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Construction of main-effect plans orthogonal through the block factor[☆]

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ABSTRACT

Bagchi (2010) proposed main-effect plans orthogonal through the block factor (POTB), in which the treatment factors are pairwise orthogonal through the block factor. However, not many construction methods are available in the literature. In this paper, we present several new construction approaches for saturated POTBs with small runs and mixed levels. Moreover, all of them are connected and variance-balanced.

1. Introduction

Main-effect plans play an important role in many industrial experiments when interest lies only in the main effects. A main-effect plan **D** is said to be an orthogonal main-effect plan if any two of its factors are orthogonal. According to Addelman (1962), two factors, F_1 and F_2 (with p_1 and p_2 levels, respectively), of a main-effect plan **D** with n runs are said to be orthogonal (to each other) if they satisfy the *proportional frequency condition*: for every $i = 0, 1, \ldots, p_1 - 1$ and every $j = 0, 1, \ldots, p_2 - 1$, the number of runs in which F_1 is at level i and F_2 is at level j is proportional to the product of the frequencies of level i of F_1 and level j of F_2.

It is known that the only problem with an orthogonal main-effect plan is that the plan often requires a large number of runs due to the requirement of proportional frequency condition, particularly for asymmetric factorials. Mukerjee et al. (2002) provided main-effect plans on blocks of a given size, in which every treatment factor may be nonorthogonal to the block factor. The method in Mukerjee et al. (2002) relies on an existing orthogonal main-effect plan. Bose and Bagchi (2007) obtained orthogonal main-effect plans satisfying the property of plans in Mukerjee et al. (2002) but requiring fewer blocks. Recently, Bagchi (2010) obtained main-effect plans orthogonal through the block factor (POTB) for a $p^{3m}2^{3m}$ experiment in

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Table 1
A $\rho_{20}(5; 5^3 2^3)$ with five blocks of size 4 each.

1 000 000	2 111 000	3 222 000	4 333 000	5 444 000
1 011 011	2 122 011	3 233 011	4 344 011	5 400 011
1 101 101	2 212 101	3 323 101	4 434 101	5 040 101
1 110 110	2 221 110	3 332 110	4 443 110	5 004 110

pm blocks each of size 4, where m is a Hadamard number. Here, $p^{3m}2^{3m}$ indicates that the plan has $3m$ factors of p levels and $3m$ factors of two levels. Note that the run size of a POTB is much smaller than the one of an orthogonal main-effect plan. In such plans, the two-level factors are still orthogonal to the block factor, and the p-level factors are nonorthogonal to the block factor but are pairwise orthogonal through the block factor. Furthermore, Bagchi (2010) demonstrated that in any POTB, every main-effect contrast of every factor is estimable and the BLUEs of main-effect contrast for any two factors F_1 and F_2 are uncorrelated.

However, not many construction methods for POTBs are available in the literature. Bagchi (2010) obtained a construction method for saturated connected POTBs $p^{3m}2^{3m}$ in pm blocks each of size 4. Obviously, the run size of the POTBs in Bagchi (2010) must be $n = 4pm$. In this paper, several new constructions for POTBs are discussed. Then many new POTBs of run size $n \neq 4pm$ with flexible levels of factors are tabulated for practical use.

The article proceeds as follows. Section 2 introduces some basic concepts and notations of POTBs. In Section 3 we present direct as well as recursive constructions for asymmetrical saturated POTBs, which are all connected and variance-balanced. Conclusions will be drawn in Section 4.

2. POTB

We first give some notation and background. Following Bagchi (2010), consider a plan $\mathbf{D}$ for an experiment with factors $F_0, F_1, F_2, \ldots, F_m$ at $b, p_1, p_2, \ldots, p_m$ levels on n runs, respectively. The plan $\mathbf{D}$ is said to be saturated if $\sum_{i=1}^{m}(p_i - 1) + (b - 1) = n - 1$. Note that there is a block factor (represented by F_0) apart from the m treatment factors $F_1, F_2, \ldots, F_m$ and the design has b blocks each of size $k = n/b$. For $s = 0, 1, \ldots, m$, let the $n \times p_1$ matrix $\mathbf{X}_s$ be the *incidence matrix* of factor F_s, in which the (u, i)th entry is 1 if the factor F_s is set at level i in the uth run and 0 otherwise. For $s, t = 0, 1, \ldots, m$, let the $p_1 \times p_2$ matrix $\mathbf{M}_{s,t}$ be the incidence matrix of factor F_s versus factor F_t, where the (i, j)th entry is the number of runs in which F_s is set at level i and F_t is set at level j. Clearly, $\mathbf{M}_{s,0}$ represents the incidence matrix of the treatment factor F_s versus the block factor F_0 for $s = 1, \ldots, m$. For the relationship between $\mathbf{X}_s$, $\mathbf{X}_t$ and $\mathbf{M}_{s,t}$, it is easy to see that $\mathbf{M}_{s,t} = \mathbf{X}_s'\mathbf{X}_t$, where $\mathbf{X}'$ denotes the transpose of matrix $\mathbf{X}$. We now give the definition of a POTB in Bagchi (2010).

Definition 2.1. For $1 \le s \neq t \le m$, two factors F_s and F_t are said to be orthogonal through the block factor F_0 if they satisfy

$$\boldsymbol{M}_{s,0}\boldsymbol{M}_{0,t} = k\boldsymbol{M}_{s,t}, \quad (1)$$

where k is the block size. A design $\boldsymbol{D}$ is said to be a POTB if each pair of treatment factors of $\boldsymbol{D}$ is orthogonal through the block factor.

Morgan and Uddin (1996) studied main effect plans on a nested row–column set up satisfying condition (1) and noted their interesting properties.

Throughout the paper, a POTB with n runs, b blocks, m treatment factors of levels $p_1, p_2, \ldots, p_m$, is denoted by $\rho_n(b; p_1p_2\cdots p_m)$. Thus, a POTB with n runs, b blocks, m_i treatment factors of p_i levels, $i = 1, \ldots, s$, is denoted by $\rho_n(b; p_1^{m_1}\cdots p_s^{m_s})$. On the other hand, a POTB $\rho_n(b; p_1p_2\cdots p_m)$ can also be denoted by an $n \times (m+1)$ matrix $\mathbf{D} = [\mathbf{F}_0\ \mathbf{F}_1\ \mathbf{F}_2 \cdots \mathbf{F}_m]$, where the first column vector $\mathbf{F}_0$ represents the block factor F_0 of b levels and the subsequent columns $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_m$ represent the treatment factors $F_1, F_2, \ldots, F_m$. Thus, the ith run of the POTB is in the lth block if the ith row of $\mathbf{F}_0$ is set at level l, where $1 \le i \le n$, $1 \le l \le b$. The arrangement of level combinations of factors are represented by rows and can be found in the corresponding constructions. Now we present a design for illustration.

Example 2.1. Consider the design $\rho_{20}(5; 5^3 2^3)$ constructed in Bagchi (2010). It contains three treatment factors of five levels and three treatment factors of two levels. To save space we have broken the 20×6 array $D = [F_0 \cdots F_5]$ into five 4×6 arrays: the ith one being the ith block, $i = 1, 2, \ldots, 5$, and place them side-wise. See Table 1.

3. Construction of POTBs

3.1. Construction of $\rho_{2p}(2; p^2)$

Let (p) be the column vector $(0, 1, \ldots, p-1)'$ and $\mathbf{1}_n$ be an $n \times 1$ vector with all elements unity. For a constant c, let $c \times \mathbf{1}_n$ be an $n \times 1$ vector with all elements c. Define

$$\mathbf{D}^{ij} = [\mathbf{F}_0\ \mathbf{F}_1\ \mathbf{F}_2] = \begin{bmatrix} \mathbf{1}_p & (p) & i \times \mathbf{1}_p \\ 2 \times \mathbf{1}_p & j \times \mathbf{1}_p & (p) \end{bmatrix}, \quad 0 \le i, j \le p-1. \quad (2)$$

Table 2
Plans of $\rho_6(2; 3^2)$ constructed by Theorem 3.1.

	$\mathbf{D}^{01}$	$\mathbf{D}^{02}$	$\mathbf{D}^{10}$	$\mathbf{D}^{11}$	$\mathbf{D}^{12}$	$\mathbf{D}^{20}$	$\mathbf{D}^{21}$	$\mathbf{D}^{22}$
Block 1	1 0 0	1 0 0	1 0 1	1 0 1	1 0 1	1 0 2	1 0 2	1 0 2
	1 1 0	1 1 0	1 1 1	1 1 1	1 1 1	1 1 2	1 1 2	1 1 2
	1 2 0	1 2 0	1 2 1	1 2 1	1 2 1	1 2 2	1 2 2	1 2 2
Block 2	2 1 0	2 2 0	2 0 0	2 1 0	2 2 0	2 0 0	2 1 0	2 2 0
	2 1 1	2 2 1	2 0 1	2 1 1	2 2 1	2 0 1	2 1 1	2 2 1
	2 1 2	2 2 2	2 0 2	2 1 2	2 2 2	2 0 2	2 1 2	2 2 2

Theorem 3.1. *The plan* $\mathbf{D}^{ij}$ *constructed in (2) is a saturated POTB* $\rho_{2p}(2; p^2)$.

Proof. Without loss of generality, we only give a proof for the case of $i = j = 0$ and other cases can be obtained similarly. For $i = j = 0$, it can be verified that the incidence matrices of factors F_1 versus F_2, F_1 versus F_0, and F_0 versus F_2 are

$$\mathbf{M}_{1,2} = \begin{bmatrix} 2 & \mathbf{1}'_{p-1} \\ \mathbf{1}_{p-1} & \mathbf{0}_{p-1,p-1} \end{bmatrix}, \qquad \mathbf{M}_{1,0} = \begin{bmatrix} 1 & p \\ \mathbf{1}_{p-1} & \mathbf{0}_{p-1} \end{bmatrix}, \qquad \mathbf{M}_{0,2} = \begin{bmatrix} p & \mathbf{0}'_{p-1} \\ 1 & \mathbf{1}'_{p-1} \end{bmatrix}, \tag{3}$$

respectively, where $\mathbf{0}_{p-1,p-1}$ is a $(p-1) \times (p-1)$ matrix of all elements zero. Thus, the equation $\mathbf{M}_{1,0}\mathbf{M}_{0,2} = k\mathbf{M}_{1,2}$ holds, where $k = p$. This completes the proof.

Example 3.1. A $\rho_6(2; 3^2)$ with two blocks each of size 3 can be constructed from (2) by putting $i = j = 0$ as

$$\mathbf{D}^{00} = [\mathbf{F}_0\, \mathbf{F}_1\, \mathbf{F}_2] = \begin{bmatrix} \mathbf{1}_3 & (3) & \mathbf{0}_3 \\ 2 \times \mathbf{1}_3 & \mathbf{0}_3 & (3) \end{bmatrix}.$$

The first column $\mathbf{F}_0$ corresponds to the block factor and the second and third columns correspond to two three-level treatment factors. Hence the first block of $\mathbf{D}^{00}$ contains the first three runs and the second block of $\mathbf{D}^{00}$ contains the last three runs. It is easy to verify that $3\mathbf{M}_{1,2} = \mathbf{M}_{1,0}\mathbf{M}_{0,2}$.

We now list the other plans ($\mathbf{D}^{ij}$) in Table 2

We now present several recursive constructions based on the existing POTBs.

3.2. Adding one treatment factor to an existing POTB

We now give a method for constructing POTBs by adding a new treatment factor of k levels. For a vector $\mathbf{x}$, let "$\mathbf{x}$ mod y" be the vector obtained from $\mathbf{x}$ by taking the modular operation for each entry. Let $\mathbf{D}_1 = [\mathbf{F}_0^1\, \mathbf{F}_1^1\, \mathbf{F}_2^1 \cdots \mathbf{F}_m^1]$ be a $\rho_n(b; p_1 \cdots p_m)$ with b blocks each of size k, where $\mathbf{F}_0^1$ is the block factor and $\mathbf{F}_1^1, \mathbf{F}_2^1, \ldots, \mathbf{F}_m^1$ are the treatment factors of $p_1, \ldots, p_m$ levels, respectively. Define

$$\mathbf{D}_2 = [\mathbf{F}_0^2\, \mathbf{F}_1^2 \cdots \mathbf{F}_m^2\, \mathbf{F}_{m+1}^2] = \begin{bmatrix} \mathbf{F}_0^1 & \mathbf{F}_1^1 & \cdots & \mathbf{F}_m^1 & c \times \mathbf{1}_n \\ (b+1) \times \mathbf{1}_k & \mathbf{e}_1 & \cdots & \mathbf{e}_m & (k) \end{bmatrix}, \tag{4}$$

where $\mathbf{e}_i = i \times \mathbf{1}_k \pmod{p_i}$, $i = 1, 2, \ldots, m$, and c is an integer such that $0 \le c \le k-1$.

Theorem 3.2. *Design* $\mathbf{D}_2$ *in (4) is a* $\rho_{n+k}(b+1; p_1 \cdots p_m p_{m+1})$ *in* $b+1$ *blocks each of size* k*, where* $p_{m+1} = k$.

Proof. Let $c = 1$ and $\mathbf{M}_{i,j}$ be the incidence matrix of factor $\mathbf{F}_i^2$ versus factor $\mathbf{F}_j^2$ in $\mathbf{D}_2$, where $0 \le i, j \le m+1$, $i \ne j$.

We first consider the pair of factors $\mathbf{F}_i^2$ and $\mathbf{F}_j^2$ for $1 \le i, j \le m$, $i \ne j$. For a fixed l, $1 \le l \le b+1$, consider the k runs in the lth block. Let $\mathbf{M}_{i,j,l}$ be the incidence matrix of factor $\mathbf{F}_i^2$ versus factor $\mathbf{F}_j^2$ within the lth block and $\mathbf{r}_{i,l}$ be the replicate vector of factor $\mathbf{F}_i^2$ within the lth block. Thus by the definition of incidence matrix $\mathbf{M}_{i,j}$, we have

$$\mathbf{M}_{i,j} = \sum_{l=1}^{b+1} \mathbf{M}_{i,j,l}, \quad \text{and} \quad \mathbf{M}_{i,0}\mathbf{M}_{0,j} = \sum_{l=1}^{b+1} \mathbf{r}_{i,l}\mathbf{r}'_{l,l}. \tag{5}$$

Since $\mathbf{D}_1$ is a $\rho_n(b; p_1 \cdots p_s)$ in b blocks each of size k, then for the first b blocks of $\mathbf{D}_2$, we have

$$k \sum_{l=1}^{b} \mathbf{M}_{i,j,l} = \sum_{l=1}^{b} \mathbf{r}_{i,l}\mathbf{r}'_{l,l}. \tag{6}$$

Moreover, for the $(b+1)$th block of $\mathbf{D}_2$, it can be verified that

$$\mathbf{r}_{i,(b+1)}\mathbf{r}'_{j,(b+1)} = k\mathbf{M}_{i,j,(b+1)}. \tag{7}$$

Thus by formulas (5)–(7), it follows that

$$\begin{aligned} k\mathbf{M}_{i,j} &= k\sum_{l=1}^{b+1}\mathbf{M}_{i,j,l} \\ &= k\sum_{l=1}^{b}\mathbf{M}_{i,j,l} + k\mathbf{M}_{i,j,(b+1)} \\ &= k(1/k)\sum_{l=1}^{b}\mathbf{r}_{i,l}r'_{j,l} + \mathbf{r}_{i,(b+1)}\mathbf{r}'_{j,(b+1)} \\ &= \sum_{l=1}^{b+1}\mathbf{r}_{i,l}\mathbf{r}'_{j,l}. \end{aligned}$$

By similar method described above, we have $k\mathbf{M}_{i,m+1} = \mathbf{M}_{i,0}\mathbf{M}_{0,m+1}$ for each pair of factors $\mathbf{F}_i^2$ and $\mathbf{F}_{m+1}^2$, $i = 1, \ldots, m$. Hence, $\mathbf{D}_2$ is a $\rho_{n+k}(b+1; p_1 \cdots p_m p_{m+1})$ where $p_{m+1} = k$.

Remark 3.1. The POTB $\mathbf{D}_2$ constructed by Theorem 3.2 is saturated if $\mathbf{D}_1$ is saturated.

Example 3.2. Let $\mathbf{D}_1 = [\mathbf{F}_0^1\,\mathbf{F}_1^1\,\mathbf{F}_2^1 \cdots \mathbf{F}_6^1]$ be a $\rho_{12}(3; 3^3 2^3)$ with three blocks each of size four. Then the following design $\mathbf{D}_2$ is a $\rho_{16}(4; 3^3 2^3 4)$ constructed by Theorem 3.2, which has four blocks each of size four. Let $c = 1$ in Theorem 3.2.

$$\mathbf{D}_2 = [\mathbf{F}_0^2\,\mathbf{F}_1^2\,\mathbf{F}_2^2 \cdots \mathbf{F}_7^2] = \begin{bmatrix} \mathbf{F}_0^1 & \mathbf{F}_1^1 & \mathbf{F}_2^1 & \mathbf{F}_3^1 & \mathbf{F}_4^1 & \mathbf{F}_5^1 & \mathbf{F}_6^1 & \mathbf{1}_{12} \\ 4\times\mathbf{1}_4 & \mathbf{1}_4 & 2\times\mathbf{1}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{1}_4 & \mathbf{0}_4 & (4) \end{bmatrix}.$$

According to the levels of the block factor $\mathbf{F}_0^2$, the first three blocks of $\mathbf{D}_2$ contain the first 12 runs and the fourth block of $\mathbf{D}_2$ contains the last four runs.

3.3. Adding more treatment factors to an existing POTB

We propose a generalization of the method in Section 3.2, which can lead to POTBs containing more treatment factors. Let $\mathbf{D}_1 = [\mathbf{F}_0^1\,\mathbf{F}_1^1\,\mathbf{F}_2^1 \cdots \mathbf{F}_{m_1}^1]$ be a POTB $\rho_n(b; p_1 p_2 \cdots p_{m_1})$ with b blocks each of size k. Let $\mathbf{D}_2 = [\mathbf{F}_1^2\,\mathbf{F}_2^2 \cdots \mathbf{F}_{m_2}^2]$ be an orthogonal array with k runs and m_2 factors of $p_{m_1+1}, \ldots, p_{m_1+m_2}$ levels respectively. Define $\mathbf{D}_3 = [\mathbf{F}_0^3\,\mathbf{F}_1^3 \cdots \mathbf{F}_{m_1}^3\,\mathbf{F}_{m_1+1}^3 \cdots \mathbf{F}_{m_1+m_2}^3]$ to be

$$\begin{bmatrix} \mathbf{F}_0^1 & \mathbf{F}_1^1 & \cdots & \mathbf{F}_{m_1}^1 & \mathbf{e}_{m_1+1} & \cdots & \mathbf{e}_{m_1+m_2} \\ \mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_{m_1} & \mathbf{F}_1^2 & \cdots & \mathbf{F}_{m_2}^2 \end{bmatrix}, \tag{8}$$

where

$$\mathbf{e}_i = \begin{cases} (b+1)\times\mathbf{1}_k, & i = 0; \\ i\times\mathbf{1}_k \pmod{p_i}, & i = 1, \ldots, m_1; \\ i\times\mathbf{1}_n \pmod{p_i}, & i = m_1+1, \ldots, m_1+m_2. \end{cases}$$

Theorem 3.3. *The design $\boldsymbol{D}_3$ in (8) is a $\rho_{n+k}(b+1; p_1 \cdots p_{m_1} p_{m_1+1} \cdots p_{m_1+m_2})$.*

Proof. We first consider the pair of factors $\mathbf{F}_i^3$ and $\mathbf{F}_j^3$ for $m_1 + 1 \le i,\ j \le m_1 + m_2,\ i \ne j$. By the definition of incidence matrix, we have

$$\begin{aligned} k\mathbf{M}_{i,j} &= k\sum_{l=1}^{b+1}\mathbf{M}_{i,j,l} \\ &= k\sum_{l=1}^{b}\mathbf{M}_{i,j,l} + k\mathbf{M}_{i,j,(b+1)} \\ &= k\sum_{l=1}^{b}\mathbf{r}_{i,l}\mathbf{r}'_{j,l}/k + \mathbf{r}_{i,b+1}\mathbf{r}'_{j,b+1} \\ &= \sum_{l=1}^{b+1}\mathbf{r}_{i,l}\mathbf{r}'_{j,l} \\ &= \mathbf{M}_{i,0}\mathbf{M}_{0,j}. \end{aligned}$$

The second term of the third equation above, $k\mathbf{M}_{i,j,(b+1)} = \mathbf{r}_{i,.b+1}\mathbf{r}'_{j,b+1}$, is based on the fact that $[\mathbf{F}^2_1\,\mathbf{F}^2_2 \cdots \mathbf{F}^2_{m_2}]$ is an orthogonal array. For the first b blocks of $\mathbf{D}_3$, it can be verified that the (u, v)th element of $\mathbf{M}_{i,j,l}$ or $\frac{\mathbf{r}_{i,l}\mathbf{r}'_{j,l}}{k}$ equals

$$\begin{cases} k, & \text{if } \mathbf{e}_i = u \times \mathbf{1}_n \quad \text{and} \quad \mathbf{e}_j = v \times \mathbf{1}_n; \\ 0, & \text{otherwise}, \end{cases}$$

where $0 \le u \le p_i - 1$ and $0 \le v \le p_j - 1$. Thus, $\sum_{l=1}^{b} \mathbf{M}_{i,j,l} = \sum_{l=1}^{b} \mathbf{r}_{i,l}\mathbf{r}'_{j,l}/k$. The same results can be obtained for pairs of factors $\mathbf{F}^3_i$ and $\mathbf{F}^3_j$, where $1 \le i,\ j \le m_1 + 1$. Hence, design $\mathbf{D}_3$ in (8) is a $\rho_{n+k}(b + 1; p_1 \cdots p_{m_1} p_{m_1+1} \cdots p_{m_1+m_2})$.

Remark 3.2. The POTB $\mathbf{D}_3$ constructed by Theorem 3.3 is saturated if $\mathbf{D}_1$ and $\mathbf{D}_2$ in (8) are saturated.

Example 3.3. Let $\mathbf{D}_1 = [\mathbf{F}^1_0\,\mathbf{F}^1_1 \cdots \mathbf{F}^1_6]$ be a $\rho_{12}(3; 3^3 2^3)$ constructed by Bagchi (2010) and $\mathbf{D}_2 = [\mathbf{F}^2_1\,\mathbf{F}^2_2\,\mathbf{F}^2_3]$ be an orthogonal array with three two-level factors in 4 runs. From Theorem 3.3,

$$\begin{aligned} \mathbf{D}_3 &= [\mathbf{F}^3_0\,\mathbf{F}^3_1\,\mathbf{F}^3_2 \cdots \mathbf{F}^3_9] \\ &= \begin{bmatrix} \mathbf{F}^1_0 & \mathbf{F}^1_1 & \mathbf{F}^1_2 & \mathbf{F}^1_3 & \mathbf{F}^1_4 & \mathbf{F}^1_5 & \mathbf{F}^1_6 & \mathbf{1}_{12} & \mathbf{0}_{12} & \mathbf{1}_{12} \\ 4 \times \mathbf{1}_4 & \mathbf{1}_4 & 2 \times \mathbf{1}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{1}_4 & \mathbf{0}_4 & \mathbf{F}^2_1 & \mathbf{F}^2_2 & \mathbf{F}^2_3 \end{bmatrix} \end{aligned}$$

is a $\rho_{16}(4; 3^3 2^6)$. According to the levels of the block factor $\mathbf{F}^3_0$, the first three blocks of $\mathbf{D}_3$ correspond to the first 12 runs and the fourth block corresponds to the last four runs. Every pair of treatment factors from $\mathbf{F}^3_1, \ldots, \mathbf{F}^3_9$ is orthogonal through the block factor $\mathbf{F}^3_0$.

3.4. Construction by combining two POTBs with the same block size

Let $\mathbf{D}_1 = [\mathbf{F}^1_0\,\mathbf{F}^1_1\,\mathbf{F}^1_2 \cdots \mathbf{F}^1_{m_1}]$ and $\mathbf{D}_2 = [\mathbf{F}^2_0\,\mathbf{F}^2_1\,\mathbf{F}^2_2 \cdots \mathbf{F}^2_{m_2}]$ be two POTBs $\rho_{n_1}(b_1; p_1, \ldots, p_{m_1})$ and $\rho_{n_2}(b_2; p_{m_1+1}, \ldots, p_{m_1+m_2})$, respectively. Both designs have block size k. Define $\mathbf{D}_3 = [\mathbf{F}^3_0\ \mathbf{F}^3_1\ \cdots\ \mathbf{F}^3_{m_1}\ \mathbf{F}^3_{m_1+1}\ \cdots\ \mathbf{F}^3_{m_1+m_2}]$ to be

$$\begin{bmatrix} \mathbf{F}^1_0 & \mathbf{F}^1_1 & \cdots & \mathbf{F}^1_{m_1} & \mathbf{e}_{m_1+1} & \cdots & \mathbf{e}_{m_1+m_2} \\ \mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_{m_1} & \mathbf{F}^2_1 & \cdots & \mathbf{F}^2_{m_2} \end{bmatrix}, \tag{9}$$

where

$$\mathbf{e}_i = \begin{cases} \mathbf{F}^2_0 \oplus b_1, & i = 0; \\ i \times \mathbf{1}_{n_2} \pmod{p_i}, & i = 1, \ldots, m_1; \\ i \times \mathbf{1}_{n_1} \pmod{p_i}, & i = m_1 + 1, \ldots, m_1 + m_2. \end{cases}$$

Here, notation $\mathbf{F}^2_0 \oplus b_1$ means all entries of the vector $\mathbf{F}^2_0$ are added by b_1.

Theorem 3.4. *The design $\mathbf{D}_3$ in (9) is a $\rho_{n_1+n_2}(b_1 + b_2; p_1 \cdots p_{m_1} p_{m_1+1} \cdots p_{m_1+m_2})$.*

Proof. The proof is omitted here since the technique is similar to Theorem 3.3.

Remark 3.3. Theorem 3.4 tells us that the $\mathbf{D}_3$ in (9) is a POTB with $m_1 + m_2$ treatment factors in $n_1 + n_2$ runs. Each pair of treatment factors are orthogonal through the block factor $\mathbf{F}^3_0$. All $n_1 + n_2$ runs can be divided into $b_1 + b_2$ blocks each of size k according to the levels of the block factor $\mathbf{F}^3_0$.

The following Theorem 3.5 gives an alternative construction approach by combining two POTBs together. The main differences between Theorems 3.4 and 3.5 are the values of $\mathbf{e}_i$ in (9) and (10), where $1 \le i \le m_1 + m_2$. See Example 3.4.

Theorem 3.5. *Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be defined as in Theorem 3.4. Then the following design is a $\rho_{n_1+n_2}(b_1 + b_2; p_1 \cdots p_{m_1} p_{m_1+1} \cdots p_{m_1+m_2})$ in $b_1 + b_2$ blocks each of size k.*

$$\begin{aligned} D_3 &= [\boldsymbol{F}^3_0\ \ \boldsymbol{F}^3_1\ \ \cdots\ \ \boldsymbol{F}^3_{m_1}\ \ \boldsymbol{F}^3_{m_1+1}\ \ \cdots\ \ \boldsymbol{F}^3_{m_1+m_2}] \\ &= \begin{bmatrix} \boldsymbol{F}^1_0 & \boldsymbol{F}^1_1 & \cdots & \boldsymbol{F}^1_{m_1} & \boldsymbol{e}_{m_1+1} & \cdots & \boldsymbol{e}_{m_1+m_2} \\ \boldsymbol{e}_0 & \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{m_1} & \boldsymbol{F}^2_1 & \cdots & \boldsymbol{F}^2_{m_2} \end{bmatrix}, \end{aligned} \tag{10}$$

where

$$\boldsymbol{e}_i = \begin{cases} \boldsymbol{F}^2_0 \oplus b_1, & i = 0; \\ \boldsymbol{F}^2_0 \oplus i \pmod{p_i}, & i = 1, \ldots, m_1; \\ \boldsymbol{F}^1_0 \oplus i \pmod{p_i}, & i = m_1 + 1, \ldots, m_1 + m_2. \end{cases}$$

According to the levels of the block factor $\boldsymbol{F}_0^3$, $[\boldsymbol{F}_1^1 \ \cdots \ \boldsymbol{F}_{m_1}^1 \ \boldsymbol{e}_{m_1+1} \ \cdots \ \boldsymbol{e}_{m_1+m_2}]$ represents the first b_1 blocks, and $[\boldsymbol{e}_1 \ \cdots \ \boldsymbol{e}_{m_1} \ \boldsymbol{F}_1^2 \ \cdots \ \boldsymbol{F}_{m_2}^2]$ represents the remaining b_2 blocks of $\boldsymbol{D}_3$.

Proof. We only demonstrate that each pair of factors $\mathbf{F}_i^3$ and $\mathbf{F}_j^3$ of $\mathbf{D}_3$ are orthogonal through the block factor for $1 \le i, j \le m_1, \ i \ne j$. By dividing all $b_1 + b_2$ blocks into two parts: one containing the first b_1 blocks and the other containing the remaining b_2 blocks, we have

$$\begin{aligned} k\mathbf{M}_{i,j} &= k \sum_{l=1}^{b_1+b_2} \mathbf{M}_{i,j,l} \\ &= k \sum_{l=1}^{b_1} \mathbf{M}_{i,j,l} + k \sum_{l=b_1+1}^{b_1+b_2} \mathbf{M}_{i,j,l} \\ &= k \sum_{l=1}^{b_1} \mathbf{r}_{i,l}\mathbf{r}_{j,l}'/k + k \sum_{l=b_1+1}^{b_1+b_2} \mathbf{r}_{i,l}\mathbf{r}_{j,l}'/k \\ &= \sum_{l=1}^{b_1+b_2} \mathbf{r}_{i,l}\mathbf{r}_{j,l}' \\ &= \mathbf{M}_{i,0}\mathbf{M}_{0,j}. \end{aligned}$$

Thus factors $\mathbf{F}_i^3$ and $\mathbf{F}_j^3$ are orthogonal through the block factor. Other cases of factors $\mathbf{F}_i^3$ and $\mathbf{F}_j^3$ can be obtained similarly.

Remark 3.4. The POTBs $\mathbf{D}_3$ constructed by Theorems 3.4 and 3.5 are saturated if $\mathbf{D}_1$ and $\mathbf{D}_2$ used in the construction are both saturated.

Example 3.4. Let $\mathbf{D}_1 = [\mathbf{F}_0^1 \, \mathbf{F}_1^1 \cdots \mathbf{F}_6^1]$ and $\mathbf{D}_2 = [\mathbf{F}_1^2 \, \mathbf{F}_2^2 \, \mathbf{F}_3^2]$ be defined as in Example 3.3. Then the following design

$$\mathbf{D}_3 = [\mathbf{F}_0^3 \, \mathbf{F}_1^3 \, \mathbf{F}_2^3 \cdots \mathbf{F}_9^3] = \begin{bmatrix} \mathbf{F}_0^1 & \mathbf{F}_1^1 & \mathbf{F}_2^1 & \mathbf{F}_3^1 & \mathbf{F}_4^1 & \mathbf{F}_5^1 & \mathbf{F}_6^1 & \mathbf{e}_7 & \mathbf{e}_8 & \mathbf{e}_9 \\ 4 \times \mathbf{1}_4 & 2 \times \mathbf{1}_4 & \mathbf{0}_4 & \mathbf{1}_4 & \mathbf{0}_4 & \mathbf{1}_4 & \mathbf{0}_4 & \mathbf{F}_1^2 & \mathbf{F}_2^2 & \mathbf{F}_3^2 \end{bmatrix}$$

is a $\rho_{16}(4; 3^3 2^6)$ constructed by Theorem 3.5, where

$$\mathbf{e}_7 = \begin{bmatrix} \mathbf{0}_4 \\ \mathbf{1}_4 \\ \mathbf{0}_4 \end{bmatrix}, \quad \mathbf{e}_8 = \begin{bmatrix} \mathbf{1}_4 \\ \mathbf{0}_4 \\ \mathbf{1}_4 \end{bmatrix}, \quad \text{and} \quad \mathbf{e}_9 = \begin{bmatrix} \mathbf{0}_4 \\ \mathbf{1}_4 \\ \mathbf{0}_4 \end{bmatrix}.$$

For the design $\mathbf{D}_3$ in Example 3.3, $\mathbf{e}_7 = \mathbf{1}_{12}$, $\mathbf{e}_8 = \mathbf{0}_{12}$, and $\mathbf{e}_9 = \mathbf{1}_{12}$. Clearly, both designs $\mathbf{D}_3$ and $\mathbf{D}_4$ are saturated.

Theorem 3.6. *All POTBs constructed in this paper are connected and variance-balanced.*

Proof. Consider a POTB constructed above. Let $\mathbf{X}_i$ and $\mathbf{X}_0$ be the incidence matrices of the treatment factor F_i and the block factor F_0, respectively. It can be checked that the C-matrix of F_i, which is $\mathbf{C}_i = \mathbf{X}_i'\mathbf{X}_i - \mathbf{X}_i'\mathbf{X}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0'\mathbf{X}_i$, has the form $\mathbf{C}_i = \theta(\mathbf{I}_p - \frac{1}{p}\mathbf{1}_p\mathbf{1}_p')$, where $\theta > 0$ is a scalar and $\mathbf{I}_p$ is the identity matrix of order p. Hence, by Theorem 2.2.1 and Corollary 2.3.1 in Dey (2010), all of them are connected and variance-balanced.

Finally, we summarize the results for POTBs with $n \le 40$ in Table 3. Notation "B" in the last column indicates that the corresponding design can be constructed by Bagchi (2010), "Th3.1" means that the corresponding design can be constructed by Theorem 3.1, and so on.

4. Conclusions

In this paper, several new approaches have been developed for constructing main-effect plans orthogonal through the block factor (Definition 2.1). The proposed methods are easy to implement, and many new POTBs with flexible levels and small runs are obtained. Moreover, all POTBs in Table 3 are saturated, connected and variance-balanced.

However, Example 3.1, Theorems 3.4 and 3.5 indicate that most of POTBs are not unique for given parameters $n, b, p_1, \ldots, p_m$. Thus, uniformity or space-filling property (Fang et al., 2006; Yang et al., 2010; Ai et al., 2014; Tang et al., 2012; Zhou and Xu, 2014) via level permutation can be used to further distinguish different POTBs. Some progress in this direction is made in Chen et al. (2015).

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Table 3
POTBs with run size $n \leq 40$.

Run size	Number of blocks	Block size	Experiment	Reference
4	2	2	2^2	Th3.1
6	3	2	2^3	Th3.1
6	2	3	3^2	Th3.1
8	2	4	4^2	Th3.1
10	2	5	5^2	Th3.1
12	3	4	$3^3 \cdot 2^3$	B
12	6	2	4^2	B
14	7	2	$4^2 \cdot 2^1$	Th3.2
16	4	4	$4^1 \cdot 3^3 \cdot 2^3$	Th3.2
16	4	4	$3^3 \cdot 2^6$	Th3.3
20	5	4	$3^3 \cdot 2^9$	Th3.3
20	5	4	$4^1 \cdot 3^3 \cdot 2^6$	Th3.3
20	5	4	$4^2 \cdot 3^3 \cdot 2^3$	Th3.2
20	5	4	$5^3 \cdot 2^3$	B
24	6	4	$4^1 \cdot 3^3 \cdot 2^9$	Th3.2
24	6	4	$4^2 \cdot 3^3 \cdot 2^6$	Th3.2
24	6	4	$4^3 \cdot 3^3 \cdot 2^3$	Th3.2
24	6	4	$5^3 \cdot 4^1 \cdot 2^3$	Th3.2
24	6	4	$5^3 \cdot 2^6$	Th3.3
24	6	4	$3^6 \cdot 2^6$	B
28	7	4	$4^1 \cdot 3^6 \cdot 2^6$	Th3.2
28	7	4	$4^2 \cdot 3^3 \cdot 2^9$	Th3.2
28	7	4	$4^3 \cdot 3^3 \cdot 2^6$	Th3.2
28	7	4	$4^4 \cdot 3^3 \cdot 2^3$	Th3.2
28	7	4	$5^3 \cdot 4^2 \cdot 2^3$	Th3.2
28	7	4	$5^3 \cdot 4^1 \cdot 2^6$	Th3.2
28	7	4	$4^1 \cdot 3^3 \cdot 2^{12}$	Th3.3
28	7	4	$5^3 \cdot 2^9$	Th3.3
28	7	4	$3^6 \cdot 2^9$	Th3.3
32	8	4	$5^3 \cdot 3^3 \cdot 2^6$	Th3.4
32	8	4	$4^1 \cdot 3^6 \cdot 2^6$	Th3.2
32	8	4	$4^2 \cdot 3^3 \cdot 2^9$	Th3.2
32	8	4	$4^3 \cdot 3^3 \cdot 2^6$	Th3.2
32	8	4	$4^4 \cdot 3^3 \cdot 2^3$	Th3.2
32	8	4	$5^3 \cdot 4^2 \cdot 2^3$	Th3.2
32	8	4	$5^3 \cdot 4^1 \cdot 2^6$	Th3.2
32	8	4	$4^1 \cdot 3^3 \cdot 2^{12}$	Th3.3
32	8	4	$5^3 \cdot 2^9$	Th3.3
32	8	4	$3^6 \cdot 2^9$	Th3.3
36	9	4	$4^1 \cdot 3^6 \cdot 2^{12}$	Th3.4
36	9	4	$4^2 \cdot 3^6 \cdot 2^9$	Th3.4
36	9	4	$4^3 \cdot 3^6 \cdot 2^6$	Th3.4
36	9	4	$5^3 \cdot 4^1 \cdot 3^3 \cdot 2^6$	Th3.4
36	9	4	$3^6 \cdot 2^{15}$	Th3.4
36	9	4	$5^3 \cdot 3^3 \cdot 2^9$	Th3.4
40	10	4	$4^1 \cdot 3^6 \cdot 2^{15}$	Th3.4
40	10	4	$4^2 \cdot 3^6 \cdot 2^{12}$	Th3.4
40	10	4	$5^3 \cdot 3^3 \cdot 2^{12}$	Th3.4
40	10	4	$4^3 \cdot 3^6 \cdot 2^9$	Th3.4
40	10	4	$5^3 \cdot 4^1 \cdot 3^3 \cdot 2^9$	Th3.4
40	10	4	$5^3 \cdot 4^2 \cdot 3^3 \cdot 2^6$	Th3.4

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