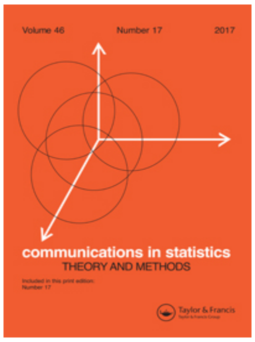

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Construction of some mixed two- and four-level regular designs with GMC criterion

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ABSTRACT
General minimum lower-order confounding (GMC) criterion is to choose optimal designs, which are based on the aliased effect-number pattern (AENP). The AENP and GMC criterion have been developed to form GMC theory. Zhang et al. (2015) introduced GMC 2^n4^m criterion for choosing optimal designs and constructed all GMC 2^n4^1 designs with $N/4+1 \le n+2 \le 5N/16$. In this article, we analyze the properties of 2^n4^1 designs and construct GMC 2^n4^1 designs with $5N/16+1 \le n+2 < N-1$, where n and N are, respectively, the numbers of two-level factors and runs. Further, GMC 2^n4^1 designs with 16-run, 32-run are tabulated.

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1. Introduction

Factorial experiments have broad applications in agricultural, engineering, and scientific studies. A factorial design in which the numbers of levels of the factors are not all equal is called a mixed-level or asymmetrical factorial design. Mixed-level factorial designs were first introduced by Yates (1937). Factorial experiments with mixed levels are often encountered in practice because the choice of factor levels may vary with the nature of the factor, especially, mixed two- and four-level fractional factorial designs, see Wang and Wu (1991), Wu and Zhang (1993), and Wu and Hamada (2000).

One of main tasks in mixed-level factorial designs is to find optimal designs and analyze experimental data effectively, so that more important effects and more possible models related to the effects in experiments can be estimated. Under the effect hierarchy principle (EHP) that lower-order factorial effects are more important than higher-order ones and that factorial effects of the same order are equally important, many criteria have been employed to compare fractions, such as maximum resolution (MR) (Box and Hunter, 1961), minimum aberration (MA) (Fries and Hunter, 1980), clear effects (CE) (Wu and Chen, 1992), maximum estimation capacity (MEC) (Sun, 1993), and general minimum lower-order confounding (GMC) (Zhang et al., 2008) criteria. Among these criteria, MA criterion is the most popular one, which is based on the word-length pattern (WLP). Many related topics have been studied to gain insights into selecting MA designs, see Franklin (1984), Butler (2003), and Cheng and Tang (2005). A nice summary of optimal criteria is given in Mukerjee and Wu (2006).

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However, MA criterion cannot reveal the relationships of the above criteria. Moreover, two designs with the same WLP cannot be distinguished by MA criterion, and most of MA designs are obtained by computer searching. In order to solve these problems, Zhang et al. (ZLZA for short, 2008) introduced the aliased effect-number pattern (AENP) to judge two-level designs, which contain the basic information of all factorial effects aliased with other effects at varying severity degrees in a design. Based on the AENP, they proposed GMC criterion and proved that the MR, MA, CE, MEC, and GMC criteria can each be viewed as sequentially minimizing or maximizing the components of a corresponding vector function of the AENP. GMC criterion treats the AENP as a set to compare designs and provides a unified approach for the other criteria. Now the AENP and GMC criterion widely apply in two-level regular designs, two-level block designs, two-level split-plot designs, see Zhang and Cheng (2010), Cheng and Zhang (2010), Wei et al. (2010), Hu and Zhang (2011), Li et al. (2011), Zhang et al. (2011), and Zhao et al. (2013). Zhang and Mukerjee (2009a, 2009b) extended GMC theory to general-level designs by complementary sets.

In this work, we apply GMC criterion to select optimal 2^n4^m design, which is a simple type of mixed-level designs. Zhang et al. (ZYLZ for short, 2015) proposed GMC 2^n4^m criterion, which is an extension of GMC criterion to the case of mixed-level designs. They also constructed GMC 2^n4^1 designs for $N/4 + 1 \le n + 2 \le 5N/16$, where n is the number of two-level factors and N is the number of runs. Li et al. (2011) provided a solution to construct two-level GMC designs and obtained the fact that every GMC design was formed by the last n columns of the saturated design in the Yates order under some conditions. Based on some results of ZYLZ (2015) and Li et al. (2011), we aim to solve the construction of GMC 2^n4^1 designs with $5N/16 + 1 \le n + 2 < N - 1$. The remainder of the article is organized as follows. In the next section, we review the definition of GMC 2^n4^m criterion and some notation. In Sections 3 and 4, we respectively construct GMC 2^n4^1 designs with $5N/16 + 1 \le n + 2 \le N/2 + 2$ and $N/2 + 3 \le n + 2 \le N - 1$. Proofs of some lemmas and theorems are given in Appendix A. GMC 2^n4^1 designs with 16 runs ($n = 4, \ldots, 11$) and GMC 2^n4^1 designs with 32 runs ($n = 10, \ldots, 14$) are tabulated in Appendix B for practical use.

2. Preliminaries

A design with n two-level factors and m four-level factors is said to be a 2^n4^m design, which can be constructed from the corresponding symmetrical orthogonal arrays through the method of replacement (Addelman, 1962). Wu (1989) improved the construction method by introducing the method of grouping. Wu et al. (1992) and Wu and Zhang (1993) further applied the grouping method to general designs. To explain the grouping method, we consider a saturated 2^{N-1} design with k independent columns denoted by $1, 2, \ldots, k$ and all possible interactions, where $N = 2^k$. Any three columns of the form (a_1, a_2, a_3) with $a_3 = a_1a_2$ can be replaced by a four-level factor without affecting orthogonality. We call a_1, a_2, and a_3 the three components of the four-level factor. If we can find m triplets of columns of the form (a_1, a_2, a_3), we can obtain m four-level columns by repeated using of replacement. By grouping scheme, a large class of $OA(2^k, 2^n4^m)$ can be constructed.

We first review some definitions and notation of GMC theory. A 2^{n-p} regular design T is called *a GMC design* if it sequentially maximizes the following sequence:

$$^{\#}C = \left({}_{1}^{\#}C_2, {}_{2}^{\#}C_2, {}_{1}^{\#}C_3, {}_{2}^{\#}C_3, {}_{3}^{\#}C_2, {}_{3}^{\#}C_3, \ldots\right), \tag{1}$$

where ${}^{\#}_{i}C_j = ({}^{\#}_{i}C_j^{(0)}, {}^{\#}_{i}C_j^{(1)}, \ldots, {}^{\#}_{i}C_j^{(K_j)})$, $K_j = \binom{n}{j}$, and ${}^{\#}_{i}C_j^{(k)}$ denote the number of ith-order effects aliased with jth-order effects at degree k. We call (1) AENP of the two-level regular design.

For any 2^n4^m design, the interaction effects have three different cases: *(i) two-level factors and two-level factors*, *(ii) two-level factors and four-level components*, and *(iii) four-level components and four-level components*. ZYLZ (2015) proposed GMC 2^n4^m criterion as follows. Let ${}^{\#}_{i,i_0}C^{(k)}_{j,j_0}$ be the number of ith-order effects aliased with jth-order effects at degree k, where each ith-order effect contains i_0 components of four-level factors and each jth-order effect contains j_0 components of four-level factors for $i_0 \le \min\{i, m\}$ and $j_0 \le \min\{j, m\}$. The larger the degree k is, the more severely the effect is aliased. For every pair of $\{(i, i_0), (j, j_0)\}$, let

$$
{}^{\#}_{i,i_0}C_{j,j_0} = \left({}^{\#}_{i,i_0}C^{(0)}_{j,j_0}, {}^{\#}_{i,i_0}C^{(1)}_{j,j_0}, \ldots, {}^{\#}_{i,i_0}C^{(K_j)}_{j,j_0}\right), \tag{2}
$$

where $K_j = \binom{n}{j}$. A 2^n4^m design D is called a *GMC* 2^n4^m *design* if it sequentially maximizes the following sequence:

$$
{}^{\#}C = \left({}^{\#}_{1,0}C_{2,0}, {}^{\#}_{1,0}C_{2,1}, {}^{\#}_{1,1}C_{2,0}, {}^{\#}_{2,0}C_{2,0}, {}^{\#}_{2,0}C_{2,1}, {}^{\#}_{2,1}C_{2,0}, {}^{\#}_{2,1}C_{2,1}, \ldots\right), \tag{3}
$$

which is called the AENP of the 2^n4^m design.

The following example is used to illustrate the above definition.

Example 1. Consider a 2^34^1 design $D = \{A, 3, 4, 1234\}$ with 16 runs, where $A = (1, 2, 12)$. Under the assumption that the interactions involving three or more factor effects are absent, the components of ${}^{\#}C$ are

$$
{}^{\#}_{1,0}C_{2,0}(D) = (3), \quad {}^{\#}_{1,0}C_{2,1}(D) = (3), \quad {}^{\#}_{1,1}C_{2,0}(D) = (3), \quad {}^{\#}_{2,0}C_{2,0}(D) = (3),
$$
$$
{}^{\#}_{2,0}C_{2,1}(D) = (0, 3), \quad {}^{\#}_{2,1}C_{2,0}(D) = (6, 3), \quad {}^{\#}_{2,1}C_{2,1}(D) = (9).
$$

Then, ${}^{\#}C = ((3), (3), (3), (3), (0, 3), (6, 3), (9))$.

In order to get GMC 2^n4^m designs with given n and m, we need to obtain all the confounding information among factors of 2^n4^m designs. Generally, the confounding of effects is given by computer algorithm. However, it is hard to get the alias relations of designs if the number of factors and runs is large. Therefore, it is necessary to develop the construction method of GMC 2^n4^m designs. For $5N/16 + 1 \le n + 2 < N - 1$, we construct GMC 2^n4^1 designs for the following two cases: (i) $5N/16 + 1 \le n + 2 \le N/2 + 2$, and (ii) $N/2 + 3 \le n + 2 < N - 1$, where n is the number of two-level factors and N is the number of runs.

3. GMC 2^n4^1 designs with $5N/16 + 1 \le n + 2 \le N/2 + 2$

For convenience, we introduce some notation as follows. Let H_r be a set containing all main effects and all interactions between two-level factors $1, \ldots, r$, and $H_1 = \{1\}$, $H_r = \{H_{r-1}, r, rH_{r-1}\}$. For example, $H_3 = \{1, 2, 12, 3, 13, 23, 123\}$. Denote $S_{qr} = H_q \backslash H_r$ for $r < q$ and $F_{rr} = \{r, rH_{r-1}\}$. For any 2^{n-p} design $T \subset H_q$ and $\gamma \in H_q$, define

$$
B_2(T, \gamma) = \#\{(d_1, d_2) : d_1, d_2 \in T,\ d_1d_2 = \gamma\},
$$

which is the number of two-factor interactions (2fi's) in T aliased with γ. Based on Li et al. (2011), the useful lemmas are shown below.

Lemma 1. *Suppose T is a 2^{n-p} design with $N = 2^{n-p}$ and $T \subseteq F_{qq}$. Then*

$$B_2(T,\gamma) = \begin{cases} 0, & \gamma \in F_{qq}, \\ B_2(F_{qq}\backslash T,\gamma) + n - N/4, & \gamma \in H_{q-1}. \end{cases} \quad (4)$$

Lemma 2. *Suppose T is a 2^{n-p} design with $N = 2^{n-p}$. If $S_{qr} \subset T$ $(r < q)$, then*

$$B_2(T,\gamma) = \begin{cases} n - N/2, & \gamma \in S_{qr}, \\ B_2(T\backslash S_{qr},\gamma) + N/2 - 2^{r-1}, & \gamma \in H_r. \end{cases}$$

By Lemmas 1 and 2, we obtain the following result.

Lemma 3. *Suppose T is a 2^{n-p} design with $N = 2^{n-p}$. If $qS_{q-1,r} \subseteq T \subset qS_{q-1,r-1}$ $(r < q)$, then*

$$B_2(T,\gamma) = \begin{cases} n - N/4, & \gamma \in S_{q-1,r}, \\ B_2(T\backslash qS_{q-1,r},\gamma) + N/4 - 2^{r-1}, & \gamma \in H_r. \end{cases}$$

Let D be a 2^n4^1 design which combines a two-level regular design T and a four-level factor $A = (a_1, a_2, a_1a_2)$, where a_1, a_2 are not in T. Denote $D = \{T, A\}$ and $T_0 = \{T, a_1, a_2\}$. A 2^n4^1 design can be generated from T_0 by grouping method if a_1a_2 is not in T. Hence, it is easy to see that, to construct a GMC 2^n4^1 design D, we first have to consider the regular two-level design T_0 and then select two different factors a_1 and a_2 with a_1a_2 not in T to form the four-level factor. It is key to select T, as well as a_1 and a_2 from T_0, such that D has GMC. For $5N/16 + 1 \le n + 2 \le N/2$, the maximal resolution of $T_0 \subset F_{qq}$ is at least IV. In this case, we have ${}^{\#}_{1,0}C_{2,0}(D) = {}^{\#}_{1}C_2(T) = (n)$. Next ${}^{\#}_{1,0}C_{2,1}(D)$ and ${}^{\#}_{1,1}C_{2,0}(D)$ need to be maximized.

Lemma 4. *Suppose D is a 2^n4^1 design with N runs. For $5N/16 + 1 \le n + 2 \le N/2$, up to isomorphism, D is a GMC design if $T_0 \subseteq F_{qq}$ and $B_2(F_{qq}\backslash T_0, a_3) = 0$. Further, ${}^{\#}_{1,0}C_{2,1}(D)$, ${}^{\#}_{1,1}C_{2,0}(D)$ are sequentially maximized and*

$${}^{\#}_{1,0}C_{2,1}(D) = (N/2 - n - 2, 2n - N/2 + 2), \quad {}^{\#}_{1,1}C_{2,0}(D) = (2, 0^{n-N/4}, 1),$$

where $0^{n-N/4}$ denotes $n - N/4$ continuous zeros.

Proof. Consider $D = \{T, A\}$ and $T_0 = \{T, a_1, a_2\}$. If $T_0 \subseteq F_{qq}$, clearly, $a_1, a_2 \in F_{qq}$ and $a_3 \in H_{q-1}$. By Lemma 1, we know that $B_2(T, a_i) = 0$ for $i = 1, 2$ and $B_2(T, a_3) = B_2(F_{qq}\backslash T, a_3) + n - N/4$. Moreover, $B_2(F_{qq}\backslash T, a_3) = B_2(F_{qq}\backslash T_0, a_3) + B_2(\{a_1, a_2\}, a_3) \ge 1$, we have

$$B_2(T, a_3) \ge n - N/4 + 1.$$

The lower bound is achieved if $B_2(F_{qq} \setminus T_0, a_3) = 0$. Without loss of generality, let $a_1 = q$, $a_2 = 12\dots q$ and T be the last n columns of $F_{qq}\backslash a_2$. Then $\{qF_{q-1,q-1}\backslash a_2\} \subseteq T \subset F_{qq}$ and $a_3 = 12\dots(q-1)$. Thus, $F_{qq}\backslash T_0 \subseteq qH_{q-2}$, and 2fi of $F_{qq}\backslash T_0$ must be in H_{q-2}. So $B_2(F_{qq}\backslash T_0, a_3) = 0$, and $B_2(F_{qq}\backslash T, a_3) = 1$. As a result, $B_2(T, a_3) = n - N/4 + 1$.

Since ${}^{\#}_{1,0}C^{(1)}_{2,1}(D) = \#\{\alpha : \alpha a_3 = \beta, \alpha, \beta \in T\} = 2B_2(T, a_3)$, we have ${}^{\#}_{1,0}C^{(1)}_{2,1}(D) = 2(n - N/4 + 1)$ and ${}^{\#}_{1,0}C^{(0)}_{2,1}(D) = n - {}^{\#}_{1,0}C^{(1)}_{2,1}(D) = N/2 - n - 2$. Thus, ${}^{\#}_{1,0}C_{2,1}(D) = (N/2 - n - 2, 2n - N/2 + 2)$. Consider ${}^{\#}_{1,1}C^{(k)}_{2,0}(D) = \#\{a_i : B_2(T, a_i) = k\}$ for $k \ge 0$. When $k = 0$, it is easy to get ${}^{\#}_{1,1}C^{(0)}_{2,0}(D) = \#\{a_1, a_2\} = 2$. For $k = n - N/4 + 1$, we have ${}^{\#}_{1,1}C^{(k)}_{2,0}(D) = 1$. Then, ${}^{\#}_{1,1}C_{2,0}(D) = (2, 0^{n-N/4}, 1)$. □

By Lemma 4, ${}^{\#}_{1,0}C_{2,1}(D)$ and ${}^{\#}_{1,1}C_{2,0}(D)$ are sequentially maximized. Next we consider ${}^{\#}_{2,0}C_{2,0}(D)$.

Lemma 5. *Suppose D is a 2^n4^1 design, $T_0 \subset F_{qq}$ and $B_2(F_{qq}\backslash T_0, a_3) = 0$. Then*

$$
{}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \begin{cases} (n - N/4)(n - \bar{g}(F_{qq}\backslash T_0)), & k = n - N/4 - 1, \\ (n - N/4 + 1)(N/2 - n - 1), & k = n - N/4, \\ \frac{k+1}{k+1-n+N/4-\delta}\ {}^{\#}_{2}C^{(k-n+N/4-\delta)}_{2}(F_{qq}\backslash T_0), & k > n - N/4, \\ 0, & otherwise, \end{cases}
$$

where $\delta = 0$ or 1 and

$$
\bar{g}(S) = \#\{\gamma : \gamma \in H_q\backslash S, B_2(S, \gamma) > 0\}. \tag{5}
$$

The proof of Lemma 5 is in Appendix A. By Lemmas 4 and 5, the following result is obvious.

Theorem 1. *Suppose D is a 2^n4^1 design with $5N/16 + 1 \le n + 2 \le N/2$. The design D has GMC if $\{-\bar{g}(F_{qq}\backslash T_0), {}^{\#}_{2}C_2(F_{qq}\backslash T_0)\}$ are sequentially maximized.*

Since $\#\{F_{qq}\backslash T_0\} = N/2 - (n + 2)$, according to Li et al. (2011), when $2^{r-2} + 1 \le N/2 - (n + 2) \le 2^{r-1}$ for some $r \le q$, $\{-\bar{g}(F_{qq}\backslash T_0), {}^{\#}_{2}C_2(F_{qq}\backslash T_0)\}$ are sequentially maximized in all the designs with $N/2 - (n + 2)$ factors and resolution at least IV, if and only if $F_{qq}\backslash T_0$ consists of the first $N/2 - (n + 2)$ elements of F_{qq}. Thus, T_0 consists of the last $n + 2$ columns of F_{qq}. We can obtain that $qS_{q-1,r} \subseteq T_0 \subseteq qS_{q-1,r-1}$. If $T_0 = qS_{q-1,r}$, then $a_1, a_2 \in qS_{q-1,r}$. Otherwise, there are two different cases for a_1, a_2: (i) both a_1 and a_2 are in $qS_{q-1,r}$, (ii) for a_1 and a_2, one is in $qS_{q-1,r}$ and the other is in $T_0\backslash qS_{q-1,r}$. The following theorem shows us that the latter is better than the former.

Theorem 2. *Suppose D is a 2^n4^1 design and $qS_{q-1,r} \subset T_0 \subset qS_{q-1,r-1}$. If $a_1 \in T_0\backslash qS_{q-1,r}$ and $a_2 \in qS_{q-1,r}$, then the design D has less GLOC than others and*

$$
{}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \begin{cases} (n - 2^r + 1)(n - N/4), & k = n - N/4 - 1, \\ (N/2 - n - 1)(n - N/4 + 1), & k = n - N/4, \\ (k + 1)\#\{\gamma : \gamma \in H_{q-1}, n^\gamma = v\}, & k = N/4 - 2^{r-1} - 2 + v, \\ & v = 0, 1, \ldots, \lfloor g/2 \rfloor, \\ 0, & otherwise, \end{cases}
$$

where n^γ is the number of 2fi's of $T_0\backslash\{qS_{q-1,r}, a_1\}$ aliased with some $\gamma \in H_{q-1}$ and $g = \#\{T_0\backslash qS_{q-1,r}\}$, $\lfloor g/2 \rfloor$ is the integer part of $g/2$.

The proof of Theorem 2 is in Appendix A.

Based on the above theorem, the best choices for a_1 and a_2 are obtained and the GMC designs can be constructed.

Theorem 3. *Suppose D is a 2^n4^1 design with $5N/16 + 1 \le n + 2 \le N/2$. Up to isomorphism, the design D has GMC if T_0 consists of the last $n + 2$ columns of F_{qq} and a_1, a_2 are the first and last columns of T_0, respectively.*

Proof. Denote $G = T_0\backslash qS_{q-1,r}$ and $g = \#\{G\}$. Consider two cases of G.

(a) $G \ne \Phi$. By Theorem 2, maximizing ${}^{\#}_{2,0}C_{2,0}(D)$ equals to maximize $\#\{\gamma : \gamma \in H_{q-1}, n^\gamma = v\}$, i.e., maximizing ${}^{\#}_{2}C_2(G\backslash a_1)$. On the other hand, $\bar{g}(G\backslash a_1) = \{\gamma : \gamma \in H_q\backslash\{G, a_1\}$, and $B_2(G\backslash a_1, \gamma) > 0\} = 2^{r'+1} - 1$ for some $r' < q$. We only need to sequentially maximize the components of $\{-\bar{g}(G\backslash a_1), {}^{\#}_{2}C_2(G\backslash a_1)\}$, where $G\backslash a_1$ is a $(g - 1)$-factor design with resolution at least IV and $2^{r'} < g - 1 \le 2^{r'+1}$. According to Li et al. (2011), $\{-\bar{g}(G\backslash a_1), {}^{\#}_{2}C_2(G\backslash a_1)\}$ is maximized if and only if $G\backslash a_1$ consists of the last $g - 1$ columns of F_{qr}. That is to say, a_1 must be the first column of T_0. For any $a_2 \in qS_{q-1,r}$, the design is isomorphic. Without loss of generality, a_2 can be the last column of T_0, that is, $a_2 = 12\ldots q$.

(b) $G = \Phi$, i.e., $T_0 = qS_{q-1,r}$. By Lemma 3, for $a_1, a_2 \in T_0$, the design with $a_3 \in S_{q-1,r}$ has less $GLOC$ than a design with $a_3 \in H_r$. Further, for any $a_1, a_2 \in T_0$, we have $a_3 \in S_{q-1,r}$. Up to isomorphism, a_1 and a_2 can be the first and last columns of T_0. □

Example 2. Construct a GMC $2^4 4^1$ design D with 16 runs. Obviously, $n = 4$ and $N = 16$. By Theorem 3, take T_0 to be the last six columns of F_{44}, that is,

$$T_0 = \{24, 124, 34, 134, 234, 1234\}.$$

Moreover, $a_1 = 24$, $a_2 = 1234$, $a_3 = 13$, and $T = \{124, 34, 134, 234\}$. Thus, the GMC $2^4 4^1$ design $D = \{A, 34, 134, 234, 1234\}$, where $A = (24, 1234, 13)$. Up to isomorphism, factors 24, 13, 124, 34 are replaced by factors 1, 2, 3, and 4, respectively. Then,

$$\widetilde{D} = \{(1, 2, 12), 3, 4, 134, 23\},$$

which is just an MA design given in Table 2 ($n = 4$) of Wu and Zhang (1993).

4. GMC $2^n 4^1$ designs with $N/2 + 3 \le n + 2 < N - 1$

For $n > N/2$, the resolution of a 2^{n-p} design T is III. Since ${}^{\#}_{1,0}C_{2,0}(D) = {}^{\#}_{1}C_2(T)$, ${}^{\#}_{1}C_2(T)$ should be maximized first. If $2^{r-1} \le N - 3 - n \le 2^r - 1$ for some r, $H_q \backslash T$ has r independent factors. Therefore, by Theorem 1 of Li et al. (2011), ${}^{\#}_{1}C_2(T)$ is maximized if $S_{qr} \subseteq T \subset S_{q,r-1}$. As a result, D can be rewritten as $D = \{D\backslash S_{qr}, S_{qr}\}$ and ${}^{\#}_{1,0}C_{2,0}(D)$ is maximized. Next, we will maximize ${}^{\#}_{1,0}C_{2,1}(D)$, ${}^{\#}_{1,1}C_{2,0}(D)$, and ${}^{\#}_{2,0}C_{2,0}(D)$. For simplification, let $n_1 = \#\{T\backslash S_{qr}\}$.

Lemma 6. *Suppose $D = \{D\backslash S_{qr}, S_{qr}\}$, where $S_{qr} \subseteq T \subset S_{q,r-1}$. Then*

$${}^{\#}_{1,0}C^{(k)}_{2,1}(D) = \begin{cases} {}^{\#}_{1,0}C^{(k)}_{2,1}(D\backslash S_{qr}), & k < 3, \\ {}^{\#}_{1,0}C^{(k)}_{2,1}(D\backslash S_{qr}) + N - 2^r, & k = 3. \end{cases}$$

Proof. Since $a_1, a_2, a_3 \in H_r$, for any $\gamma \in S_{qr}$, there exists $d_1, d_2, d_3 \in S_{qr}$ satisfying $a_1 d_1 = a_2 d_2 = a_3 d_3 = \gamma$. Each $\gamma \in S_{qr}$ is added to the class of ${}^{\#}_{1,0}C^{(3)}_{2,1}(D)$, the total number is $N - 2^r$. Only $D\backslash S_{qr}$ needs to be considered for other cases. The result is obtained. □

Since ${}^{\#}_{2,0}C_{2,0}(D) = {}^{\#}_{2}C_2(T)$, the following result is obvious from Lemma 4 of Li et al. (2011).

Lemma 7. *Suppose D is a $2^n 4^1$ design and $D = \{D\backslash S_{qr}, S_{qr}\}$. Then,*

$${}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \begin{cases} constant, & k < N/2 - 2^{r-1} - 1, \\ -(k+1)\bar{g}(T\backslash S_{qr}) + {}^{\#}_{1}C^{(0)}_2(T), & k = N/2 - 2^{r-1} - 1, \\ \frac{k+1}{\upsilon+1}\,{}^{\#}_{2}C^{(\upsilon)}_2(T\backslash S_{qr}), & k = N/2 - 2^{r-1} + \upsilon, \\ & \upsilon = 0, 1, \ldots, \lfloor n_1/2 \rfloor, \\ 0, & otherwise. \end{cases}$$

Two different cases are respectively discussed in the following.

4.1. GMC $2^{n_1} 4^1$ designs with $2^{r-2} + 1 \le n_1 + 2 \le 2^{r-1}$

Based on Lemma 4, we can get

$${}^{\#}_{1,0}C_{2,1}(D\backslash S_{qr}) = (2^{r-1} - n_1 - 2, 2n_1 - 2^{r-1} + 2).$$

By Lemma 6,

$$ {}^{\#}_{1,0}C_{2,1}(D) = (2^{r-1} - n_1 - 2, 2n_1 - 2^{r-1} + 2, 0, N - 2^r). $$

For any $a_i \in H_r$, there exist $N/2 - 2^{r-1}$ pairs of 2fi's aliased with a_i for $i = 1, 2, 3$. Thus, ${}^{\#}_{1,1}C_{2,0}(D) = (0^{N/2-2^{r-1}}, {}^{\#}_{1,1}C_{2,0}(D\backslash S_{qr}))$. As a result ${}^{\#}_{1,0}C_{2,1}(D)$ is maximized. By Lemma 4, we have ${}^{\#}_{1,1}C_{2,0}(D\backslash S_{qr}) = (2, 0^{n_1-2^{r-2}}, 1)$. Further, ${}^{\#}_{1,1}C_{2,0}(D)$ is maximized. Next we will maximize ${}^{\#}_{2,0}C_{2,0}(D)$, which is completely determined by the small design $D\backslash S_{qr}$.

According to Cheng and Zhang (2010), let

$$ \mathbf{X} = \begin{pmatrix} \mathbf{1} & S_1(t) & S_2(t) \\ -\mathbf{1} & -S_1(t) & S_2(t) \end{pmatrix}, \quad (6) $$

where $\mathbf{1}$ is a $2^{t-1} \times 1$ vector of 1's. $S(t) = (S_1(t), S_2(t))$ is the resolution IV design with 2^{t-1} runs and 2^{t-2} factors, $S_1(t)$ is any column of $S(t)$. Thus, (6) can be rewritten as

$$ \mathbf{X} = (X_1, X_2, \ldots, X_{2^{t-2}+1}), $$

where X_i is the ith column of $\mathbf{X}$. Doubling $\mathbf{X}$ $q - t$ times, we can obtain

$$ D^{(q-t)}(\mathbf{X}) = \left(D^{(q-t)}(X_1), D^{(q-t)}(X_2), \ldots, D^{(q-t)}(X_{2^{t-2}+1})\right). $$

Denote $D^{(q)}(1) = (D^{(q-t)}(D^t(1)\backslash\mathbf{X}), D^{(q-t)}(\mathbf{X}))$, which is said to have *RC Yates order.*

Theorem 4. *Suppose $D\backslash S_{qr}$ is a $2^{n_1}4^1$ design with 2^r runs and $D\backslash S_{qr} = \{\hat{T}, A\}$.*

- *(a) For $5 \cdot 2^r/16 + 1 \le n_1 + 2 \le 2^{r-1}$, $D\backslash S_{qr}$ is a GMC design, up to isomorphism, if $\hat{T}$ consists of the last $n_1 + 2$ columns of F_{rr} and a_1, a_2 are the first and last column of $\hat{T}$ respectively.*
- *(b) For $(2^{t-1} + 1)2^r/2^{t+1} + 1 \le n_1 + 2 \le (2^{t-2} + 1)2^r/2^t$ $(4 \le t \le q)$, $D\backslash S_{qr}$ is a GMC design, up to isomorphism, if a_1 is the first column of $D^{q-t}(\mathbf{X})$ and a_2 is the first column of $D^{q-t}(X_2)$, where $\{\hat{T}, a_2\}$ consists of the last $n_1 + 1$ columns of $D^{q-t}(\mathbf{X})$.*

Based on Theorem 3, Theorem 4 (a) is obtained. Meanwhile, by ZYLZ (2015), Theorem 4 (b) is obtained. Combining $D\backslash S_{qr}$ and S_{qr}, the GMC 2^n4^1 design D is obtained.

4.2. GMC $2^{n_1}4^1$ designs with $n_1 + 2 \le 2^{r-2}$

In this case, the resolution of $T\backslash S_{qr}$ is at least IV, then ${}^{\#}_{1}C_2^{(k)}(T\backslash S_{qr}) = 0$ for $k > 0$. Thus, ${}^{\#}_{1,0}C_{2,0}(D)$ is maximized. Up to isomorphism, let $T\backslash S_{qr} \subseteq F_{rr}$. For $n_1 + 2 \le 2^{r-2}$, $F_{rr}\backslash\{T\backslash S_{qr}\}$ has no less than 2^{r-2} factors. It is easy to find $a_1, a_2 \in F_{rr}\backslash\{T\backslash S_{qr}\}$ such that $B_2(T\backslash S_{qr}, a_i) = 0$ $(i = 1, 2, 3)$. For example, let $a_1 = r$, $a_2 = (r-1)r$, and $\{T\backslash S_{qr}\}$ be the last n_1 columns of F_{rr}. By Lemma 2, $B_2(T, a_i) = N/2 - 2^{r-1}$, so ${}^{\#}_{1,1}C_{2,0}(D) = (0^{N/2-2^{r-1}}, 3)$. For any $\gamma \in S_{qr}$ there exist $d_1, d_2, d_3 \in S_{qr}$ satisfying $a_1d_1 = a_2d_2 = a_3d_3 = \gamma$ when $a_1, a_2, a_3 \in H_r$. Thus, ${}^{\#}_{1,0}C_{2,1}(D) = (n - N + 2^r, 0, 0, N - 2^r)$.

By computation and three steps (see the proof in Appendix A), the following result is given.

Theorem 5. *Suppose D is a 2^n4^1 design with N runs. Then, for $n > N/2$ and $n + 2^r - N < 2^{r-2}$, the design D has GMC if T consists of the last n columns of H_q and $a_1 = r$, $a_2 = r(r-1)$.*

The proof of Theorem 5 is given in Appendix A. The following example is used for illustration of Theorem 5.

Example 3. Construct a GMC $2^{18}4^1$ design D with 32 runs. Since $n = 18$, $N = 32$, $q = 5$, $r = 4$, one has $n > N/2$, $n + 2^r - N < 2^{r-2}$. By Theorem 5, $D = \{A, 234, 1234, S_{54}\}$ is a GMC

$2^{18}4^1$ design, where $A = (3, 4, 34)$ and $S_{54} = H_5 \setminus H_4$. Clearly, $D \setminus S_{54}$ is an MA design with 32 runs. By Theorem 1 of Li, Liu and Zhang (2007), D is also an MA design with 32 runs.

5. Conclusions

Zhang et al. (2015) extended the GMC theory to mixed-level case. Under this theory, a new criterion is established for choosing optimal regular two- and four-level designs, and all the GMC 2^n4^1 designs for $N/4 + 1 \le n + 2 \le 5N/16$ are constructed. In this article, a construction method is proposed to obtain all the GMC 2^n4^1 designs for $5N/16 + 1 \le n + 2 \le N - 1$. Tables 1 and 2 in Appendix B provide GMC 2^n4^1 designs with 16 and 32 runs for practical use. This method is possible to be extended to the construction of GMC 2^n4^m designs, and further for GMC $s^n(s^2)$ or $s^n(s^2)^2$ designs with general level s. This is an open problem for further study.

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Appendix A: Proofs

Proof of Lemma 5:

By Lemma 1, we get $B_2(T,\gamma) = B_2(F_{qq}\backslash T,\gamma) + n - N/4$. The connection between $F_{qq}\backslash T$ and $F_{qq}\backslash T_0$ is very important, we study it now.

$$\begin{aligned}B_2(F_{qq}\backslash T,\gamma) = B_2(F_{qq}\backslash T_0 \cup \{a_1,a_2\},\gamma) = B_2(F_{qq}\backslash T_0,\gamma)\\ + \#\{d : a_i d = \gamma, d \in F_{qq}\backslash T_0, i = 1,2\} + \#\{(a_1,a_2) : a_1a_2 = \gamma\}.\end{aligned}$$

If $\gamma = a_3$, then $\#\{(a_1,a_2) : a_1a_2 = \gamma\} = 1$, $\#\{d : a_i d = \gamma, d \in F_{qq}\backslash T_0, i = 1,2\} = 0$, and $B_2(F_{qq}\backslash T_0, a_3) = 0$. Further, $B_2(T, a_3) = n - N/4 + 1$. If $\gamma \neq a_3$, we have $\#\{(a_1,a_2) : a_1a_2 = \gamma\} = 0$. Let $\delta = \#\{d : a_i d = \gamma, d \in F_{qq}\backslash T_0, i = 1,2\}$, then

$$B_2(T,\gamma) = B_2(F_{qq}\backslash T_0,\gamma) + n - N/4 + \delta, \tag{A1}$$

where δ is no more than 1. Otherwise, there exist $d_1, d_2 \in F_{qq}\backslash T_0$ such that $a_1d_1 = a_2d_2 = \gamma$. Then $a_3 = d_1d_2$ and $B_2(F_{qq}\backslash T_0, a_3) > 0$, which contradicts to the condition $B_2(F_{qq}\backslash T_0, a_3) = 0$.

When $B_2(F_{qq}\backslash T_0,\gamma) = 0$, there are three different cases: (i) $\delta = 0$, that is to say, $d_1, d_2 \in T$, (ii) $\gamma = a_3$, and (iii) $\delta = 1$, that is, one of d_1, d_2 is in T, the other is in $F_{qq}\backslash T_0$. Without loss of generality, suppose d_1 is from T and d_2 is from $F_{qq}\backslash T_0$. It can be written in the form:

$$\begin{aligned}&\#\{\gamma : \gamma \in H_{q-1}, B_2(F_{qq}\backslash T_0,\gamma) = 0\}\\ &\quad = \#\{\gamma : \gamma \in H_{q-1}, a_1d_1 = a_2d_2 = \gamma, B_2(T,\gamma) = n - N/4, d_1, d_2 \in T\} + \#\{a_3\}\\ &\qquad + \#\{\gamma : \gamma \in H_{q-1}, a_1d_1 = a_2d_2 = \gamma, B_2(T,\gamma) = n - N/4 + 1, d_1 \in T, d_2 \in F_{qq}\backslash T_0\}.\end{aligned} \tag{A2}$$

For any $d_2 \in F_{qq}\backslash T_0$, we can find $d_1 \in T$ satisfying $a_1 d_1 = a_2 d_2$ and $B_2(T,\gamma) = n - N/4 + 1$. Then,

$$\begin{aligned}&\#\{\gamma : \gamma \in H_{q-1}, a_1 d_1 = a_2 d_2 = \gamma, B_2(T,\gamma) = n - N/4 + 1, d_1 \in F_{qq}\backslash T_0, d_2 \in T\}\\ &\quad = \#\{d_i : d_i \in F_{qq}\backslash T_0\} = N/2 - (n+2). \end{aligned} \tag{A3}$$

On the other hand, we have

$$\#\{\gamma : \gamma \in H_{q-1}, B_2(F_{qq}\backslash T_0, \gamma) = 0\} = N/2 - 1 - \bar{g}(F_{qq}\backslash T_0). \tag{A4}$$

By Equations (A2)–(A4),

$$\#\{\gamma : \gamma \in H_{q-1}, a_1 d_1 = a_2 d_2 = \gamma, B_2(T,\gamma) = n - N/4, d_1, d_2 \in T\} = n - \bar{g}(F_{qq}\backslash T_0).$$

Then, for $k = n - N/4 - 1$,

$${}^{\#}_{2,0}C^{(n-N/4-1)}_{2,0}(D) = (n - N/4)(n - \bar{g}(F_{qq}\backslash T_0)).$$

When $k = n - N/4$, by Equation (A3),

$$\begin{aligned}{}^{\#}_{2,0}C^{(n-N/4)}_{2,0}(D) &= (n - N/4 + 1)(N/2 - (n+2) + \#\{a_3\})\\ &= (n - N/4 + 1)(N/2 - n - 1).\end{aligned}$$

When $k > n - N/4$, by Equation (A1), the following equation is obtained.

$${}^{\#}_{2,0}C^{(k)}_{2,0}(D) = \frac{k+1}{k+1-n+N/4-\delta}\ {}^{\#}_{2}C^{(k-n+N/4-\delta)}_{2}(F_{qq}\backslash T_0).$$

□

Proof of Theorem 2:

There are two cases to be considered.

Case 1: If $a_1 \in T_0\backslash qS_{q-1,r}$, $a_2 \in qS_{q-1,r}$ and $k = n - N/4 - 1$, by Lemma 3, γ must be in $S_{q-1,r}$. In this case, the alias set has the form $\{\gamma = a_1 t_1 = a_2 t_2, t_1, t_2 \in T\}$, then $t_1 \in qS_{q-1,r}$. Since $a_2 \in qS_{q-1,r}$ and $t_2 \in F_{qq} \setminus T_0$ also leads to $a_2 t_2 \in S_{q-1,r}$. However, this case should be excluded. Moreover,

$$\begin{aligned}\#\{\gamma : B_2(T,\gamma) = n - N/4, \gamma \in S_{q-1,r}\} &= \#\{\gamma : \gamma = a_1 t_1 \in S_{q-1,r}, t_1 \in qS_{q-1,r}\}\\ &\quad - \#\{\gamma : \gamma = a_2 t_2 \in S_{q-1,r}, t_2 \in F_{qq}\backslash T_0\} - \#\{a_3\}\\ &= (N/2 - 2^r) - (N/2 - n - 2) - 1 = n - 2^r + 1.\end{aligned}$$

When $k = n - N/4$, by Lemma 5, the result is obvious. If $k > n - N/4$, γ must be in H_r. The smallest integer n is $N/4 - 2^{r-1} - 2$ such that $B_2(T,\gamma) > n - N/4$. By Lemma 3, for $k \geq N/4 - 2^{r-1} - 2$,

$$\begin{aligned}&\#\{\gamma : B_2(T,\gamma) = k+1, \gamma \in H_r\}\\ &\quad = \#\{\gamma : B_2(qS_{q-1,r}\backslash\{a_2\}, \gamma) + B_2(T_0\backslash\{qS_{q-1,r}, a_1\}, \gamma) = k+1, \gamma \in H_r\}\\ &\quad = \#\{\gamma : N/4 - 2^{r-1} - 1 + B_2(T_0\backslash\{qS_{q-1,r}, a_1\}, \gamma) = k+1, \gamma \in H_r\}.\end{aligned}$$

Denote $n^\gamma = B_2(T_0\backslash\{qS_{q-1,r}, a_1\}, \gamma)$ and $k = N/4 - 2^{r-1} - 2 + \upsilon$, we can get the result

$${}^{\#}_{2,0}C^{(k)}_{2,0}(D) = (k+1)\#\{\gamma : \gamma \in H_r, n^\gamma = \upsilon\}.$$

Case 2: Suppose a_1, a_2 are in $qS_{q-1,r}$, s.t., $a_3 \in S_{q-1,r}$. In the case $k = n - N/4 - 1$, γ must be in $S_{q-1,r}$ and the alias set has the form $\{\gamma = a_1 t_1 = a_2 t_2, t_1, t_2 \in T\}$, then

$$\begin{aligned}
&\#\{\gamma : B_2(T_0\backslash\{a_1, a_2\}, \gamma) = n - N/4, \gamma \in S_{q-1,r}\} \\
&= \#\{\gamma : \gamma \in S_{q-1,r}\} - \#\{\gamma : \gamma = a_1 t_1 \in S_{q-1,r}, t_1 \in F_{qq}\backslash T_0\} \\
&\quad -\#\{\gamma : \gamma = a_2 t_2 \in S_{q-1,r}, t_2 \in F_{qq}\backslash T_0\} - \#\{a_3\} \\
&= (N/2 - 2^r) - 2(N/2 - n - 2) - 1 \\
&= 2n - N/2 - 2^r + 3 < n - 2^r + 1 \quad (for \quad n + 2 < N/2).
\end{aligned}$$

So Case 1 has less *GLOC* than Case 2. □

Proof of Theorem 5:

By Lemma 7, the GMC design must maximize $\{-\bar{g}(T\backslash S_{qr}), {}^{\#}_{2}C_2(T\backslash S_{qr})\}$. Since the resolution of $T\backslash S_{qr}$ is at least IV and $2^{s-2} + 1 < \#\{T\backslash S_{qr}\} \leq 2^{s-1}$ for some $s < r$. $\{-\bar{g}(T\backslash S_{qr}), {}^{\#}_{2}C_2(T\backslash S_{qr})\}$ is maximized if and only if $T\backslash S_{qr}$ consists of the last $n - (N - 2^r)$ columns of F_{rr} by Li et al. (2011). Combining S_{qr} with $T\backslash S_{qr}$, T consists of the last n columns of H_q. According to the above proof, since $a_1, a_2 \in F_{rr}\backslash\{T\backslash S_{qr}\}$ satisfying $a_1 a_2 \in (r-1)H_{r-2}$, we have ${}^{\#}_{1,0}C_{2,0}, {}^{\#}_{1,0}C_{2,1}, {}^{\#}_{1,1}C_{2,0}, {}^{\#}_{2,0}C_{2,0}$ of D are maximized. Without loss of generality, let $a_1 \in rH_{r-2}$ and $a_2 \in r(r-1)F_{r-2,r-2}\backslash\{T\backslash S_{qr}\}$, then $a_3 \in (r-1)H_{r-2}$.

Next we will maximize ${}^{\#}_{2,0}C_{2,1}(D)$. Only if $t_1 \in S_{qr}, t_2 \in T\backslash S_{qr}$ (or $t_2 \in S_{qr}, t_1 \in T\backslash S_{qr}$), the alias set has the form $a_1 d_1 = a_2 d_2 = a_3 d_3 = t_1 t_2$, where $d_1, d_2, d_3 \in T$. Then

$${}^{\#}_{2,0}C^{(3)}_{2,1}(D) = \#\{(t_1, t_2) : t_1 \in S_{qr}, t_2 \in T\backslash S_{qr}\} = (n - (N - 2^r))(N - 2^r).$$

There are three cases of ${}^{\#}_{2,0}C_{2,1}(D)$ for $0 < k < 3$.

(i) If $d_1 \in T\backslash S_{qr}$, d_2 must be in rH_{r-2} and d_3 must be in H_{r-1}. Then $B_2(T\backslash S_{qr}, a_1 d_1) = 0$, $B_2(T, a_1 d_1) = B_2(S_{qr}, a_1 d_1) = N/2 - 2^r$.

(ii) If $d_2 \in T\backslash S_{qr}$, d_1 must be in rH_{r-2} and d_3 must be in H_{r-1}. Then $B_2(T, a_2 d_2) = B_2(T\backslash S_{qr}, a_2 d_2) + N/2 - 2^r$.

(iii) If $d_3 \in T\backslash S_{qr}$, d_1, d_2 must be in H_{r-1}. Then $B_2(T\backslash S_{qr}, a_3 d_3) = 0$, $B_2(T, a_3 d_3) = B_2(S_{qr}, a_3 d_3) = N/2 - 2^r$.

All of the above cases are $k = 1$, so ${}^{\#}_{2,0}C^{(2)}_{2,1}(D) = 0$, ${}^{\#}_{2,1}C^{(1)}_{2,1}(D) = 0$, and

$$\begin{aligned}
{}^{\#}_{2,0}C^{(1)}_{2,1}(D) &= \sum_{i=1,2,3} \sum_{d_i \in T\backslash S_{qr}} B_2(T, a_1 d_1) \\
&= 3(n - (N - 2^r))(N/2 - 2^r) + \sum_{d_2 \in T\backslash S_{qr}} B_2(T\backslash S_{qr}, a_2 d_2), \\
{}^{\#}_{2,0}C^{(0)}_{2,1}(D) &= n(n-1)/2 - \sum_{k>0} {}^{\#}_{2,0}C^{(k)}_{2,1}(D) = constant - \sum_{d_2 \in T\backslash S_{qr}} B_2(T\backslash S_{qr}, a_2 d_2).
\end{aligned}$$

We will minimize $\sum_{d_2 \in T\backslash S_{qr}} B_2(T\backslash S_{qr}, a_2 d_2)$. Let $G = T\backslash S_{qr}$ and $g = \#\{G\} = n - (N - 2^r)$. Since G consists of the last g columns of F_{rr}, we only need to minimize $\sum_{d_2 \in G} B_2(G, a_2 d_2)$ for $a_2 \in r(r-1)H_{r-2}\backslash G$. There is a simple method to find d_2.

First set $N_1 = 2^{r-1}$, then go to P1.

P1: If $g \leq N_1/4$, let $a_2 = r(r-1)$, by Lemma 1, $\sum_{d_2 \in G} B_2(G, a_2 d_2) = 0$. Stop the computation.

P2: If $g > N_1/4$, $B_2(G, a_2 d_2) = B_2(G\backslash r(r-1)(r-2)H_{r-3}, a_2 d_2) + n_0 - 2^{r-4}$. Go to P3.

P3: Replace G as $G\backslash r(r-1)(r-2)H_{r-3}$, g as $\#\{G\backslash r(r-1)(r-2)H_{r-3}\}$ and N_1 as $N_1/2$. Repeat P1 and P2.

The above method shows $a_2 = r(r-1)$ is the best one in any cases. For any $a_1 \in F_{r,r-2}$, it leads to the isomorphic design, without loss of generality, let $a_1 = r$. □

Appendix B: GMC 2^n4^1 designs with 16 and 32 runs

Table 1. GMC 2^n4^1 designs with 16 runs.

n	Columns	${}^{\#}_{1,0}C_{2,0}$, ${}^{\#}_{1,0}C_{2,1}$, ${}^{\#}_{1,1}C_{2,0}$, ${}^{\#}_{2,0}C_{2,0}$, ${}^{\#}_{2,0}C_{2,1}$, ${}^{\#}_{2,1}C_{2,0}$, ${}^{\#}_{2,1}C_{2,1}$	WLP $(A_{30}, A_{31}), \ldots$	$c_1, \bar{c}_1$ $c_2, \bar{c}_2$
4	(24, 1234, 13), 124, 34, 134, 234	(4), (2, 2), (2, 1), (6), (1, 4, 1), (6, 6), (11, 1)	(0, 1), (0, 2)	2, 2 0, 4
5	(14, 1234, 23), 24, 124, 34, 134, 234	(5), (1, 4), (2, 0, 1), (4, 6), (2, 4, 4), (5, 8, 2), (7, 8)	(0, 2), (1, 4)	1, 2 0, 5
6	A_1, 14, 24, 124, 34, 134, 234	(6), (0, 6), (2, 0, 0, 1) (0, 12, 3), (3, 0, 12), (6, 0, 12), (6, 12)	(0, 3), (3, 8), (0, 0), (0, 0), (0, 1)	0, 2 0, 6
7	A_0, $F_{44}\backslash\{4\}$	(7), (0, 0, 3, 4), (0, 0, 0, 3), (0, 0, 21), (21), (21), (6, 15)	(0, 9), (7, 0), (0, 12), (0, 0), (0, 3)	0, 0 0, 0
8	A_0, F_{44}	(8), $(0^3, 8)$, $(0^4, 3)$, $(0^3, 28)$, (28), (28), (0, 0, 24)	(0, 12), (14, 0), (0, 24), (0, 0), (0, 12), (1, 0)	0, 0 0, 0
9	A_2, 23, F_{44}	(0, 8, 0, 0, 1), (1, 0, 0, 8), $(0^3, 3)$, (8, 0, 0, 28), (8, 12, 0, 8), (0, 24, 0, 0, 3), (3, 0, 24)	(4, 12), (14, 12), (8, 24), (0, 24), (4, 12), (1, 12)	0, 0 0, 0
10	A_2, 13, 23, F_{44}	(0, 0, 8, 0, 2), (0, 2, 8), $(0^4, 2, 1)$, (0, 16, 0, 24, 5), (13, 8, 8, 16), (0, 0, 24, 0, 6), (2, 4, 24)	(8, 13), (18, 24), (16, 32), (8, 48), (8, 42), (5, 24) (0, 8), (0, 0), (0, 1)	0, 0 0, 0
11	A_0, 3, 13, 23, F_{44}	$(0^3, 8, 3)$, (0, 0, 3, 8), $(0^5, 3)$ $(0^3, 24, 16, 15)$, (15, 0, 12, 28), $(0^4, 30, 3)$, (0, 30, 3)	(12, 115), (26, 36), (28, 48), (24, 84), (20, 102), (13, 60) (4, 24), (0, 12) (0, 3)	0, 0 0, 0

Note: 1. $A_0 = (1, 2, 12)$, $A_1 = (4, 1234, 123)$, $A_2 = (3, 123, 12)$, $F_{44} = \{4, 14, 24, 124, 34, 134, 234, 1234\}$.
2. c_1 and c_2 are the numbers of clear two-level main effects and 2fi's respectively, and $\bar{c}_1$ and $\bar{c}_2$ are the numbers of clear four-level components and clear 2fi's containing four-level components.

Table 2. GMC $2^n 4^1$ designs with 32 runs.

n	Columns	${}^{\#}_{1,0}C_{2,0}$, ${}^{\#}_{1,0}C_{2,1}$, ${}^{\#}_{1,1}C_{2,0}$, ${}^{\#}_{2,0}C_{2,0}$, ${}^{\#}_{2,0}C_{2,1}$, ${}^{\#}_{2,1}C_{2,0}$, ${}^{\#}_{2,1}C_{2,1}$	WLP $(A_{30}, A_{31}), \ldots$	$c_1, \bar{c}_1$ $c_2, \bar{c}_2$
10	A_1, 135 ∼ 2345	(10), (4, 6), (2, 0, 0, 1), (0, 6, 27, 12), (3, 27, 15), (10, 0, 6, 11, 3), (18, 12)	(0, 3), (16, 19), (0, 13), (12, 27), (0, 13), (3, 17), (0, 3), (0, 1)	4, 2 0, 10
11	A_1, 35 ∼ 2345	(11), (3, 8), (2, 0^3, 1), (0, 0, 24, 16, 15), (4, 27, 24), (11, 0, 0, 16, 3, 3), (17, 16)	(0, 4), (26, 25), (0, 20), (24, 52), (0, 28), (13, 46) (0, 12), (0, 4), (0, 0), (0, 1)	3, 2 0, 11
12	A_1, 125 ∼ 2345	(12), (2, 10), (2, 0^4, 1), (0^3, 36, 15, 15), (5, 20, 41), (21, 0^4, 12, 12), (12, 24)	(0, 5), (38, 34), (0, 28), (52, 88), (0, 62), (33, 108) (0, 28), (4, 24), (0, 5), (0, 2)	2, 2 0, 12
13	A_1, 25 ∼ 2345	(13), (1, 12), (2, 0^5, 1), (0^5, 60, 18), (6, 6, 66), (13, 0^4, 24, 2), (15, 24)	(0, 6), (55, 44), (0, 40), (96, 144), (0, 116), (87, 232) (0, 72), (16, 80),	1, 2 0, 13
14	A_1, 15 ∼ 2345	(14), (0, 14), (2, 0^6, 1), (0^5, 84, 7), (7, 0, 87), (14, 0^6, 28), (14, 28)	(0, 7), (77, 56), (0, 56), (168, 224), (0, 203), (203, 464) (0, 168), (56, 224), (0, 77), (7, 56) (0, 0), (0, 0), (0, 1)	1, 2 0, 13

Note: 1. $A_1 = (5, 12345, 1234)$, $F_{55} = \{5, 15, 25, 125, 35, 45, 145, 245, 1245, 345, 1345, 2345, 12345\}$.
2. $\{a \sim b\}$ means all columns from a to b of F_{55}.