Taylor & Francis
Taylor & Francis Group

Communications in Statistics - Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: http://www.tandfonline.com/loi/lsta20

Construction of regular 2^n4^1 designs with general minimum lower-order confounding

Tian-Fang Zhang, Jian-Feng Yang, Zhi-Ming Li & Run-Chu Zhang

To cite this article: Tian-Fang Zhang, Jian-Feng Yang, Zhi-Ming Li & Run-Chu Zhang (2017) Construction of regular 2^n4^1 designs with general minimum lower-order confounding, Communications in Statistics - Theory and Methods, 46:6, 2724-2735, DOI: 10.1080/03610926.2015.1048887

To link to this article: http://dx.doi.org/10.1080/03610926.2015.1048887

Accepted author version posted online: 08 Apr 2016.
Published online: 08 Apr 2016.

COMMUNICATIONS IN STATISTICS—THEORY AND METHODS
2017, VOL. 46, NO. 6, 2724–2735
http://dx.doi.org/10.1080/03610926.2015.1048887

Construction of regular 2^n4^1 designs with general minimum lower-order confounding

Tian-fang Zhang[a], Jian-Feng Yang[b], Zhi-ming Li[c], and Run-chu Zhang[d,b,e]

[a]College of Mathematical and Informational Sciences, Jiangxi Normal University, Nanchang, China; [b]LPMC and Institute of Statistics, Nankai University, Tianjin, China; [c]School of Mathematical Sciences, Xinjiang University, Urumqi, China; [d]KLAS and School of Mathematics and Statistics, Northeast Normal University, Changchun, China; [e]Department of Statistics, University of British Columbia, BC, Canada

ABSTRACT

Mixed-level designs, especially two- and four-level designs, are very useful in practice. In the last two decades, there are quite a few literatures investigating the selection of this kind of optimal designs. Recently, the general minimum lower-order confounding (GMC) criterion (Zhang et al., 2008) gave a new approach for choosing optimal factorials. It is proved that the GMC designs are more powerful than other criteria in the widely practical situations. In this paper, we extend the GMC theory to the mixed-level designs. Under the theory we establish a new criterion for choosing optimal regular two- and four-level designs. Further, a construction method is proposed to obtain all the 2^n4^1 GMC designs with $N/4 + 1 \leq n + 2 \leq 5N/16$, where N is the number of runs and n is the number of two-level factors.

ARTICLE HISTORY
Received 18 November 2014
Accepted 1 May 2015

KEYWORDS
Aliased effect-number pattern; effect hierarchy principle; factorial design; general minimum lower-order confounding (GMC); mixed-level.

MATHEMATICS SUBJECT CLASSIFICATION
62K15

1. Introduction

Mixed-level designs have been widely used in practice. The important case of mixed two- and four-level designs is firstly discussed for their practical use. The design with n two-level factors and m four-level factors is said to be a 2^n4^m design. A 2^n4^m design can be easily constructed from the corresponding symmetrical orthogonal arrays through the method of replacement (Addelman, 1962). The replacement rule is that any three two-level factors of the form (a_1, a_2, a_3) can be replaced by a four-level factor without affecting orthogonality, where a_3 is the interaction between a_1 and a_2. We call a_1, a_2 and a_3 the three components of the four-level factor. Wu (1989) improved Addelman's construction method by introducing the method of grouping. Wu et al. (1992) further applied the grouping method to general designs. By their method, a large class of asymmetric designs can be constructed.

For a 2^n4^m design D constructed by the above method, there are two types of words in the defining contrast group. The first type involves only the two-level factors, which is called type 0. The second type involves at least one of the four-level components and some of the two-level factors, which is called type 1. Let $A_{i0}(D)$ and $A_{i1}(D)$ be the number of words with length i of type 0 and type 1 of design D, respectively. Under the assumption that the component of four-level factor is not as important as two-level factors, Wu and Zhang (1993) defined the

CONTACT Jian-Feng Yang jfyang@nankai.edu.cn LPMC and Institute of Statistics, Nankai University, Tianjin 300071, China.

word length-pattern (WLP) of D as follows:

$$W(D) = \{(A_{i0}(D), A_{i1}(D))\}_{i\geq 1}. \quad (1)$$

The minimum aberration (MA) criterion is to sequentially minimize the terms of the WLP.

There are some developments of the mixed-level optimal designs in recent years. Zhang and Shao (2001) extended the results of Wu and Zhang (1993) to general cases. Mukerjee and Wu (2001) used the projective geometry theory and complementary theory to discuss MA mixed-level designs. Ai and Zhang (2004) later established the general rules to identify MA mixed-level factorial designs by the coding theory. Li et al. (2007) studied 2^m4^1 designs with MA or weak MA. Zhao and Zhang (2008) considered 2^m4^n designs with resolution III or IV containing clear two-factor interaction (2fi) components. Joseph et al. (2009) used the Bayesian method to measure the mixed-level designs. Note that all of them are based on WLP. However, two designs with the same WLP cannot be distinguished by any of the above criteria.

Example 1. Consider two $2^{13}4^1$ designs D_1 and D_2 with 128 runs:

$$D_1 : I = 126 = 137 = 238 = 12349 = 1235t_0 = 45t_1 = 12345t_2 = 12a_1t_3,$$
$$D_2 : I = 126 = 137 = 248 = 349 = 125t_0 = 135t_1 = 145t_2 = 12a_1t_3,$$

where t_0, t_1, t_2, t_3, respectively, denote the factors 10, 11, 12, 13, and a_1 is the component of the four-level factor A $(= (a_1, a_2, a_3))$. Both designs have the same WLP

$$W = ((0, 0), (0, 0), (8, 1), (15, 2), (24, 5), (32, 16), (24, 26), (15, 28), (8, 26), (0, 16), (0, 5), (1, 2), (0, 1)),$$

so they cannot be distinguished by any criterion based on WLP.

In order to solve the above problem, Zhang et al. (2008) first introduced the general minimum lower-order confounding (GMC) criterion to choose two-level regular optimal designs, based on aliased effect-number pattern (AENP). It was proved that, the GMC theory can manage nearly all the existing criteria through functions of the AENP, and under having priori information, the optimal designs selected by the GMC criterion are better than the ones selected by other existing criteria. These results can be extended to the mixed-level cases.

In this paper, we extend GMC theory to the mixed-level cases. In Section 2, a new criterion for 2^n4^1 design is given. In Section 3, construction methods are proposed and all 2^n4^1 GMC designs with $N/4 + 1 \leq n + 2 \leq 5N/16$ are obtained, where N is the number of runs. The conclusion and discussions are given in Section 4. Some proofs and tables are deferred to the Appendix.

2. A new criterion for 2^n4^m design

For a 2^{n-p} regular design, the degree of an ith-order effect being aliased with the jth-order effects is k if the ith-order effect is aliased with k jth-order effects simultaneously. Zhang et al. (2008) defined a 2^{n-p} regular design as a GMC design if it sequentially maximizes

$$^{\#}C = \left({}_1^{\#}C_2, {}_2^{\#}C_2, {}_1^{\#}C_3, {}_2^{\#}C_3, {}_3^{\#}C_2, {}_3^{\#}C_3, \ldots\right),$$

where ${}_i^{\#}C_j = ({}_i^{\#}C_j^{(0)}, {}_i^{\#}C_j^{(1)}, \ldots, {}_i^{\#}C_j^{(K_j)})$, $K_j = \binom{n}{j}$ and ${}_i^{\#}C_j^{(k)}$ denotes the number of ith-order effects aliased with jth-order effects at degree k. However, we cannot directly apply GMC criterion of two-level case to mixed-level designs. For a 2^n4^m design, if the ith-order effects contain i_0 components of four-level factors for $i_0 \leq \min\{i, m\}$, then we add i_0 besides i. Denote by ${}^{\#}_{i,i_0}C^{(k)}_{j,j_0}$ the number of ith-order effects aliased with jth-order effects at degree k, where the ith-order effects contain i_0 components of four-level factors and the jth-order effects contain j_0 components of four-level factors for $i_0 \leq \min\{i, m\}$ and $j_0 \leq \min\{j, m\}$. For every pair of $\{(i, i_0), (j, j_0)\}$, the vector

$$ {}^{\#}_{i,i_0}C_{j,j_0} = \left({}^{\#}_{i,i_0}C^{(0)}_{j,j_0}, {}^{\#}_{i,i_0}C^{(1)}_{j,j_0}, \ldots, {}^{\#}_{i,i_0}C^{(K_j)}_{j,j_0} \right), \tag{2} $$

reflects the confounding information among mixed-level designs and the larger the degree k is, the more severely the effect is aliased. We still use ${}^{\#}_{i}C_j$ to denote the set

$$ \left\{ {}^{\#}_{i,i_0}C_{j,j_0}, i_0 = 0, 1, \ldots, \min\{i, m\}, j_0 = 0, 1, \ldots, \min\{j, m\} \right\}. $$

Based on the assumptions (i) lower-order interactions are more likely to be important than higher-order interactions; (ii) interactions of the same order and of the same type are equally likely to be important; and (iii) the components of four-level factors are less important than two-level factors, the elements of ${}^{\#}_{i}C_j$ will be ordered according to Rule 2.1 for the fixed (i, j), and ${}^{\#}_{i}C_j$, as elements of ${}^{\#}C$, will be ordered according to Rule 2.2.

Rule 2.1: *The term* ${}^{\#}_{i,i_1}C_{j,j_1}$ *is put ahead of* ${}^{\#}_{i,i_2}C_{j,j_2}$ *if it satisfies one of the following three conditions:*(i)$\max(i_1, j_1) < \max(i_2, j_2)$,(ii)$\max(i_1, j_1) = \max(i_2, j_2)$ *and* $i_1 < i_2$, *and*(iii)$\max(i_1, j_1) = \max(i_2, j_2)$, $i_1 = i_2$ *and* $j_1 < j_2$. We obtain the new ranked vector as follows:

$$ {}^{\#}_{i}C_j = \left({}^{\#}_{i,0}C_{j,0}, {}^{\#}_{i,0}C_{j,1}, {}^{\#}_{i,1}C_{j,0}, \ldots \right). \tag{3} $$

For example, for $i = 1$ and $j = 2$, we have ${}^{\#}_{1}C_2 = ({}^{\#}_{1,0}C_{2,0}, {}^{\#}_{1,0}C_{2,1}, {}^{\#}_{1,1}C_{2,0}, {}^{\#}_{1,1}C_{2,1}, {}^{\#}_{1,0}C_{2,2}, {}^{\#}_{1,1}C_{2,2})$.

Rule 2.2: *The term* ${}^{\#}_{i}C_j$ *is put ahead of* ${}^{\#}_{s}C_t$ *if it satisfies one of the following three conditions:*(i)$\max(i, j) < \max(s, t)$,(ii)$\max(i, j) = \max(s, t)$ *and* $i < s$,(iii)$\max(i, j) = \max(s, t)$, $i = s$ *and* $j < t$.

Through Rule 2.2, we obtain the ordering of ${}^{\#}_{i}C_j$'s:

$$ {}^{\#}C = \left({}^{\#}_{1}C_1, {}^{\#}_{0}C_2, {}^{\#}_{1}C_2, {}^{\#}_{2}C_1, {}^{\#}_{2}C_2, {}^{\#}_{0}C_3, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, {}^{\#}_{3}C_1, {}^{\#}_{3}C_2, {}^{\#}_{3}C_3, \ldots \right). \tag{4} $$

Similar to Zhang and Mukerjee (2009), by deleting some terms that can be determined by their previous terms, (4) is simplified as

$$ {}^{\#}C = \left({}^{\#}_{1}C_2, {}^{\#}_{2}C_2, {}^{\#}_{1}C_3, {}^{\#}_{2}C_3, {}^{\#}_{3}C_1, {}^{\#}_{3}C_2, {}^{\#}_{3}C_3, \ldots \right) \tag{5} $$

for designs with resolution at least III. Combining (3) and (5), we obtain the ordering of ${}^{\#}_{i,i_0}C_{j,j_0}$'s,

$$ {}^{\#}C = \left({}^{\#}_{1,0}C_{2,0}, {}^{\#}_{1,0}C_{2,1}, {}^{\#}_{1,1}C_{2,0}, \ldots, {}^{\#}_{2,0}C_{2,0}, {}^{\#}_{2,0}C_{2,1}, {}^{\#}_{2,1}C_{2,0}, {}^{\#}_{2,1}C_{2,1}, \ldots \right), $$

which is called the AENP of a mixed two- and four-level design. Then the GMC criterion for such designs can be defined as follows.

Definition 1. Let ${}^{\#}C_l$ be the lth component of ${}^{\#}C$. Let ${}^{\#}C(D_1)$ and ${}^{\#}C(D_2)$ be the AENPs of mixed two- and four-level designs D_1 and D_2, respectively. Suppose that ${}^{\#}C_t$ is the first component such that ${}^{\#}C_t(D_1)$ and ${}^{\#}C_t(D_2)$ are different. If ${}^{\#}C_t(D_1) > {}^{\#}C_t(D_2)$, then D_1 is said to have less general lower-order confounding (GLOC) than D_2. A design D is said to have general minimum lower-order confounding if no other design has less GLOC than D and such a design is called a GMC mixed two- and four-level design.

To explain the above definition, we give an example.

Example 2. Consider two $2^4 4^1$ designs D_1 and D_2 with 16 runs,

$$D_1 : A, 3, 4, 134, 23, \quad D_2 : A, 3, 4, 13, 34,$$

where $A = (1, 2, 12)$. For the two designs, the first different components of ${}^{\#}C$ are

$$ {}^{\#}_{1,0}C^{(0)}_{2,0}(D_1) = 4 \text{ and } {}^{\#}_{1,0}C^{(0)}_{2,0}(D_2) = 1.$$

So D_1 has less *GLOC* than D_2.

The following theorem is obtained directly by the definition of GMC.

Theorem 1. *For a $2^n 4^1$ design with the resolution $R \geq III$, the WLP in (1) is a function of $\{{}^{\#}_{i,i_0}C^{(k)}_{j,j_0} : i, j = 0, \ldots, n, i_0, j_0 = 0, 1, k = 1, \ldots, K_j\}$ in the following form:*

$$A_{i,0} = {}^{\#}_{i,0}C^{(1)}_{0,0} \text{ and } A_{i,1} = {}^{\#}_{i,1}C^{(1)}_{0,0}.$$

From Theorem 1, the designs with different WLPs must have different AENPs. However, designs with the same WLPs may have different AENPs. The next example shows this point.

Example 3. (Example 1 continued) The $2^{13}4^1$ designs D_1 and D_2 have the same WLP, but different AENPs. The first different items are ${}^{\#}_{2,0}C^{(0)}_{2,0}(D_1) = 60$ and ${}^{\#}_{2,0}C^{(0)}_{2,0}(D_2) = 54$. Under the GMC criterion for mixed two- and four-level designs, it is obvious that D_1 has less *GLOC* than D_2.

The optimal designs under the MA and GMC criteria are often consistent especially for designs with small runs. However, there are a significant number of cases where the two criteria yield different optimal designs. Here is another example.

Example 4. Consider two $2^{15}4^1$ designs with 64 runs D_1 and D_2,

$$\begin{aligned} D_1 : I &= 12347 = 12358 = 13459 = 245t_0 = 1236t_1 = 1346t_2 = 246t_3 \\ &= 1356t_4 = 256t_5 = 456t_6 = 123456t_7, \\ D_2 : I &= 12347 = 2358 = 13459 = 245t_0 = 1236t_1 = 1346t_2 = 246t_3 \\ &= 1356t_4 = 256t_5 = 456t_6 = 123456t_7, \end{aligned}$$

where $A = (1, 2, 12)$ and $t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7$, respectively, denote the factors 10, 11, 12, 13, 14, 15, 16, 17. The WLPs $((A_{30}, A_{31}), (A_{40}, A_{41}), (A_{50}, A_{51}), \ldots)$ of D_1 and D_2 are, respectively, $((0, 0), (77, 35), (0, 28), \ldots)$ and $((0, 0), (61, 35), (0, 44), \ldots)$. According to the MA criterion, design D_2 is better than D_1. However, their first different items of ${}^{\#}C$ are ${}^{\#}_{2,0}C^{(0)}_{2,0}(D_1) = 14$ and ${}^{\#}_{2,0}C^{(0)}_{2,0}(D_2) = 0$. By GMC criterion, design D_1 is better than D_2. Both designs have all clear main effects for two-level factors and the components of four-level factor. Further, design D_1 has 14 clear 2fi's but D_2 has 0 clear 2fi's. In this case, the design D_2 is not a good choice.

3. Construction of GMC $2^n 4^1$ designs

This section will mainly discuss the construction of GMC $2^n 4^m$ designs with $m = 1$, i.e., GMC $2^n 4^1$ designs. Let H_r be the set containing all main effects $1, \ldots, r$ and all possible interactions with *Yates* order, that is,

$$H_1 = \{1\} \text{ and } H_r = \{H_{r-1}, r, rH_{r-1}\} \text{ for } r = 2, \ldots, q.$$

Denote $S_{qr} = H_q \backslash H_r$, $F_{qr} = \{q, qH_{r-1}\}$ and $T_r = \{r, rH_{r-1}\}$, where $qH_{r-1} = \{qd : d \in H_{r-1}\}$ and $1H_{1-1} = \{1\}$. Obviously, the designs F_{qr} and T_r with $r \geq 3$ are the saturated resolution IV designs with r independent factors, which are unique up to isomorphism.

For a given design $D \subset H_q$ and $\gamma \in H_q$, define

$$B_2(D, \gamma) = \#\{(d_1, d_2) : d_1, d_2 \in D,\ d_1 d_2 = \gamma\}$$

to be the number of 2fi's in D aliased with γ. Let $D \subseteq F_{qq}$ be a 2^{n-p} design with N runs. From Li et al. (2011), we have

$$B_2(D, \gamma) = \begin{cases} 0, & \gamma \in F_{qq}, \\ B_2(F_{qq} \backslash D, \gamma) + n - N/4, & \gamma \in H_{q-1}. \end{cases} \tag{6}$$

Lemma 1. *Suppose D is a 2^{n-p} design with N runs. If $qS_{q-1,r} \subseteq D \subset qS_{q-1,r-1}$ $(r < q)$, then*

$$B_2(D, \gamma) = \begin{cases} n - N/4, & \gamma \in S_{q-1,r}, \\ B_2(D \backslash qS_{q-1,r}, \gamma) + N/4 - 2^{r-1}, & \gamma \in H_r. \end{cases}$$

Let $D_0 = \{D_1, a_1, a_2\}$ for convenience. A $2^n 4^1$ design $D = (D_1, A)$ with $A = (a_1, a_2, a_1 a_2)$ being the four-level factor can be generated from D_0 by grouping and combining method if $a_1 a_2$ is not in D_1. Hence, it is easy to see that, to construct a GMC $2^n 4^1$ design D, we have to first consider the regular two-level design D_0 and then select two different factors a_1 and a_2 with $a_1 a_2$ not in D_1 to form the four-level factor. The problem here is how to select D_0, as well as a_1 and a_2 from D_0, such that $D = (D_1, A)$ has GMC. In the following, we will discuss the construction of GMC $2^n 4^1$ designs for $9N/32 + 1 \leq n + 2 \leq 5N/16$ and $N/4 + 1 \leq n + 2 \leq 9N/32$, respectively.

3.1. GMC $2^n 4^1$ designs with $9N/32 + 1 \leq n + 2 \leq 5N/16$

A 2^{n-p} design is called second-order saturated (SOS) if all degrees of freedom can be used to estimated main effects and 2fi's (Block and Mee, 2003). For the case $9N/32 + 1 \leq n + 2 \leq 5N/16$, let $\mathbf{X} = (X_1, X_2, X_3, X_4, X_5)$ be the 2^{5-1} design with $I = X_1 X_2 X_3 X_4 X_5$. Then $D^{q-4}(\mathbf{X})$ is the unique SOS design of $5N/16$ (Chen and Cheng, 2006), and

$$D^{q-4}(\mathbf{X}) = (D^{q-4}(X_1), D^{q-4}(X_2), D^{q-4}(X_3), D^{q-4}(X_4), D^{q-4}(X_5)), \tag{7}$$

where $D^{q-4}(X_i) = (1, 2, 12, \ldots, 12\ldots(q-4)) \otimes X_i$ for $i = 1, 2, 3, 4, 5$. The columns in (7) are said to have *RC (rechanged) Yates order.*

Block and Mee (2003) proved that every non SOS design is a projection of at least one SOS design. For $9N/32 \leq n \leq 5N/16$, a 2^{n-p} design with resolution IV must be an n-projection of $D^{q-4}(\mathbf{X})$ (Zhang and Cheng, 2010). Although the AENP of $2^n 4^1$ is different from that of 2^{n-p}, we still can easily obtain the result that D_0 is an $(n+2)$-projection of $D^{q-4}(\mathbf{X})$. Since the $(n+2)$-projection of $D^{q-4}(\mathbf{X})$ is not unique, we have to find the projection design D_0 to obtain the GMC $2^n 4^1$ design.

Theorem 2. *For $9N/32 + 1 \leq n + 2 \leq 5N/16$, if D is a GMC design then D_0 must be an $(n+2)$-projection of $D^{q-4}(\mathbf{X})$ and $\bar{D}_0 = D^{q-4}(\mathbf{X}) \backslash D_0 \subset D^{q-4}(X_p)$ for some p, $a_1 \in D^{q-4}(X_p) \cap D_0$ and $a_2 \in D^{q-4}(X_i)$, $i \neq p$. Furthermore, we have*

(1) ${}^{\#}_{1,0}C_{2,0}(D) = (n)$, ${}^{\#}_{1,0}C_{2,1}(D) = (N/2 - n - 2, 2n - N/2 + 2)$, ${}^{\#}_{1,1}C_{2,0}(D) = (2, 0^{n-N/4}, 1)$,
where $0^{n-N/4}$ denotes $n - N/4$ zeros,

$$(2)\ {}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \begin{cases} (n-N/4+1)(n-N/4), & k=n-N/4-1, \\ (N/2-n-1)(n-N/4+1), & k=n-N/4, \\ 3N/16(N/16-1), & k=N/16-2, \\ 3(N/16)^2, & k=N/16-1, \\ (k+1)\#\{\gamma:\gamma\in D^{q-4}(I_{16})\backslash I_N,\ n^{\gamma}=\upsilon\}, & k=N/8+\upsilon-2, \\ & \upsilon=0,1\ldots,\lfloor g/2\rfloor, \\ 0, & \text{other } k\text{'s}, \end{cases}$$

where n^{γ} is the number of 2fi's aliased with some $\gamma \in D^{q-4}(I_{16})\backslash I_N$ of $D^{q-4}(X_p)\cap\{D_0\backslash a_1\}$, and $g=\#\{D^{q-4}(X_p)\cap\{D_0\backslash a_1\}\}$ and

$$(3)\ {}^{\#}_{2,0}C^{(k)}_{2,1}(D) = \begin{cases} n-N/4+1+{}^{\#}_{2,0}C^{(N/16-1)}_{2,0}(D), & k=0, \\ n(n+1)/2-{}^{\#}_{2,0}C^{(0)}_{2,1}(D)-{}^{\#}_{2,0}C^{(2)}_{2,1}(D), & k=1, \\ {}^{\#}_{2,0}C^{(n-N/4-1)}_{2,0}(D)+{}^{\#}_{2,0}C^{(1)}_{2,1}(G)+m(N/8-1), & k=2, \\ 0, & \text{other } k\text{'s}, \end{cases}$$

where $G=D^{q-t}(X_p)\cap D_0$ and $m=\#\{t_1: B_2(G,a_1t_1)>0,\ t_1\in G\backslash a_1\}$.

Theorem 2 shows how to select D_0 and the possible cases of a_1 and a_2. The following theorem shows the exact choices of D_0 and a_2.

Theorem 3. *Up to isomorphism, a 2^n4^1 design D with $9N/32+1\le n+2\le 5N/16$ is a GMC design, if and only if $\{D_1\cup a_2\}$ consists of the last $n+1$ columns of $D^{q-4}(\mathbf{X})$, $a_1\in D^{q-4}(X_1)\backslash D_1$ and a_2 is the first column of $D^{q-4}(X_2)$, where $D=(D_1,A)$.*

To find a GMC 2^n4^1 design, we should determine the choice of a_1. From the above theorem, we know a_1 is one of the first $N/16-g+1$ columns of $D^{q-4}(X_1)$. Since ${}^{\#}_{2,0}C_{2,0}(D)$ is not enough to fix the best choice of a_1, we need discuss ${}^{\#}_{2,0}C_{2,1}(D)$. The term ${}^{\#}_{2,0}C^{(0)}_{2,1}(D)$ is determined by ${}^{\#}_{2,0}C_{2,0}(D)$, so ${}^{\#}_{2,0}C^{(1)}_{2,1}(D)$ should be maximized firstly. By Theorem 2, this equals to minimize $C_0={}^{\#}_{2,0}C^{(1)}_{2,1}(G)+m(N/8-1)$. Since ${}^{\#}_{2,0}C^{(1)}_{2,1}(G)=\sum_{\gamma\in G\backslash a_1}B_2(G,a_1\gamma)$, minimizing C_0 needs to consider $B_2(G,a_1\gamma)$ firstly for $\gamma\in G\backslash a_1$.

Suppose $qS_{q-1,r}\subseteq S\subset qS_{q-1,r-1}$ $(r<q)$, if the number of factors in $S\backslash S_{q-1,r}$ is small, it is convenient to construct GMC designs. The next lemma studies the connection between $B_2(S,\gamma)$ and $B_2(S\backslash qS_{q-1,r},\gamma)$.

Lemma 2. *Suppose S is a 2^{n-p} design which consists of the last n columns of F_{qr} (for $n<2^r$) and a_1 is a four-level component. We have*

(1) if $n\le 2^{r-1}$, then $S\subseteq qS_{r,r-1}$ and $B_2(S,a_1\gamma)=0$ for any $a_1\in F_{q,r-1}$, $\gamma\in S$; and

(2) if $n>2^{r-1}$, then $qS_{r,r'}\subseteq S\subset qS_{r,r'-1}$ $(r'<r)$. Moreover, for any $a_1\in F_{qr'}$, it can be obtained that

$$B_2(S,a_1\gamma) = \begin{cases} n-2^{r-1}, & \gamma\in qS_{r,r'}, \\ B_2(S\backslash qS_{r-1,r'},a_1\gamma)+2^{r-1}-2^{r'-1}, & \gamma\in qH_{r'}\cap D. \end{cases} \quad (8)$$

By Lemma 2, C_0 can be minimized. The result is shown in the following lemma.

Lemma 3. *Suppose $G\backslash a_1$ consists of the last $g-1$ columns of $D^{q-t}(X_1)$ for $g>N/2^{t+1}$, where a_1 is a four-level component and $g=\#\{G\}$. If a_1 is the first column of $D^{q-t}(X_1)$, then C_0 is minimized.*

According to the above lemmas, the best choice of a_1 is shown below by setting $t=4$.

Theorem 4. *Up to isomorphism, a 2^n4^1 design D with $9N/32+1 \le n+2 \le 5N/16$ is a GMC design, if and only if $\{D_1, a_2\}$ consists of the last $n+1$ columns of $D^{q-4}(X)$, a_1 is the first column of $D^{q-4}(X_1)$ and a_2 is the first column of $D^{q-4}(X_2)$, where $D = (D_1, A)$.*

Example 5. Consider a 2^84^1 design D with 32 runs. From Theorem 4, let $\mathbf{X}$ be a 2^{5-1} design with 16 runs and $I = X_1X_2X_3X_4X_5$. Since $D^{q-4}(\mathbf{X}) = (D^{q-4}(X_1), D^{q-4}(X_2), D^{q-4}(X_3), D^{q-4}(X_4), D^{q-4}(X_5))$ and $q = 5$, we can obtain $D^1(\mathbf{X}) = ((IX_1, 1X_1), (IX_2, 1X_2), (IX_3, 1X_3), (IX_4, 1X_4), (IX_5, 1X_5))$, where IX_i and $1X_i$, respectively, mean $I_2 \otimes X_i$ and $1_2 \otimes X_i$ for $i = 1, \dots, 5$. Let a_1 be the first column of $D^{q-4}(X_1)$ and a_2 be the first column of $D^{q-4}(X_2)$, that is to say $a_1 = IX_1$ and $a_2 = IX_2$. The rest of $D^{q-4}(\mathbf{X})$ are D_1. Then (D_1, A) is a GMC 2^84^1 design.

3.2. GMC 2^n4^1 designs with $N/4 + 1 \le n + 2 \le 9N/32$

Let

$$\mathbf{X} = \begin{pmatrix} \mathbf{1} & S_1(t) & S_2(t) \\ -\mathbf{1} & -S_1(t) & S_2(t) \end{pmatrix}, \tag{9}$$

where $\mathbf{1}$ is a $2^{t-1} \times 1$ vector of 1's, $S(t) = (S_1(t), S_2(t))$ is the resolution IV design with 2^{t-1} runs and 2^{t-2} factors and $S_1(t)$ is any column of $S(t)$. Rewrite (9) as $X = (X_1, X_2, \dots, X_{2^{t-2}+1})$. Doubling $\mathbf{X}$ $q-t$ times, we can obtain

$$D^{(q-t)}(X) = (D^{(q-t)}(X_1), D^{(q-t)}(X_2), \dots, D^{(q-t)}(X_{2^{t-2}+1})).$$

Denote $D^{(q)}(1) = (D^{(q-t)}(D^t(1)\backslash\mathbf{X}), D^{(q-t)}(\mathbf{X}))$, which is said to have *RC Yates order.* Suppose D_0 is a 2^{n+2}_{IV} design with $(2^{t-1}+1)N/2^{t+1} < n+2 \le (2^{t-2}+1)N/2^t$ $(5 \le t \le q)$. According to a discussion similar to the case of $9N/32+1 \le n+2 \le 5N/16$, D_0 must be an $(n+2)$-projection of $D^{q-t}(\mathbf{X})$. So the following result is obtained.

Theorem 5. *Suppose D_0 is an $(n+2)$-projection of $D^{q-t}(\mathbf{X})$ for $(2^{t-1}+1)N/2^{t+1} < n+2 \le (2^{t-2}+1)N/2^t$ $(5 \le t < q)$ and $\bar{D}_0 = D^{q-t}(\mathbf{X})\backslash D_0$. Up to isomorphism, if D has less GLOC than any other cases in all the $(n+2)$-projection of $D^{q-t}(\mathbf{X})$, then $\bar{D}_0 \subset D^{q-t}(X_p)$ for $p = 1, 2$, $a_1 \in D^{q-t}(X_p) \cap D_0$ and $a_2 \in D^{q-t}(X_i)$, $i = 1, 2$, $i \ne p$. ${}^{\#}_{1}C_2(D)$ is maximized, and*

(1) ${}^{\#}_{1,0}C_{2,0}(D) = (n)$, ${}^{\#}_{1,0}C_{2,1}(D) = (N/2 - n - 2, 2n - N/2 + 2)$, ${}^{\#}_{1,1}C_{2,0}(D) = (2, 0^{n-N/4}, 1)$, where $0^{n-N/4}$ denotes $n - N/4$ zeros.

(2)
$$ {}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \begin{cases} (g-1)(n-N/4), & k = n - N/4 - 1, \\ (N/4 - g + 1)(n - N/4 + 1), & k = n - N/4, \\ (N/4 - N/2^t)(N/2^t - 1), & k = N/2^t - 2, \\ (N/4 - N/2^t)(N/8 - N/2^t), & k = N/8 - N/2^t - 1, \\ (k+1)\#\{\gamma : \gamma \in D^{q-t}(I_{2^t})\backslash I_N, n^\gamma = \upsilon\}, & k = N/8 + \upsilon - 2 \\ & \upsilon = 0, 1 \dots, \lfloor g/2 \rfloor, \\ 0, & \text{other } \; k\text{'s}, \end{cases}$$

where n^γ is the number of 2fi's aliased with some $\gamma \in D^{q-t}(I_{2^t})\backslash I_N$ of $G\backslash a_1$, for $G = D^{q-t}(X_p) \cap D_0$ and $g = n + 2 - N/4$; and

$$\text{(3)}\quad {}^{\#}_{2,0}C^{(k)}_{2,1}(D) = \begin{cases} {}^{\#}_{2,0}C^{(N/8-N/2^t-1)}_{2,0}(D) + n - N/4 + 1, & k = 0, \\ n(n+1)/2 - {}^{\#}_{2,0}C^{(0)}_{2,1}(D) - {}^{\#}_{2,0}C^{(2)}_{2,1}(D), & k = 1, \\ {}^{\#}_{2,0}C^{(n-N/4-1)}_{2,0}(D) + {}^{\#}_{2,0}C^{(1)}_{2,1}(G) + (N/8-1)m, & k = 2, \\ 0, & other \quad k's, \end{cases}$$

where $m = \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in G\backslash a_1\}$.

Theorem 5 shows the choices of D_0, a_1 and a_2. Similarly to Theorem 3, a 2^n4^1 design with $(2^{t-1}+1)N/2^{t+1} < n+2 \le (2^{t-2}+1)N/2^t (5 \le n < q)$ is a GMC design, only if $\{D_1, a_2\}$ consists of the last $n+1$ columns of $D^{q-t}(\mathbf{X})$, $a_1 \in D^{q-t}(X_1)\backslash D_1$ and a_2 is the first column of $D^{q-t}(X_2)$ up to isomorphism, where $D = (D_1, A)$.

To determine the exact position of a_1, we should maximize ${}^{\#}_{2,0}C_{2,1}(D)$. By Theorem 5(3), maximizing ${}^{\#}_{2,0}C_{2,1}(D)$ equals to minimizing ${}^{\#}_{2,0}C^{(1)}_{2,1}(G) + (N/8-1)m$, denoted as C_0 in short. Noting that

$${}^{\#}_{2,0}C^{(1)}_{2,1}(G) = \sum_{\gamma \in G\backslash a_1} B_2(G, a_1\gamma) \text{ and } m = \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in G\backslash a_1\},$$

to maximize ${}^{\#}_{2,0}C_{2,1}(D)$ only needs to consider $B_2(G, a_1\gamma)$, for $\gamma \in G\backslash a_1$.

By Lemma 3, we know that if a_1 is the first column of $D^{q-t}(X_1)$, C_0 is minimized for $5 \le t < q$. And for $t = q$, by Theorem 2, $D_0 = \mathbf{X}$, $a_1 = X_1$ and $a_2 = X_j$ $(j \ne 1, 2)$ forms a GMC design. Then the following result is obvious.

Theorem 6. *Up to isomorphism, a* 2^n4^1 *design with* $(2^{t-1}+1)N/2^{t+1} < n+2 \le (2^{t-2}+1)N/2^t$ *is a GMC design if and only if*

(1) $\{D_1, a_2\}$ *consists of the last* $n+1$ *columns of* $D^{q-t}(\mathbf{X})$, a_1 *is the first column of* $D^{q-t}(X_1)$ *and* a_2 *is the first column of* $D^{q-t}(X_2)$, *where* $D = (D_1, A)$ *and* $5 \le t < q$*; and*

(2) $a_1 = X_1$, $a_2 = X_j$ $(j \ne 1, 2)$ *and* $D_1 = \mathbf{X}\backslash\{a_1, a_2\}$ *for* $t = q$.

Example 6. Consider to construct a $2^{16}4^1$ design D with 64 runs. Note that $t = 5$ and $q = 6$. Let $\mathbf{X} = \{X_1, X_2, \ldots, X_9\}$ be the *SOS* design with 9 factors and 32 runs. Since $D^{q-5}(\mathbf{X}) = (D^{q-5}(X_1), D^{q-5}(X_2), \ldots, D^{q-5}(X_9))$ and $q = 6$, we can obtain $D^1(\mathbf{X}) = ((IX_1, 1X_1), (IX_2, 1X_2), \ldots, (IX_9, 1X_9))$, where IX_i and $1X_i$, respectively, mean $I_2 \otimes X_i$ and $1_2 \otimes X_i$ for $i = 1, \ldots, 9$. According to Theorem 6, let a_1 be the first column of $D^{q-5}(X_1)$, a_2 be the first column of $D^{q-5}(X_2)$ (i.e., $a_1 = IX_1$ and $a_2 = IX_2$) and D_1 contain the rest columns of $D^{q-5}(\mathbf{X})$. Then (D_1, A) is a GMC $2^{16}4^1$ design.

4. Conclusions and discussions

In this paper, we extend the GMC theory to mixed-level case. Under this theory, a new criterion is established for choosing optimal regular two- and four-level designs. Then, a construction method is proposed to obtain all the 2^n4^1 GMC designs for $N/4+1 \le n+2 \le 5N/16$. Some GMC 2^n4^1 designs with 16 and 32 runs are listed in the tables of Appendix B. This method can be extended to the construction of 2^n4^2, and used for $s^n(s^2)$ or $s^n(s^2)^2$ designs for general s. This is an open problem for further study.

Acknowledgments

We are grateful to the editor and two referees for their insightful comments and constructive suggestions. This work was supported by the doctorial start-up fund research of Jiangxi Normal University, the

National Natural Science Foundation of China (Grant Nos. 11601201, 11471172, 11661076, 11171165, 11271205, 11271355), Scientific Research Program of the Higher Education Institution of XinJiang (Grant No. XJEDU2012S01) and Research Fund for the Doctoral Program of XinJiang University (Grant No. BS130106).

References

Addelman, S. (1962). Orthogonal main-effect plans for asymmetrical factorial experiments. *Technometrics* 4:21–46.

Ai, M.Y., Zhang, R.C. (2004). Theory of minimum aberration blocked regular mixed factorial designs. *J. Stat. Plann. Inference* 126:305–323.

Block, R., Mee, R. (2003). Second order saturated resolution IV designs. *Journal of Statistical Theory and Applications* 2:96–112.

Chen, H., Cheng, C.S. (2006). Doubling and projection: a method of construction two-level designs of resolution IV. *Ann. Stat.* 34:546–558.

Cheng, Y., Zhang, R.C. (2010). On construction of general minimum lower order confounding 2^{n-m} designs with $N/4 + 1 \le n \le 9N/32$. *J. Stat. Plann. Inference* 140:2384–2394.

Joseph, R.V., Ai, M.Y., Wu, C.F.J. (2009). Bayesian-inspired minimum aberration two- and four-level designs. *Biometrika* 96:95–106.

Li, P.F., Liu, M.Q., Zhang, R.C. (2007). 2^m4^1 designs with minimum aberration or weak minimum aberration. *Stat. Pap.* 48:235–248.

Li, P.F., Zhao, S.L., Zhang, R.C. (2011). A theory on constructing 2^{m-m} designs with general minimum lower-order confounding. *Stat. Sin.* 21:1571–1589.

Mukerjee, R., Wu, C.F.J. (2001). Minimum aberration designs for mixed Factorials in terms of complementary sets. *Stat. Sin.* 11:225–239.

Wu, C.F.J. (1989). Construction of 2^m4^n via a group scheme. *Ann. Stat.* 17:1880–1885.

Wu, C.F.J., Zhang, R.C. (1993). Minimum aberration designs with two-level and four-level factors. *Biometrika* 80:203–209.

Wu, C.F.J., Zhang R.C., Wang, R.G. (1992). Construction of asymmetrical orthogonal arrays of the type $OA(s^k, (s^{r_1})^{n_1} \ldots (s^{r_l})^{n_l}$. *Stat. Sin.* 2:203–219.

Zhang, R.C., Cheng, Y. (2010). General minimum lower order confounding designs: An overview and a construction theory. *J. Stat. Plann. Inference* 140:1719–1730.

Zhang, R.C., Li, P.F., Zhao, S.L., Ai, M.Y. (2008). A general minimum lower order confounding criterion for two-level regular design. *Stat. Sin.* 18:1689–1705.

Zhang, R.C., Mukerjee, R. (2009). Characterization of general minimum lower order confounding via complementary sets. *Stat. Sin.* 19:363–375.

Zhang, R.C., Shao, Q. (2001). Minimum aberration $(s^2)s^{n-k}$ designs. *Stat. Sin.* 11:213–223.

Zhao, S.L., Zhang, R.C. (2008). 2^m4^n designs with resolution III or IV containing clear two-factor interaction components. *Stat. Pap.* 49:441–454.

Appendix A: Proofs

Proof of Theorem 3. By Theorem 2, $\bar{D}_0 \subset D^{q-4}(X_p)$ for some $p, a_1 \in D^{q-4}(X_p) \cap D_0$ and $a_2 \in D^{q-4}(X_i)$, $i \neq p$, ${}^{\#}_{1,0}C_{2,0}(D)$, ${}^{\#}_{1,0}C_{2,1}(D)$, ${}^{\#}_{1,1}C_{2,0}(D)$ are maximized. Without loss of generality, let $p = 1$ and $i = 2$. Now we maximize ${}^{\#}_{2,0}C_{2,0}(D)$. Based on Theorem 2, ${}^{\#}_{2,0}C^{(k)}_{2,0}(D)$ is maximized when $k \le N/16 - 1$. For $k = N/8 + \upsilon - 2$ $(\upsilon = 0, 1 \ldots, \lfloor g/2 \rfloor)$,

$$ {}^{\#}_{2,0}C^{(k)}_{2,0}(D) = (k+1)\#\{\gamma : \gamma \in D^{q-4}(I_{16} \backslash I_N), n^{\gamma} = \upsilon\}, $$

where n^{γ} is the number of 2fi's aliased with some $\gamma \in D^{q-4}(I_{16} \backslash I_N)$ of $D^{q-4}(X_1) \cap \{D_0 \backslash a_1\}$.

Since $G \backslash a_1 = D^{q-4}(X_1) \cap \{D_0 \backslash a_1\}$, then maximizing ${}^{\#}_{2,0}C^{(k)}_{2,0}(D)$ (for $k = N/8 + \upsilon - 2$) equals to maximizing $\#\{\gamma : \gamma \in D^{q-4}(I_{16}) \backslash I_N,\ n^{\gamma} = \upsilon\}$ of $\{G \backslash a_1\}$, i.e., maximizing ${}^{\#}_{2,0}C^{(\upsilon)}_{2,0}(G \backslash a_1)$, which is equal to ${}^{\#}_{2}C^{(\upsilon)}_{2}(G \backslash a_1)$.

$D^{q-4}(X_p)$ can be seen as some F_{qr} for $r \le q$, where $r = q - 3$ and $2^{r-2} < g \le 2^{r-1}$. By the computation, we have

$$\bar{g}(G\backslash a_1) = \{\gamma : \gamma \in H_q\backslash\{G\backslash a_1\}, B_2(G\backslash a_1, \gamma) > 0\} = N/16 - 1,$$

so maximizing ${}^{\#}_{2}C_2(G\backslash a_1)$ equals to maximizing $\{-\bar{g}(G\backslash a_1), {}^{\#}_{2}C_2(G\backslash a_1)\}$. By Li et al. (2011) $\{-\bar{g}(G\backslash a_1), {}^{\#}_{2}C_2(G\backslash a_1)\}$ is maximized if and only if $G\backslash a_1$ consists of the last $g - 1$ columns of $D^{q-4}(X_p)$. Thus ${}^{\#}_{2}C_2(G\backslash a_1)$ is maximized if and only if $G\backslash a_1$ consists of the last $g - 1$ columns of $D^{q-4}(X_1)$. Note that for any $a_2 \in D^{q-4}(X_i)$ $(i \ne 1)$, (D_1, A) is isomorphic. For convenience, let a_2 be the first column of $D^{q-4}(X_2)$. □

Proof of Lemma 2. S is a 2^{n-p} design which consists of the last n columns of F_{qr}. From the above notation, F_{qr} can be rewritten as $\{F_{q,r'-1}, qS_{r,r'-1}\}$ where $r' \le r$.

(a) If $n \le 2^{r-1}$, it is obvious that $S \subseteq qS_{r,r-1}$. For any $a_1 \in F_{q,r-1}$ and $\gamma \in S$, $a_1\gamma \in rH_{r-1}$. But for any two factors $\alpha, \beta \in S$, the interaction $\alpha\beta$ must be in H_{r-1}, that is to say $B_2(S, a_1\gamma) = 0$.

(b) If $n > 2^{r-1}$, there exists $r' < r$ such that $qS_{r,r'} \subseteq S \subset qS_{r,r'-1}$, then a_1 must be in $F_{qr'}$. There are two different cases for $\gamma \in S$. If $\gamma \in qS_{r,r'}$, we have $a_1\gamma \in S_{r,r'}$ and if $\gamma \in qH_{r'} \cap S$, we have $a_1\gamma \in H_{r'}$. From Lemma 1, by replacing $q - 1$ as r and r as r', we can obtain (8). □

Proof of Lemma 3. The proof uses searching method by the following three steps.

Step 1: Find the proper r'.

Since $g > N/2^{t+1}$, there exists $r' < r$ such that $S_{r,r'} \otimes X_1 \subseteq G\backslash a_1 \subset S_{r,r'-1} \otimes X_1$ $(r = q - t)$. For convenience, rewrite $S_{r,r'} \otimes X_1$ as $qS_{r,r'}$ and $S_{r,r'-1} \otimes X_1$ as $qS_{r,r'-1}$, then $qS_{r,r'} \subseteq G\backslash a_1 \subset qS_{r,r'-1}$.

Step 2: Calculate the components of C_0.

By Lemma 2 (b), we obtain

$$\begin{aligned}\sum_{\gamma\in G\backslash a_1} B_2(G, a_1\gamma) &= \sum_{\gamma\in qS_{r,r'}} B_2(G, a_1\gamma) + \sum_{\gamma\in G\backslash\{qS_{r,r'},a_1\}} B_2(G, a_1\gamma)\\ &= (n - 2^{r-1})\#\{\gamma : \gamma \in qS_{r,r'}\} + \sum_{\gamma\in G\backslash\{qS_{r,r'},a_1\}} B_2(G\backslash qS_{r-1,r'}, a_1\gamma).\end{aligned}$$

Since

$$\begin{aligned} m &= \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in G\backslash a_1\}\\ &= \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in qS_{r,r'}\} + \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in G\backslash\{qS_{r,r'}, a_1\}\}\\ &= 2^r - 2^{r'} + \#\{\gamma : B_2(G, a_1\gamma) > 0, \gamma \in G\backslash\{qS_{r,r'}, a_1\}\}.\end{aligned}$$

Step 3: Minimize C_0 and find the best position of a_1.

If $\#\{G\backslash\{qS_{r-1,r'}, a_1\}\} \le 2^{r'-1}$, we have $B_2(G\backslash qS_{r-1,r'}, a_1\gamma) = 0$ for $\gamma \in G\backslash\{qS_{r,r'}, a_1\}$ by Lemma 2 (a). Let $a_1 = q$, the first column of $D^{q-4}(X_1)$, then C_0 is minimized. Otherwise, replace $G\backslash\{qS_{r-1,r'}\backslash a_1\}$ by $G\backslash a_1$ and return to Step 1.

After the searching procedure, we find no other column of $D^{q-t}(X_1)$ is better than the first column of $D^{q-t}(X_1)$ to minimize C_0. □

Proof of Theorem 5. All clear 2fi's of design $\mathbf{X}$ are either X_1X_j $(j \ne 1)$ or X_2X_j $(j > 2)$ (Cheng and Zhang, 2010). Similar to the case of $9N/32 + 1 \le n + 2 \le 5N/16$, ${}^{\#}_{1,0}C_{2,0}$, ${}^{\#}_{1,0}C_{2,1}$, ${}^{\#}_{1,1}C_{2,0}$ of D

is maximized, if and only if $\bar{D}_0 \subseteq D^{q-t}(X_p)$ $(p = 1, 2)$, $a_1 \in D^{q-t}(X_p) \cap D_0$ and $a_2 \in D^{q-t}(X_i)$ $(i \neq p)$. The result is the same to Theorem 2 (2). Next we maximize ${}^{\#}_{2,0}C_{2,0}(D)$.

Let $G = D^{q-t}(X_p) \cap D_0$ and $g = \#\{G\} = n + 2 - N/4$. The smallest positive $B_2(D_0\backslash\{a_1, a_2\}, \gamma)$ is $n - N/4$. These alias sets have the form $\gamma = a_1t_1 = a_2t_2$, where $t_1 \in D^{q-t}(X_i)$ $(i \neq p)$ and $t_2 \in D^{q-t}(X_p) \cap D_0$. Then we obtain ${}^{\#}_{2,0}C^{(k)}_{2,0}(D) = 0$, for $k < n - N/4 - 1$, ${}^{\#}_{2,0}C^{(n-N/4-1)}_{2,0}(D) = (g - 1)(n - N/4)$.

For $B_2(D_0\backslash\{a_1, a_2\}, \gamma) = n - N/4 + 1$, the alias sets contain a_1t_1 but exclude a_2t_2, where $t_1, t_2 \in D_0$ or $\gamma = a_3$. The number of such alias sets is $N/4 - g + 1$. Then ${}^{\#}_{2,0}C^{(n-N/4)}_{2,0}(D) = (N/4 - g + 1)(n - N/4 + 1)$.

For $k > n - N/4$ and $a_2 \in D^{q-t}(X_j)$ $(j \neq p)$, the smallest positive $B_2(D_0\backslash\{a_1, a_2\}, \gamma) > n - N/4 + 1$ is $N/2^t - 1$. The alias sets have the form $\gamma = a_2t_2$, where $t_2 \in D^{q-t}(X_k)$ $(k \neq j, p)$. There are two different cases that make ${}^{\#}_{2,0}C^{(k)}_{2,0}(D)$ different.

Case I: $a_2 \in D^{q-t}(X_i)$, $i = 1, 2$ but $i \neq p$.

Since X_1X_l and X_2X_l are clear 2fi's $(l \neq 1, 2)$, any factor in $D^{q-t}(X_l)$ $(l \neq 1, 2)$ combined with $a_2 \in D^{q-t}(X_i)$ is only confounded with some $\gamma \in D^{q-t}(X_iX_l)$, then for $k = N/2^t - 2$

$$\begin{aligned}
&\#\{\gamma : \gamma \in D^q(1), B_2(D_0\backslash\{a_1, a_2\}, \gamma) = N/2^t - 1\} \\
&\quad = \#\{(t_1, t_2) : t_1t_2 = \gamma, t_1 \in D^{q-t}(X_i), t_2 \in D^{q-t}(X_l)(l \neq 1, 2)\} \\
&\quad = \sum_{l\neq 1,2} \#\{t_2 : t_2 \in D^{q-t}(X_l)\} = N/4 - N/2^t,
\end{aligned}$$

it is obtained

$${}^{\#}_{2,0}C^{(N/2^t-2)}_{2,0}(D) = (N/4 - N/2^t)(N/2^t - 1).$$

Case II: $a_2 \in D^{q-t}(X_i)$, $i \neq 1, 2$.

If $t_2 \in D^{q-t}(X_l)$ $(l = 1, 2$ but $l \neq p)$, the alias sets containing a_2t_2 make $B_2(D_0\backslash\{a_1, a_2\}, \gamma) = N/2^t - 1$. Then the 2fis have the form $D^{q-t}(X_iX_l)$ for $k = N/2^t - 2$, and

$$\begin{aligned}
&\#\{\gamma : \gamma \in D^{q-t}(X_iX_l), B_2(D_0\backslash\{a_1, a_2\}, \gamma) = N/2^t - 1\} \\
&\quad = \#\{(t_1, t_2) : t_1t_2 = \gamma, t_1 \in D^{q-t}(X_i), t_2 \in D^{q-t}(X_l)(l = 1, 2, l \neq p)\} \\
&\quad = \#\{t_2 : t_2 \in D^{q-t}(X_l)(l = 1, 2, l \neq p)\} = N/2^t.
\end{aligned}$$

So

$${}^{\#}_{2,0}C^{(N/2^t-2)}_{2,0}(D) = N/2^t(N/2^t - 1).$$

□

Note that $N/4 - N/2^t > N/2^t$ for $t \geq 5$, so Case I has less *GLOC* than Case II. This completes the proof.

Appendix B: Two tables of GMC 2^n4^1 designs

Table B.1. GMC 2^n4^1 designs with 16 runs.

n	Columns	${}^{\#}_{1,0}C_{2,0}$, ${}^{\#}_{1,0}C_{2,1}$, ${}^{\#}_{1,1}C_{2,0}$, ${}^{\#}_{2,0}C_{2,0}$, ${}^{\#}_{2,0}C_{2,1}$, ${}^{\#}_{2,1}C_{2,0}$, ${}^{\#}_{2,1}C_{2,1}$	WLP $(A_{30}, A_{31}), \ldots$	$c_1, \bar{c}_1$ $c_2, \bar{c}_2$
3	A_0, 3, 4, 1234	(3), (3), (3), (3), (0, 3), (6, 3), (9)	(0, 0), (0, 1)	3, 6 0, 6
4	A_1, 124, 34, 134, 234	(4), (2, 2), (2, 1), (6), (1, 4, 1), (6, 6), (11, 1)	(0, 1), (0, 2)	2, 2 0, 4
5	A_1, 24, 124, 34, 134, 234	(5), (1, 4), (2, 0, 1), (4, 6), (2, 4, 4), (5, 8, 2), (7, 8)	(0, 2), (1, 4)	1, 2 0, 5

Note: 1. $A_0 = (1, 2, 12), A_1 = (4, 1234, 123), A_2 = (3, 123, 12)$.
2. c_1 and c_2 are the numbers of clear two-level main effects and 2fi's, respectively, while $\bar{c}_1$, $\bar{c}_2$ is the number of clear four-level component and clear 2fi's containing four-level component.

Table B.2. GMC 2^n4^1 designs with 32 runs.

n	Columns	${}^{\#}_{1,0}C_{2,0}$, ${}^{\#}_{1,0}C_{2,1}$, ${}^{\#}_{1,1}C_{2,0}$, ${}^{\#}_{2,0}C_{2,0}$, ${}^{\#}_{2,0}C_{2,1}$, ${}^{\#}_{2,1}C_{2,0}$, ${}^{\#}_{2,1}C_{2,1}$	WLP $(A_{30}, A_{31}), \ldots$	$c_1, \bar{c}_1$ $c_2, \bar{c}_2$
7	A_0, 3, 4, 5, 1345, 245, 235, 234	(7), (7), (3), (0, 0, 21), (21), (21), (21)	(0, 0), (3, 7), (0, 4), (0, 0), (0, 0), (0, 1)	6, 2 0, 0
8	A_2, $\mathbf{S}_{5N/16}\backslash\{I_2 \otimes \{\mathbf{1}, \mathbf{2}\}_{16}\}$	(8), (6, 2), (2, 1), (13, 12, 3), (13, 15), (10, 12, 0, 2), (22, 2)	(0, 1), (3, 10), (4, 8), (0, 0), (0, 3), (0, 2)	7, 3 0, 21
9	A_1, $235 \sim 2345$	(9), (5, 4), (2, 0, 1), (2, 12, 18, 4), (2, 26, 8), (9, 4, 5, 8, 1), (19, 8)	(0, 2), (9, 14), (0, 9), (6, 12), (0, 4), (0, 6), (1, 0)	5, 2 0, 9

Note: 1. $A_0 = (1, 2, 12), A_1 = (5, 12345, 1234), A_2 = (I_2 \otimes \mathbf{1}_{16}, I_2 \otimes \mathbf{2}_{16}, I_2 \otimes \mathbf{12}_{16}), F_{55} = \{5, 15, 25, 125, 35, 45, 145, 245, 1245, 345, 1345, 2345, 12345\}$.
2. $\{a \sim b\}$ means containing the columns from a to b of F_{55}.
3. c_1 and c_2 are the numbers of clear two-level main effects and 2fi's, respectively, while $\bar{c}_1$, $\bar{c}_2$ is the number of clear four-level component and clear 2fi's containing four-level component.