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Construction of (nearly) orthogonal sliced Latin hypercube designs

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ABSTRACT

Sliced Latin hypercube designs have found a wide range of applications. Such a design is a special Latin hypercube design that can be partitioned into slices which are still LHDs when the levels of each slices are collapsed properly. In this paper we propose a method for constructing sliced Latin hypercube designs with second-order orthogonality. The resulting designs are further augmented to be nearly orthogonal sliced Latin hypercube designs which have much more columns. Also, two methods of generating nearly orthogonal sliced Latin hypercube designs are proposed. The methods are convenient, efficient and capable of accommodating any number of slices.

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1. Introduction

A sliced Latin hypercube design (SLHD), proposed by Qian (2012), is a special Latin hypercube design (LHD) that can be partitioned into slices each of which is a smaller LHD when levels of each slice are collapsed properly. Such a design is attractive because of its two features: (a) each slice of the design achieves maximum uniformity in any one-dimensional projection, and (b) when collapsed over all the slices, the whole design possesses maximum stratification in any one-dimensional projection. This type of designs is useful for computer experiments with qualitative and quantitative factors (Qian et al., 2008; Han et al., 2009; Qian and Wu, 2009; Zhou et al., 2011; Deng et al., 2015; Huang et al., 2016), validating a computer model (Bayarri et al., 2007), cross-validation (Zhang and Qian, 2013), data pooling and stochastic optimization (Qian, 2012; Yang et al., 2013; Chen et al., 2014).

An $n \times f$ LHD is a matrix in which each column is a permutation of n equally-spaced levels. For convenience, the n levels are taken to be $\{-(n-1), -(n-3), \ldots, (n-3), (n-1)\}$. For integers m and s, an SLHD of $n = ms$ runs and s slices is an LHD that can be divided into s smaller slices of m levels such that the m levels of each slice correspond to the m equally-spaced intervals $[-n, -n+2s), [-n+2s, -n+4s), \ldots, [n-4s, n-2s), [n-2s, n)$ respectively. Here, such a design is denoted by SLHD(m, s, f) with $n = ms$.

Qian (2012) developed a computational algorithm for generating SLHDs, and Yang et al. (2013) was the first to give systematic constructions of SLHDs with first- or second-order orthogonality. Recent studies on SLHDs mainly focus on orthogonality and space-filling properties, see Yin et al. (2014), Yang et al. (2014), Cao and Liu (2015) and Yang et al. (2016), among others. Ba et al. (2015) constructed SLHDs in terms of optimality, where a powerful R package "SLHD" is provided for generating corresponding designs in practice. An SLHD is called *first-order orthogonal* if any two columns of each slice are orthogonal. An SLHD is called *second-order orthogonal* if each slice satisfies: (a) any two columns are orthogonal; and (b) any column is orthogonal to the elementwise product of any two columns, identical and distinct. From the modeling

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perspective, the first-order orthogonality of a design ensures independent estimates of linear effects when a first-order model is fitted, while the second-order orthogonality of a design ensures that all linear effects are orthogonal not only to each other, but also to the quadratic effects when a second-order model is fitted. If the number of factors to study is more than those orthogonal SLHDs can afford, nearly orthogonal SLHDs are good choices. In this paper, we propose a construction method for (i) second-order orthogonal SLHDs, and two construction methods for (ii) nearly orthogonal SLHDs.

The rest of the paper is organized as follows. In Section 2, the method of constructing second-order orthogonal SLHDs is proposed. In Section 3, one method is first provided to add more columns to the designs obtained in Section 2 to form nearly orthogonal SLHDs. Then another method is proposed to construct nearly orthogonal SLHDs for more cases. Concluding remarks are in Section 4. Some of the proofs are given in the Appendix.

2. Construction of second-order orthogonal SLHDs

In this section, we will give the construction of second-order orthogonal SLHDs. Before giving the construction method, we first recall some work on orthogonal matrices given in Yang and Liu (2012). For a matrix X of an even number of rows, let X^* denote the matrix obtained by swapping the signs of the top half of X. For any x and y, let

$$S_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad R_1(x, y) = \begin{pmatrix} x+y & 2x+y \\ 2x+y & -x-y \end{pmatrix}.$$

For $c \geq 2$, define

$$S_c = \begin{pmatrix} S_{c-1} & -S^*_{c-1} \\ S_{c-1} & S^*_{c-1} \end{pmatrix} \quad \text{and}$$

$$R_c(x, y) = \begin{pmatrix} R_{c-1}(x, y) & -(R^*_{c-1}(x, y) + 2^{c-1}xS^*_{c-1}) \\ R_{c-1}(x, y) + 2^{c-1}xS_{c-1} & R^*_{c-1}(x, y) \end{pmatrix}. \tag{1}$$

The $R_c(x, y)$ in (1) is called an orthogonal matrix. Yang and Liu (2012) gave the following result.

Lemma 1. *For the $R_c(x, y)$ defined in (1), we have*

$$R^{\mathrm{T}}_c(x, y)R_c(x, y) = \gamma I_{2^c},$$

where $\gamma = x^2 2^c(2^c + 1)(2^{c+1} + 1)/6 + (2^{2c} + 2^c)xy + 2^c y^2$ is a constant depending on x, y and c.

Now, we propose the algorithm for constructing the second-order orthogonal SLHDs.

Algorithm 1 (*Construction of Second-Order Orthogonal* SLHD$(2^{c+1}, s, 2^c)$).

Step 1. Given any positive integers c and s, for $p = 1, \dots, s$, take $x_s = 2s$, $y_{s,p} = -(2s - 2p + 1)$ and

$$D_{c,p} = \begin{pmatrix} R_c(x_s, y_{s,p}) \\ -R_c(x_s, y_{s,p}) \end{pmatrix}. \tag{2}$$

Step 2. Define

$$D = (D^{\mathrm{T}}_{c,1}, \dots, D^{\mathrm{T}}_{c,s})^{\mathrm{T}}. \tag{3}$$

Then we have the following result.

Theorem 1. *The D defined in (3) is a second-order orthogonal* SLHD$(2^{c+1}, s, 2^c)$ *with s slices $D_{c,1}, \dots, D_{c,s}$ each of which is a $2^{c+1} \times 2^c$ matrix.*

Proof. For $p = 1, \dots, s$, the levels of each column of $D_{c,p}$ are $\{\pm(kx_s + y_{s,p}) : k = 1, \dots, 2^c\} = \{\pm(2ks + 2p - 1) : k = 0, 1, \dots, 2^c - 1\}$. The levels of each column of D are $\bigcup_{p=1}^{s}\{\pm(2ks + 2p - 1) : k = 0, 1, \dots, 2^c - 1\} = \{\pm(2k - 1) : k = 1, \dots, s2^c\}$. This implies that D is an SLHD with slices $D_{c,1}, \dots, D_{c,s}$. The orthogonality and second-order orthogonality can be easily verified by Lemma 1 and the structure of $D_{c,p}$'s. □

The above method constructs second-order orthogonal SLHDs, where for each slice the run size is twice as many as the number of factors. Unlike the second-order orthogonal SLHDs constructed in Yang et al. (2013), Algorithm 1 can construct second-order orthogonal SLHDs with any number of slices. Table 1 presents the second-order orthogonal SLHD(8, 3, 4) for $c = 2$ and $s = 3$, where the three slices correspond to runs 1–8; runs 9–16; and runs 17–24, respectively.

Table 1
The second-order orthogonal SLHD(8, 3, 4).

Run	$D_{2,1}$				Run	$D_{2,2}$				Run	$D_{2,3}$			
1	1	7	13	19	9	3	9	15	21	17	5	11	17	23
2	7	−1	−19	13	10	9	−3	−21	15	18	11	−5	−23	17
3	13	19	−1	−7	11	15	21	−3	−9	19	17	23	−5	−11
4	19	−13	7	−1	12	21	−15	9	−3	20	23	−17	11	−5
5	−1	−7	−13	−19	13	−3	−9	−15	−21	21	−5	−11	−17	−23
6	−7	1	19	−13	14	−9	3	21	−15	22	−11	5	23	−17
7	−13	−19	1	7	15	−15	−21	3	9	23	−17	−23	5	11
8	−19	13	−7	1	16	−21	15	−9	3	24	−23	17	−11	5

3. Construction of nearly orthogonal SLHDs with more columns

In this section we will provide two methods of constructing nearly orthogonal SLHDs with much more columns and quite small correlations. The basic idea here is to add more columns to the existing (nearly) orthogonal SLHDs. Suppose $v = (v_1, \ldots, v_n)^T$ and $u = (u_1, \ldots, u_n)^T$ are two vectors. The correlation between vectors v and u is defined as

$$\rho(v, u) = \frac{\sum_{i=1}^{n}(v_i - \bar{v})(u_i - \bar{u})}{\sqrt{\sum_{i=1}^{n}(v_i - \bar{v})^2(u_i - \bar{u})^2}}.$$

3.1. *Construction of nearly orthogonal* SLHD$(2^{c+1}, s, 2^c + 2^{c-1})$

In this section, we will give a construction method for generating nearly orthogonal SLHDs based on the second-order orthogonal SLHDs generated by Algorithm 1. Note that each slice of the designs generated by Algorithm 1 has foldover structure. Let w denote a slice of an added column. To guarantee small correlations among columns of each slice of the nearly orthogonal SLHDs, the levels of w should be arranged in a careful way such that $|w_i - w_{2^c+i}| \leq 2s$, where w_i and w_{2^c+i} are the ith and $(2^c + i)$th rows of w respectively. Now we give the specific construction method as follows.

Algorithm 2 (*Construction of Nearly Orthogonal* SLHD$(2^{c+1}, s, 2^c + 2^{c-1})$).

Step 1. For $p = 1, \ldots, 2s$, take $x_{2s} = 4s$, $y_{2s,p} = -(4s - 2p + 1)$ and

$$E_{c-1,p} = \begin{pmatrix} R_{c-1}(x_{2s}, y_{2s,p}) \\ -R_{c-1}(x_{2s}, y_{2s,p}) \end{pmatrix}.$$

Step 2. For $p = 1, \ldots, s$, define

$$F_p = (E^{\mathrm{T}}_{c-1,i_p}, E^{\mathrm{T}}_{c-1,j_p})^{\mathrm{T}},$$

where $\{i_p, j_p\}$ is a permutation on $\{p, s + p\}$.

Step 3. Put

$$F = (F^{\mathrm{T}}_{k_1}, \ldots, F^{\mathrm{T}}_{k_s})^{\mathrm{T}}, \tag{4}$$

where $\{k_1, \ldots, k_s\}$ is a permutation on $\{1, \ldots, s\}$.

Step 4. Form the matrix

$$G = (D, F). \tag{5}$$

Some theoretical properties of G in (5) can be stated as follows.

Theorem 2. *For G in (5), we have*

(i) *G is an* SLHD$(2^{c+1}, s, 2^c + 2^{c-1})$ *with s slices $G_1, \ldots, G_s$, where $G_p = (D_{c,p}, F_{k_p})$ for $p = 1, \ldots, s$; and*
(ii) *for each slice G_p, the upper bound of absolute correlation coefficient between columns is bounded by*

$$\rho_u = 3(2^c + 1)/\{2(2^c - 1)(2^{c+1} - 1)\}. \tag{6}$$

Proof. To show Part (i), we only need to verify that for $p = 1, \ldots, s$, the slice F_p is a $2^{c+1} \times 2^{c-1}$ LHD. From Theorem 1, F is an SLHD$(2^c, 2s, 2^{c-1})$ of slices $E_{c-1,1}, \ldots, E_{c-1,2s}$. For $p = 1, \ldots, s$, the levels of each column of $E_{c-1,p}$ are $\{\pm(4ks + 2p - 1) : k = 0, 1, \ldots, 2^{c-1} - 1\}$ and the levels of each column of $E_{c-1,s+p}$ are $\{\pm\{(4k + 2)s + 2p - 1\} : k = 0, 1, \ldots, 2^{c-1} - 1\}$, so the levels of each column of F_p are $\{\pm(2ks + 2p - 1) : k = 0, 1, \ldots, 2^c - 1\}$, which implies that F_p is a $2^{c+1} \times 2^{c-1}$ LHD.

Table 2
The values of ρ_u in (6) when $c = 2, 3, \ldots, 7$.

c	2	3	4	5	6	7
Upper bound	0.3571	0.1286	0.0548	0.0253	0.0122	0.0060

Table 3
Nearly orthogonal SLHD(8, 3, 6) in Example 1.

Run	D_1				F_1		Run	D_2				F_2		Run	D_3				F_3	
1	1	7	13	19	1	13	9	3	9	15	21	3	15	17	5	11	17	23	5	17
2	7	−1	−19	13	13	−1	10	9	−3	−21	15	15	−3	18	11	−5	−23	17	17	−5
3	13	19	−1	−7	−1	−13	11	15	21	−3	−9	−3	−15	19	17	23	−5	−11	−5	−17
4	19	−13	7	−1	−13	1	12	21	−15	9	−3	−15	3	20	23	−17	11	−5	−17	5
5	−1	−7	−13	−19	7	19	13	−3	−9	−15	−21	9	21	21	−5	−11	−17	−23	11	23
6	−7	1	19	−13	19	−7	14	−9	3	21	−15	21	−9	22	−11	5	23	−17	23	−11
7	−13	−19	1	7	−7	−19	15	−15	−21	3	9	−9	−21	23	−17	−23	5	11	−11	−23
8	−19	13	−7	1	−19	7	16	−21	15	−9	3	−21	9	24	−23	17	−11	5	−23	11

Now, let us prove Part (ii). Let $A = (a_1, \ldots, a_{2^{c+1}})^{\mathrm{T}}$ be any column of $D_{c,p}$ and $B = (b_1, \ldots, b_{2^{c+1}})^{\mathrm{T}}$ be any column of F_{k_p}. From the structures of $D_{c,p}$ in (2) and F_{k_p} in (4), we have (i) for $i = 1, \ldots, 2^c$, $a_{2^c+i} = -a_i$ and $|b_i - b_{2^c+i}| = 2s$; and (ii) the levels of A are $\{\pm(2ms + 2p - 1) : m = 0, 1, \ldots, 2^c - 1\}$ and the levels of B are $\{\pm(2ms + 2k_p - 1) : m = 0, 1, \ldots, 2^c - 1\}$. Thus,

$$
\begin{aligned}
|\rho(A, B)| &\le \frac{2s\sum_{i=1}^{2^c}\{2is - (2s - 2p + 1)\}}{\left[2\sum_{i=1}^{2^c}\{2(i-1)s + 2p - 1\}^2\right]^{1/2}\left[2\sum_{i=1}^{2^c}\{2(i-1)s + 2k_p - 1\}^2\right]^{1/2}} \\
&\le \frac{2s\sum_{i=1}^{2^c} 2is}{2\sum_{i=1}^{2^c}\{2(i-1)s\}^2} \\
&= \frac{3(2^c + 1)}{2(2^c - 1)(2^{c+1} - 1)}.
\end{aligned}
$$

This completes the proof. □

The nearly orthogonal SLHD generated by Algorithm 2 is composed of two parts, D and F, both of which are second-order orthogonal SLHDs. Because the levels of each slice of F are specially assigned, the absolute correlation coefficients between D_i and F_i are well bounded. Furthermore, the correlation coefficients between D_i and F_i decrease dramatically to almost zero as the value of c increases. The upper bounds ρ_u for the cases $c = 2, 3, \ldots, 7$ are listed in Table 2. Next we will give an example to illustrate Algorithm 2.

Example 1. For $c = 2$ and $s = 3$, without loss of generality, we can take $i_p = p, j_p = 3 + p$ and $k_p = p$ for $p = 1, 2, 3$. By Algorithm 2 we obtain a nearly orthogonal SLHD(8, 3, 6) with three slices listed in Table 3. The first, second and third slices correspond to runs 1–8, 9–16 and 17–24, respectively. The maximum correlation between the columns of these three slices is 0.2069, 0.1905 and 0.1743, respectively.

3.2. Construction of nearly orthogonal SLHDs for more cases

The nearly orthogonal SLHDs generated in Section 3.1 has 2^{c+1} runs in each of their slices. In this section we will provide a more general algorithm to generate nearly orthogonal SLHDs. Let $L = (L_1^T, \ldots, L_s^T)^T$ be an original (nearly) orthogonal SLHD$(2m, s, f)$ in which each slice L_p has foldover structure. Here, the number of runs for each slice to be enlarged is not necessary to be a power of 2. Just foldover structure is needed. The following algorithm is used to add more columns to L to form a larger SLHD$(2m, s, f + h)$. Similar to the idea in Algorithm 2, the levels of the ith and $(m + i)$th rows in each slice of each added column are carefully arranged to guarantee small correlations among columns of each slice of the nearly orthogonal SLHDs.

Algorithm 3 (*Construction of Nearly Orthogonal* SLHD$(2m, s, f + h)$)**.**

Step 1. Let X be a (nearly) orthogonal LHD(m, h) and E be a (nearly) orthogonal LHD(s, h).
Step 2. Define $M = ((2X + 1)^T, (2X - 1)^T)^T$ and $H_p = E(p, :) \oplus_c sM$ for $p = 1, \ldots, s$, where the operator $\oplus_c$ is defined to be $a \oplus_c B = (a_j + b_{ij})_{n\times m}$ for a vector $a = (a_1, \ldots, a_m)$ and a matrix $B = (b_{ij})_{n\times m}$.

Step 3. Take $H = (H_1^T, \ldots, H_s^T)^T$.
Step 4. Take

$$Q = (L, H). \tag{7}$$

Let $\rho_M(D)$ represent the maximum correlation between columns of matrix D. Then, we have the following results.

Theorem 3. *For the Q in (7), we have*

(i) *Q is an* SLHD$(2m, s, f+h)$ *with s slices $Q_1, \ldots, Q_s$, where $Q_p = (L_p, H_p)$;*
(ii) *for each slice Q_p, the absolute correlation coefficient is upper bounded by* $\max\{\rho_M(L_p), (2m(m+1)s^2 - 2ms)/\sqrt{\gamma(m,s)\tau(m,s)}, (4(m^2-1)\rho_M(X)+3)/(4m^2-1)\}$ *for $p = 1, \ldots, s$, where $\gamma(m,s) = 2s^2m(m-1)(2m-1)/3 + 2sm(m-1) + m$ and $\tau(m,s) = 2m(2m+1)(2m-1)s^2/3$.*

The proof of Theorem 3 is deferred to Appendix. Now, we give an illustrative example.

Example 2. Let $L = (L_1^T, L_3^T, L_3^T, L_4^T)^T$ be an orthogonal SLHD(10, 4, 4) with

$$L_1 = \begin{pmatrix} -13 & -23 & -29 & -39 & -1 & 13 & 23 & 29 & 39 & 1 \\ -23 & 13 & -39 & 29 & -1 & 23 & -13 & 39 & -29 & 1 \\ -29 & 39 & 13 & -23 & 1 & 29 & -39 & -13 & 23 & -1 \\ -39 & -29 & 23 & 13 & 1 & 39 & 29 & -23 & -13 & -1 \end{pmatrix}^T,$$

$$L_2 = \begin{pmatrix} 15 & 19 & 31 & 35 & 3 & -15 & -19 & -31 & -35 & -3 \\ 19 & -15 & 35 & -31 & 3 & -19 & 15 & -35 & 31 & -3 \\ 31 & -35 & -15 & 19 & -3 & -31 & 35 & 15 & -19 & 3 \\ 35 & 31 & -19 & -15 & -3 & -35 & -31 & 19 & 15 & 3 \end{pmatrix}^T,$$

$$L_3 = \begin{pmatrix} -11 & -21 & -27 & -37 & -5 & 11 & 21 & 27 & 37 & 5 \\ -21 & 11 & -37 & 27 & -5 & 21 & -11 & 37 & -27 & 5 \\ -27 & 37 & 11 & -21 & 5 & 27 & -37 & -11 & 21 & -5 \\ -37 & -27 & 21 & 11 & 5 & 37 & 27 & -21 & -11 & -5 \end{pmatrix}^T \quad \text{and}$$

$$L_4 = \begin{pmatrix} 9 & 17 & 25 & 33 & 7 & -9 & -17 & -25 & -33 & -7 \\ 17 & -9 & 33 & -25 & 7 & -17 & 9 & -33 & 25 & -7 \\ 25 & -33 & -9 & 17 & -7 & -25 & 33 & 9 & -17 & 7 \\ 33 & 25 & -17 & -9 & -7 & -33 & -25 & 17 & 9 & 7 \end{pmatrix}^T.$$

In order to add two more columns, by taking

$$X = \begin{pmatrix} 2 & 4 & 0 & -2 & -4 \\ -4 & 2 & 0 & 4 & -2 \end{pmatrix}^T$$

and

$$E = \begin{pmatrix} -3 & -1 & 3 & 1 \\ -1 & 3 & 1 & -3 \end{pmatrix}^T,$$

we have

$$M = \begin{pmatrix} 5 & 9 & 1 & -3 & -7 & 3 & 7 & -1 & -5 & -9 \\ -7 & 5 & 1 & 9 & -3 & -9 & 3 & -1 & 7 & -5 \end{pmatrix}^T,$$

$$H_1 = \begin{pmatrix} 17 & 33 & 1 & -15 & -31 & 9 & 25 & -7 & -23 & -39 \\ -29 & 19 & 3 & 35 & -13 & -37 & 11 & -5 & 27 & -21 \end{pmatrix}^T,$$

$$H_2 = \begin{pmatrix} 19 & 35 & 3 & -13 & -29 & 11 & 27 & -5 & -21 & -37 \\ -25 & 23 & 7 & 39 & -9 & -33 & 15 & -1 & 31 & -17 \end{pmatrix}^T,$$

$$H_3 = \begin{pmatrix} 23 & 39 & 7 & -9 & -25 & 15 & 31 & -1 & -17 & -33 \\ -27 & 21 & 5 & 37 & -11 & -35 & 13 & -3 & 29 & -19 \end{pmatrix}^T \quad \text{and}$$

$$H_4 = \begin{pmatrix} 21 & 37 & 5 & -11 & -27 & 13 & 29 & -3 & -19 & -35 \\ -31 & 17 & 1 & 33 & -15 & -39 & 9 & -7 & 25 & -23 \end{pmatrix}^T.$$

Then design $Q = (L, H)$ is a nearly orthogonal SLHD(10, 4, 6) with $\rho_M(L_p, H_p) \le 0.1534$ for $p = 1, 2, 3, 4$.

4. Concluding remarks

In this work, we first propose an approach for constructing second-order orthogonal SLHDs. After that we propose two methods for constructing nearly orthogonal SLHDs by adding columns to an original (nearly) orthogonal SLHDs with foldover structure. The nearly orthogonal SLHDs generated by Algorithm 2 are SLHD(2^{c+1}, s, 2^c+2^{c-1}). More general nearly orthogonal SLHDs can be generated by Algorithm 3. The resulting nearly orthogonal SLHDs can accommodate as many factors as runs while having quite small correlation coefficients between columns of slices. The upper bounds of the correlations between columns of slices have been given in this paper. Since our approaches for generating nearly orthogonal SLHDs can only work if the original SLHDs have foldover structure, extending our work to augment LHDs without fold-over structure is an issue worth further study.

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Appendix. Proof of Theorem 3

For (i), we only need to show that H is an SLHD($2m$, s, h). From Wang et al. (2015), we know that M is a nearly orthogonal LHD($2m$, h). According to Huang et al. (2014), the result follows directly by noting that E is a (nearly) orthogonal LHD(s, h).

For (ii), we need to consider (a) the correlation between two columns of H_p and (b) the correlation between any column of H_p and any column of L_p. For $1 \le i < j \le h$, from Theorem 2(ii) of Wang et al. (2015), we have

$$\rho_{ij}(M) = \frac{4(m^2-1)\rho_{ij}(X)+3}{4m^2-1}.$$

From Theorem 2(i) of Huang et al. (2014), we can get $\rho_{ij}(H_p) = \rho_{ik}(M)$, which implies

$$\rho_M(H_p) = \frac{4(m^2-1)\rho_M(X)+3}{4m^2-1}.$$

Next, let l_i^p and h_j^p be the ith column of L_p and the jth column of H_p respectively. Note that L_p has foldover structure. Hence

$$\begin{aligned} {l_j^p}^T l_j^p &= \sum_{t=1}^{2m} {l_{tj}^p}^2 \\ &\ge 2 \times (1 + (2s+1)^2 + (4s+1)^2 + (6s+1)^2 + \cdots + (2 \times (m-1)s+1)^2) \\ &= 2s^2 m(m-1)(2m-1)/3 + 2sm(m-1) + m \triangleq \gamma(m,s). \end{aligned}$$

For ease of expression, let e_{pj} and $M(t,j)$ be the (p,j)th entry of E and (t,j)th entry of M, respectively. Let $\bar{h}_j^p = (\sum_{t=1}^{2m} h_{tj}^p)/2m$ be the average of the jth column of H_p. Note that M has the centered levels in each column. According to Step 2 of Algorithm 3, we have

$$\begin{aligned} (h_j^p - \bar{h}_j^p 1_{2m})^T (h_j^p - \bar{h}_j^p 1_{2m}) &= \sum_{t=1}^{2m} \left[\big(e_{pj} + sM(t,j)\big) - \sum_{t=1}^{2m} (e_{pj} + sM(t,j))/2m \right]^2 \\ &= s^2 \sum_{t=1}^{2m} (M(t,j))^2 \\ &= 2s^2(1 + 3^2 + 5^2 + \cdots + (2m-1)^2) \\ &= 2s^2 m(2m+1)(2m-1)/3 \\ &\triangleq \tau(m,s). \end{aligned}$$

Also, we have ${l_i^p}^T h_j^p = \sum_{t=1}^{2m} l_{ti}^p (e_{pj} + sM(t,j))$. Note that $\sum_{t=1}^{2m} l_{ti}^p = 0$. Hence

$$\begin{aligned} {l_i^p}^T h_j^p &= s \sum_{t=1}^{m} l_{ti}^p (2x_{tj}+1) + s \sum_{t=1}^{m} l_{t+q,i}^p (2x_{tj}-1) \\ &= 2s \sum_{t=1}^{m} l_{ti}^p \le 2m(m+1)s^2 - 2ms. \end{aligned}$$

This shows that $\rho(l_i^p, h_j^p) \le (2m(m+1)s^2 - 2ms)/\sqrt{\gamma(m,s)\tau(m,s)}$. Theorem 3(ii) follows from the above results together.

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