	-35
24	43	23	15	53	13	56	59	7	59	-25	-3
25	-51	-15	-49	49	43	57	-35	-31	-5	-29	-37
26	-19	-13	-19	19	11	58	-3	-29	-39	-63	-5
27	13	-11	11	-11	-21	59	29	-27	63	39	27
28	45	-9	41	-41	-53	60	61	-25	29	5	59
29	-49	-47	13	21	-55	61	-33	-63	57	-57	57
30	-17	-45	47	55	-23	62	-1	-61	27	-27	25
31	15	-43	-55	-47	9	63	31	-59	-3	3	-7
32	47	-41	-21	-13	41	64	63	-57	-33	33	-39

For $m - 1 = k(t - 1) + q$, where q is an integer with $0 \leq q < t - 1$, let

$$\begin{aligned}
 V_5 &= \begin{pmatrix} 1 & s & \dots & s^{(t-3)/2} & s^{(t+1)/2} & \dots & s^{t-1} \end{pmatrix}^T, \text{ and} \\
 R_5 &= \begin{pmatrix} s^{(t-1)/2} \mathbf{1}_2^T & s^{(t-1)/2} \mathbf{1}_2^T & \dots & s^{(t-1)/2} \mathbf{1}_2^T \\ V_5 & 0_{(t-1) \times 2} & \dots & 0_{(t-1) \times 2} \\ 0_{(t-1) \times 2} & V_5 & \dots & 0_{(t-1) \times 2} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{(t-1) \times 2} & 0_{(t-1) \times 2} & \dots & V_5 \\ 0_{q \times 2} & 0_{q \times 2} & \dots & 0_{q \times 2} \end{pmatrix}, \tag{3.3}
 \end{aligned}$$

Table 4. The values of $s^{t-1}(s^2 - 1)/(s^{2t} - 1)$ for $s = 2, \dots, 9$ and $t = 3, 5$.

t	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$	$s = 8$	$s = 9$
3	0.1905	0.0989	0.0586	0.0384	0.0270	0.0200	0.0154	0.0122
5	0.0469	0.0110	0.0037	0.0015	0.0008	0.0004	0.0002	0.0002

where V_5 occurs k times in R_5 .

Theorem 5. *Suppose A is an $OA(n, m, s, t)$ with $m - 1 = k(t - 1) + q$ and $0 \leq q < t - 1$, and R_5 is as defined in (3.3). Then $B = AR_5$ is an $SOA(n, 2k, s^t, t)$ with $\rho_{ij}(B) = s^{t-1}(s^2 - 1)/(s^{2t} - 1)$ for any $i \neq j$, strictly decreasing with respect to s and t .*

Some values of $s^{t-1}(s^2 - 1)/(s^{2t} - 1)$ are shown in Table 4. From Theorem 5 and Table 4, we know that the SOAs constructed by Theorem 5 perform well in terms of the near column-orthogonality except for the case of $s = 2$ and $t = 3$.

Remark 3.

- (i) For the case of odd t , if an $OA(n, m, s, t)$ exists, we can construct an ordinary SOA by Theorem 3. The number of columns m' can be expressed as $\lfloor 2(m - 1)/(t - 1) \rfloor$, which is the same as that of He and Tang (2013). It follows from Theorem 4 that we can construct a column-orthogonal SOA with the number of columns many fewer than that of the SOA constructed by Theorem 3. However, Theorem 5 ensures that for given parameters n, m, s and t , from the SOAs by Theorem 3, we can find nearly column-orthogonal SOAs by sacrificing at most one column.
- (ii) If $\lambda = 1$, where λ is the index of the orthogonal array, then the resulting SOA is an LHD. In particular, according to Theorems 2 and 4, we can construct column-orthogonal LHDs, even 3-orthogonal LHDs and, according to Theorem 5, we can construct nearly column-orthogonal LHDs. Furthermore, the LHDs achieve uniformity on finer grids when projected onto g dimensions for any g less than t , see Example 4 for example.
- (iii) According to Lemma 1(ii) and Theorems 1 and 3, we can obtain nearly column-orthogonal SOAs by changing the signs of the elements in matrices R_1, R_1^*, R_3 and R_3^* .

4. Construction of Sliced Strong Orthogonal Arrays

For $n = \lambda s^t$, let A be an $OA(n, m + 1, s, t)$, and a_j be the j th column of A , $j = 1, \dots, m + 1$. For any $l = 1, \dots, m + 1$, we obtain an $n \times m$ matrix B_l by permuting the rows of A in an increasing order of the elements in a_l and then omitting a_l . It is easy to see that B_l is an $OA(n, m, s, t)$ that can be divided into s slices, and each slice is an $OA(n/s, m, s, t - 1)$. According to the number of

columns and strength of B_i , we get the corresponding matrix R_i or R_i^* used in Theorem 1, 2, 3, 4 or 5, where the number of columns of R_i or R_i^* is denoted by m' . Then sliced SOAs can be produced as follows.

Theorem 6. *For any $B \in \{B_1, \dots, B_{m+1}\}$ constructed above, let R be the corresponding matrix R_i or R_i^* used in one of Theorems 1–5. Then*

- (i) $C = BR$ is a sliced SOA(n, m', s^t, t) with s slices, and each slice is an SOA($n/s, m', s^{t-1}, t - 1$) when collapsed into s^{t-1} levels, where m' is the number of columns of R ; when $t \geq 4$, the estimates of all linear main effects of each slice are uncorrelated with the estimates of all quadratic effects and bilinear interactions of the slice;
- (ii) if R is column-orthogonal, each slice of C is column-orthogonal when $t \geq 3$, and 3-orthogonal when $t \geq 4$.

For any SOA($n, m + 1, s^t, t$) with $n = \lambda s^t$, denoted by A , let a_j be the j th column, $j = 1, \dots, m + 1$. Collapse a_j into an s -level column, denoted by b_j , and obtain an $n \times m$ matrix C_j by permuting the rows of $(b_j, a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{m+1})$ in an increasing order of the elements in b_j , and then omitting b_j .

Corollary 1. *Any $C \in \{C_1, \dots, C_{m+1}\}$, constructed above, is a sliced SOA(n, m, s^t, t) with s slices, and each slice is an SOA($n/s, m, s^{t-1}, t - 1$) when collapsed into s^{t-1} levels. Furthermore, if the SOA($n, m + 1, s^t, t$) A is obtained from one of Theorems 1–5 in Section 3, and A is column-orthogonal, then each slice of C is also column-orthogonal when $t \geq 3$.*

Thus if we want to get sliced SOAs with low correlations, we can use the SOAs of Section 3 for the construction. The numbers of columns of the sliced SOAs with the same run size and strength constructed from the OA and SOA may be different.

Example 5. Suppose A is the OA(16, 8, 2, 3) of Table 2. Omit its first column and denote the resulting design by B . Let

$$R_3^* = \begin{pmatrix} 2 & -2 & 2 & -2 & 2 & -2 \\ 1 & 4 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 4 & 1 \end{pmatrix}, \text{ and } R_4 = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 0 & 0 \\ 4 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

It is easy to see that R_4 is a column-orthogonal matrix. Then $C = BR_3^*$ is a sliced SOA(16, 6, 8, 3) with two slices, $D = BR_4$ is a column-orthogonal sliced

Table 5. The sliced $SOA(16, 6, 8, 3)$ and column-orthogonal sliced $SOA(16, 3, 8, 3)$ in Example 5.

$SOA(16, 6, 8, 3)$						$SOA(16, 3, 8, 3)$		
-7	-3	-7	-3	-7	-3	-7	-5	-7
5	-5	5	-5	5	-5	3	1	3
3	7	-7	-3	3	7	5	-7	5
-1	1	5	-5	-1	1	-1	3	-1
-7	-3	3	7	3	7	-7	-1	7
5	-5	-1	1	-1	1	3	5	-3
3	7	3	7	-7	-3	5	-3	-5
-1	1	-1	1	5	-5	-1	7	1
7	3	7	3	7	3	7	5	7
-5	5	-5	5	-5	5	-3	-1	-3
-3	-7	7	3	-3	-7	-5	7	-5
1	-1	-5	5	1	-1	1	-3	1
7	3	-3	-7	-3	-7	7	1	-7
-5	5	1	-1	1	-1	-3	-5	3
-3	-7	-3	-7	7	3	-5	3	5
1	-1	1	-1	-5	5	1	-7	-1

$SOA(16, 3, 8, 3)$ with two slices, and each slice of D is also a column-orthogonal design. The designs C and D are shown in Table 5.

From the $SOA(16, 7, 8, 3)$ and column-orthogonal $SOA(16, 4, 8, 3)$ in Table 2, we can get a sliced $SOA(16, 6, 8, 3)$ with two slices and a column-orthogonal sliced $SOA(16, 3, 8, 3)$ with two slices (each slice is also a column-orthogonal design) by permuting the rows of the $SOA(16, 7, 8, 3)$ and $SOA(16, 4, 8, 3)$ according to their first columns and then omitting them, respectively.

In this example, the ordinary sliced SOAs with the same run size and strength constructed by the two methods have the same number of columns, and this is also true for the two column-orthogonal sliced SOAs.

Example 6. Suppose A is an $OA(128, 15, 2, 4)$. Permuting the rows of A in an increasing order of the elements in the first column, and then omitting this column, we obtain a matrix, B , that is an $OA(128, 14, 2, 4)$. According to Theorem 6, we can obtain a sliced $SOA(128, 7, 16, 4)$ $C = BR_1^*$ with the R_1^* of Theorem 1.

From the same A and Theorem 1, we can first construct an $SOA(128, 7, 16, 4)$, and then obtain a sliced $SOA(128, 6, 16, 4)$ according to Corollary 1. Here the sliced SOA constructed directly from the OA has one more column than that from the SOA.

Example 7. Suppose A is an $OA(2^{11}, 32, 2, 4)$ and that the matrix B is obtained as in Example 6. Then B is an $OA(2^{11}, 31, 2, 4)$ and, by Theorem 6, $C = BR_2$ is a column-orthogonal sliced $SOA(2^{11}, 14, 16, 4)$, where R_2 is defined in (3.1). Using

Theorem 2 and A, we can first construct a column-orthogonal $SOA(2^{11}, 16, 16, 4)$, and then a column-orthogonal sliced $SOA(2^{11}, 15, 16, 4)$ using Corollary 1 that has one more column than the one constructed by Theorem 6.

Remark 4. As in Theorem 6 and Corollary 1, for $i = 1, \dots, t - 1$, we can construct sliced $SOA(n, m, s^t, t)$'s, each with s^{t-i} slices, so that each slice can be collapsed into an $SOA(n/s^{t-i}, m, s^i, i)$. Since from a practical point of view, the run size of each slice is usually much larger than the number of slices, the sliced SOAs constructed by Theorem 6 and Corollary 1 suffice.

5. Concluding Remarks

The SOA can be seen as a kind of space-filling design with relatively good uniformity. We propose some methods to construct them. The methods are easy to implement, and it is easy to evaluate the orthogonality of the constructions.

Let $h(n, s, t)$ denote the largest m for which an $SOA(n, m, s^t, t)$ exists, and let $f(n, s, t)$ denote the largest m for which an $OA(n, m, s, t)$ exists. We know that if there exists an $SOA(n, m, s^t, t)$, then an $OA(n, m, s, t)$ can be constructed from it by level collapsing, so $h(n, s, t) \leq f(n, s, t)$. Remarks 2 and 3(i) imply that

$$h(n, s, t) \geq \begin{cases} \lfloor \frac{2f(n, s, t)}{t} \rfloor, & \text{for even } t; \\ \lfloor \frac{2(f(n, s, t) - 1)}{(t - 1)} \rfloor, & \text{for odd } t. \end{cases}$$

Thus for $t = 2$, $h(n, s, 2) = f(n, s, 2)$, and for $t = 3$, $f(n, s, 3) - 1 \leq h(n, s, 3) \leq f(n, s, 3)$, also obtained in He and Tang (2013).

Sliced space-filling designs have received much recent interest in computer experiments. We propose a kind of sliced space-filling design, the sliced SOA. Two methods are provided to construct such designs.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271205, 11431006 and 11401321), the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20130031110002), and the “131” Talents Program of Tianjin. The authors thank Editor Jeng-Min Chiou, an associate editor, and a referee for their valuable comments and suggestions.

Appendix: Proofs of Theorems

Lemma A.1. Let $A = (a_1, \dots, a_t)$ be an $OA(s^t, t, s, t)$, $d_k = (d_{k_1}, \dots, d_{k_k})^T$ be a vector of $(s^{k-1}, s^{k-2}, \dots, s, 1)^T$ up to sign changes, where k is a positive integer with $k \leq t$. Then

- (i) if $\sum_{r=1}^t a_r d_{t_r}$ is collapsed into s^u levels, it is $\sum_{r=1}^u \text{sgn}(d_{t_r}) a_r s^{u-r}$, $1 \leq u \leq t-1$, and if $\sum_{r=1}^t a_r s^{t-r}$ is collapsed into s^u levels, it is $\sum_{r=1}^u a_r s^{u-r}$;
- (ii) for $g \leq t$ and $u_1 + \dots + u_g = t$,

$$\left(\sum_{r=1}^{u_1} a_r d_{u_{1r}}, \sum_{r=1}^{u_2} a_{u_1+r} d_{u_{2r}}, \dots, \sum_{r=1}^{u_g} a_{\sum_{j=1}^{g-1} u_j+r} d_{u_{gr}} \right)$$

is an $OA(s^t, g, s^{u_1} \times \dots \times s^{u_g}, g)$. In particular,

$$\left(\sum_{r=1}^{u_1} a_r s^{u_1-r}, \sum_{r=1}^{u_2} a_{u_1+r} s^{u_2-r}, \dots, \sum_{r=1}^{u_g} a_{\sum_{j=1}^{g-1} u_j+r} s^{u_g-r} \right)$$

is an $OA(s^t, g, s^{u_1} \times \dots \times s^{u_g}, g)$.

A.1. Proof of Lemma A.1

Without loss of generality, we only prove the case $d_k = (s^{k-1}, s^{k-2}, \dots, s, 1)^T$, where $k \leq t$.

- (i) Obviously, $\sum_{r=1}^t a_r s^{t-r}$ yields s^t equally-spaced levels $-(s^t-1), -(s^t-3), \dots, (s^t-1)$. Then if $\sum_{r=1}^t a_r s^{t-r}$ is collapsed into s^u levels, it is

$$\begin{aligned} & 2 \left\lfloor \frac{\sum_{r=1}^t a_r s^{t-r} + s^t}{2s^{t-u}} \right\rfloor - s^u + 1 \\ &= 2 \left\lfloor \frac{\sum_{r=1}^u a_r s^{u-r} + s^u}{2} + \frac{\sum_{r=u+1}^t a_r s^{t-r}}{2s^{t-u}} \right\rfloor - s^u + 1 \\ &= 2 \left\lfloor \frac{\sum_{r=1}^{u-1} a_r s^{u-r} + a_u + s^u - 1}{2} + \frac{1}{2} + \frac{\sum_{r=u+1}^t a_r s^{t-r}}{2s^{t-u}} \right\rfloor - s^u + 1. \end{aligned}$$

If s is odd (even), the elements of a_r are even (odd), so the elements of $\sum_{r=1}^{u-1} a_r s^{u-r} + a_u + s^u - 1$ are even. Since the elements of $\sum_{r=u+1}^t a_r s^{t-r} = a_{u+1} s^{t-u-1} + a_{u+2} s^{t-u-2} + \dots + a_t$ are $-(s^{t-u}-1), -(s^{t-u}-3), \dots, (s^{t-u}-1)$, then the elements of $\sum_{r=u+1}^t a_r s^{t-r} / 2s^{t-u}$ belong to $[-1/2 + 1/2s^{t-u}, 1/2 - 1/2s^{t-u}]$. As a result, the elements of $1/2 + \sum_{r=u+1}^t a_r s^{t-r} / 2s^{t-u}$ belong to $(0, 1)$. Then it follows that if $\sum_{r=1}^t a_r s^{t-r}$ is collapsed into s^u levels, it is

$$\sum_{r=1}^u a_r s^{u-r} + (s^u - 1) - (s^u - 1) = \sum_{r=1}^u a_r s^{u-r},$$

whose elements are $-(s^u - 1), -(s^u - 3), \dots, (s^u - 1)$.

- (ii) For $l = 1, \dots, g$, let $b_l = \sum_{r=1}^{u_l} a_{\sum_{i=1}^{l-1} u_i + r} s^{u_l - r}$. We know that b_l has s^{u_l} levels with each level occurring s^{t-u_l} times. Let $(a_{i1}, a_{i2}, \dots, a_{it})$ be the i th row of A , and $(b_{i1}, b_{i2}, \dots, b_{ig})$ be the corresponding i th row of (b_1, b_2, \dots, b_g) . Then

$$(b_{i1}, b_{i2}, \dots, b_{ig}) = \left(\sum_{r=1}^{u_1} a_{ir} s^{u_1 - r}, \sum_{r=1}^{u_2} a_{i(\sum_{j=1}^{l-1} u_j + r)} s^{u_2 - r}, \dots, \sum_{r=1}^{u_g} a_{i(\sum_{j=1}^{g-1} u_j + r)} s^{u_g - r} \right).$$

Since (a_1, a_2, \dots, a_t) is an $OA(s^t, t, s, t)$, all the $(a_{i1}, a_{i2}, \dots, a_{it})$'s for $i = 1, \dots, s^t$ are different, thus all the (b_{i1}, \dots, b_{ig}) 's for $i = 1, \dots, s^t$ are different. Thus (b_1, \dots, b_g) is an $OA(s^t, g, s^{u_1} \times \dots \times s^{u_g}, g)$, noting that $\sum_{i=1}^g u_i = t$.

Remark A.1. In Lemma A.1, if we change the order of the elements in d_k , then similar conclusions can be reached. This is important for proving the theorems.

A.2. Proof of Theorem 1

Let $A = (a_1, \dots, a_m)$. Without loss of generality, let $R_1 = R_{1(-s)}$ and $R_1^* = R_{1(-s)}^*$.

- (i) For $j = 1, \dots, 2k$, let b_j be the j th column of B . Then we have

$$b_{2j-1} = \sum_{r=1}^t a_{(j-1)t+r} s^{r-1}, \text{ and } b_{2j} = \sum_{r=1}^t a_{(j-1)t+r} s^{t-r}, \quad j = 1, \dots, k.$$

According to Lemma A.1 and Remark A.1, if b_{2j-1} is collapsed into s^{u_1} levels, it is

$$2 \left\lfloor \frac{b_{2j-1} + s^t}{2s^{t-u_1}} \right\rfloor - s^{u_1} + 1 = \sum_{r=t-u_1+1}^t a_{(j-1)t+r} s^{r-t+u_1-1},$$

and if b_{2j} is collapsed into s^{u_2} levels, it is

$$2 \left\lfloor \frac{b_{2j} + s^t}{2s^{t-u_2}} \right\rfloor - s^{u_2} + 1 = \sum_{r=1}^{u_2} a_{(j-1)t+r} s^{u_2-r}.$$

If $D_j = (b_{2j-1}, b_{2j})$ is the j th block of B , $j = 1, \dots, k$, then $B = (D_1, \dots, D_k)$. It is easy to see that B is an $OA(n, 2k, s^t, 1)$. For $g \geq 2$ and $u_1 + \dots + u_g = t$, consider the case that there is at least one pair of columns which are selected from the same block. We can specially select b_{2j-1} and b_{2j} for $g = 2$ and $u_1 + u_2 = t$. In this case, if b_{2j-1} is collapsed into s^{u_1} levels, it is $\sum_{r=u_2+1}^t a_{(j-1)t+r} s^{r-u_2-1}$, while if b_{2j} is collapsed into s^{u_2} levels, it is $\sum_{r=1}^{u_2} a_{(j-1)t+r} s^{u_2-r}$. Since $(a_{(j-1)t+1}, a_{(j-1)t+2}, \dots, a_{jt})$ is an $OA(n, t, s, t)$, according to Lemma A.1 and Remark A.1 it follows that (b_{2j-1}, b_{2j}) can be collapsed into an $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$. For $g \geq 2$ and $u_1 + \dots + u_g = t$, if

we select g columns from different blocks D_{i_1}, \dots, D_{i_g} , then these g columns can be collapsed into an $OA(n, g, s^{u_1} \times \dots \times s^{u_g}, g)$ since, for any t columns $(a_{i_1}, \dots, a_{i_t})$ of A , they form an $OA(n, t, s, t)$. Other cases can be derived similarly.

- (ii) For $t/2 \leq q < t$ and $u_1 + u_2 = t$, it is enough to prove that both (b_1, b_{2k+1}) and (b_2, b_{2k+1}) can be collapsed into $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$'s, where

$$b_{2k+1} = \sum_{r=1}^{t-q} a_r s^{r-1} + \sum_{r=1}^q a_{kt+r} s^{t-q-1+r}.$$

For $u_1 + u_2 = t$ and $u_2 > q$, if b_{2k+1} is collapsed into s^{u_2} levels, it is

$$\sum_{r=t-u_2+1}^{t-q} a_r s^{r-t+u_2-1} + \sum_{r=1}^q a_{kt+r} s^{u_2-q-1+r} = \sum_{r=u_1+1}^{t-q} a_r s^{r-u_1-1} + \sum_{r=1}^q a_{kt+r} s^{u_2-q-1+r},$$

if b_1 is collapsed into s^{u_1} levels, it is $\sum_{r=t-u_1+1}^t a_r s^{r-t+u_1-1}$, and if b_2 is collapsed into s^{u_1} levels, it is $\sum_{r=1}^{u_1} a_r s^{u_1-r}$. Then both (b_1, b_{2k+1}) and (b_2, b_{2k+1}) can be collapsed into $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$'s. Since $t/2 \leq q < t$ and $u_2 = t - u_1 > q$, we have $t - q < t - u_1 + 1$ and $u_1 < u_1 + 1$, which in turn yields that both $(a_{u_1+1}, \dots, a_{t-q}, a_{t-u_1+1}, \dots, a_t, a_{kt+1}, \dots, a_{kt+q})$ and $(a_1, \dots, a_{u_1}, a_{u_1+1}, \dots, a_{t-q}, a_{kt+1}, \dots, a_{kt+q})$ are $OA(n, t, s, t)$'s.

For the case $u_2 \leq q$, we can derive the result by a similar argument. The details are omitted.

A.3. Proof of Theorem 2

According to Theorem 1, we know that $B = AR_2$ is an $SOA(n, 2k, s^t, t)$. Since R_2 is column-orthogonal, it follows from Lemma 1 that B is a column-orthogonal $SOA(n, 2k, s^t, t)$, and it is 3-orthogonal if $t > 3$.

A.4. Proof of Theorem 3

- (i) Let $A = (a_1, \dots, a_m)$. For $j = 1, \dots, 2k$, let b_j denote the j th column of B . Since the proof is similar to that of Theorem 1(i), here we only prove that (b_{2j-1}, b_{2j}) can be collapsed into an $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$, where $u_1 + u_2 = t$. For $j = 1, \dots, k$,

$$b_{2j-1} = a_1 s^{(t-1)/2} + \sum_{r=1}^{(t-1)/2} a_{(j-1)(t-1)+r+1} s^{r-1} + \sum_{r=(t+1)/2}^{t-1} a_{(j-1)(t-1)+r+1} s^r,$$

and

$$b_{2j} = a_1 s^{(t-1)/2} + \sum_{r=1}^{(t-1)/2} a_{(j-1)(t-1)+r+1} s^{t-r} + \sum_{r=(t+1)/2}^{t-1} a_{(j-1)(t-1)+r+1} s^{t-r-1}.$$

For $u_1 + u_2 = t$ and $u_2 \geq (t - 1)/2$, if b_{2j-1} is collapsed into s^{u_1} levels, it is

$$2 \left\lfloor \frac{b_{2j-1} + s^t}{2s^{t-u_1}} \right\rfloor - s^{u_1} + 1 = \begin{cases} a_1 + \sum_{r=u_2+1}^{t-1} a_{(j-1)(t-1)+r+1} s^{r-u_2}, & u_2 = \frac{t-1}{2}, \\ \sum_{r=u_2}^{t-1} a_{(j-1)(t-1)+r+1} s^{r-u_2}, & u_2 > \frac{t-1}{2}; \end{cases} \tag{A.1}$$

and if b_{2j} is collapsed into s^{u_2} levels, it is

$$2 \left\lfloor \frac{b_{2j} + s^t}{2s^{t-u_2}} \right\rfloor - s^{u_2} + 1 = \begin{cases} \sum_{r=1}^{u_2} a_{(j-1)(t-1)+r+1} s^{u_2-r}, & u_2 = \frac{t-1}{2}, \\ a_1 + \sum_{r=1}^{u_2-1} a_{(j-1)(t-1)+r+1} s^{u_2-r}, & u_2 = \frac{t+1}{2}, \\ a_1 s^{u_2-(t+1)/2} + \sum_{r=1}^{(t-1)/2} a_{(j-1)(t-1)+r+1} s^{u_2-r} \\ + \sum_{r=(t+1)/2}^{u_2-1} a_{(j-1)(t-1)+r+1} s^{u_2-r-1}, & u_2 > \frac{t+1}{2}. \end{cases} \tag{A.2}$$

From (A.1), (A.2), and Lemma A.1, it follows that (b_{2j-1}, b_{2j}) can be collapsed into an $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$.

The case $u_2 < (t - 1)/2$ can be derived similarly.

- (ii) As in the proof of Theorem 1(ii), for $(t - 1)/2 \leq q < t$ and $u_1 + u_2 = t$, it is enough to prove that both (b_1, b_{2k+1}) and (b_2, b_{2k+1}) can be collapsed into $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$'s, where

$$b_{2k+1} = \begin{cases} a_1 s^q + \sum_{r=1}^{t-q-1} a_{r+1} s^{r-1} + \sum_{r=1}^q a_{(k-1)(t-1)+r} s^{t-q-1+r}, & q = \frac{t-1}{2}, \\ a_1 s^{(t-1)/2} + \sum_{r=1}^{t-q-1} a_{r+1} s^{r-1} + \sum_{r=1}^{q-(t-1)/2} a_{(k-1)(t-1)+r} s^{t-q-2+r} \\ + \sum_{r=q-(t-3)/2}^q a_{(k-1)(t-1)+r} s^{t-q-1+r}, & q > \frac{t-1}{2}. \end{cases}$$

For $u_1 + u_2 = t$ and $u_2 > q \geq (t - 1)/2$, if b_{2k+1} is collapsed into s^{u_2} levels, it is

$$\left\{ \begin{array}{l} a_1 + \sum_{r=1}^q a_{(k-1)(t-1)+r} s^r, \quad u_2 = \frac{t+1}{2}, q = \frac{t-1}{2}, \\ a_1 s^{q-t+u_2} + \sum_{r=t-u_2+1}^{t-q-1} a_{r+1} s^{r-(t+1-u_2)} \\ + \sum_{r=1}^q a_{(k-1)(t-1)+r} s^{r-q-1+u_2}, \quad u_2 > \frac{t+1}{2}, q = \frac{t-1}{2}, \\ a_1 s^{u_2-(t+1)/2} + \sum_{r=t-u_2+1}^{t-q-1} a_{r+1} s^{r-(t+1-u_2)} \\ + \sum_{r=1}^{q-(t-1)/2} a_{(k-1)(t-1)+r} s^{r-q-2+u_2} \\ + \sum_{r=q-(t-3)/2}^q a_{(k-1)(t-1)+r} s^{r-q-1+u_2}, \quad u_2 > q \geq \frac{t+1}{2}. \end{array} \right. \quad (\text{A.3})$$

For $u_2 > (t - 1)/2$ and $u_1 + u_2 = t$, we know that if b_1 is collapsed into s^{u_1} levels, it is

$$2 \left\lfloor \frac{b_1 + s^t}{2s^{t-u_1}} \right\rfloor - s^{u_1} + 1 = \sum_{r=u_2}^{t-1} a_{r+1} s^{r-u_2} \quad (\text{A.4})$$

and, if b_2 is collapsed into s^{u_1} levels, it is

$$2 \left\lfloor \frac{b_2 + s^t}{2s^{t-u_1}} \right\rfloor - s^{u_1} + 1 = \sum_{r=1}^{t-u_2} a_{r+1} s^{t-r-u_2}. \quad (\text{A.5})$$

From (A.3), (A.4), (A.5), and Lemma A.1, it follows that both (b_1, b_{2k+1}) and (b_2, b_{2k+1}) can be collapsed into $OA(n, 2, s^{u_1} \times s^{u_2}, 2)$'s.

The other cases can be derived similarly.

A.5. Proof of Theorem 4

According to Theorem 3, we know that both AR_4 and AR_4^* are strong orthogonal arrays. Since both R_4 and R_4^* are column-orthogonal, the conclusions can be obtained through Lemma 1.

A.6. Proof of Theorem 5

Obviously, R_5 is a special case of R_3 . Then from Theorem 3, AR_5 is an $SOA(n, 2k, s^t, t)$. It can be easily calculated that the correlation between any two distinct columns of R_5 is $s^{t-1}(s^2 - 1)/(s^{2t} - 1)$. According to Lemma 1, we know that the correlation between any two distinct columns of B is also $s^{t-1}(s^2 - 1)/(s^{2t} - 1)$. It is not difficult to prove that $s^{t-1}(s^2 - 1)/(s^{2t} - 1)$ is strictly decreasing with respect to s and t , respectively. We omit the details.

A.7. Proof of Theorem 6

Without loss of generality, we only prove the case of even t with R being $R_{1(-s)}$; the proofs are similar for other cases. Let $B = (B^{1T}, \dots, B^{sT})^T$, where B^i consists of the rows of B from $(i - 1)n/s + 1$ to in/s . Then B^i is an $OA(n/s, m, s, t - 1)$. Similarly, let $C = (C^{1T}, C^{2T}, \dots, C^{sT})^T$, where C^i consists of the rows of C from $(i - 1)n/s$ to in/s . Then according to Theorem 1,

$$C^i = B^i R, \quad c_{2j-1}^i = \sum_{r=1}^t b_{(j-1)t+r}^i s^{r-1}, \quad c_{2j}^i = \sum_{r=1}^t b_{(j-1)t+r}^i s^{t-r},$$

where c_l^i is the l th column of C^i and b_v^i is the v th column of B^i , $l = 1, \dots, 2k$, $v = 1, \dots, m$, and $i = 1, \dots, s$. Let $D = (D^{1T}, D^{2T}, \dots, D^{sT})^T$ be the matrix obtained by collapsing C into s^{t-1} levels. Then

$$d_{2j-1}^i = 2 \left\lfloor \frac{c_{2j-1}^i + s^t}{2s} \right\rfloor - s^{t-1} + 1 = \sum_{r=2}^t b_{(j-1)t+r}^i s^{r-2},$$

$$d_{2j}^i = 2 \left\lfloor \frac{c_{2j}^i + s^t}{2s} \right\rfloor - s^{t-1} + 1 = \sum_{r=1}^{t-1} b_{(j-1)t+r}^i s^{t-r-1}.$$

According to the proof of Theorem 1, D^i is an $SOA(n/s, m', s^{t-1}, t - 1)$. Then C is a sliced $SOA(n, m', s^t, t)$ with s slices, and each slice is an $SOA(n/s, m', s^{t-1}, t - 1)$ when collapsed into s^{t-1} levels.

Conclusions on the orthogonality of C and its slices can be obtained through Lemma 1.

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LPMC and Institute of Statistics, Nankai University, Tianjin 300071, China.

E-mail: rain6397@163.com

LPMC and Institute of Statistics, Nankai University, Tianjin 300071, China.

E-mail: mqliu@nankai.edu.cn

(Received March 2014; accepted November 2014)