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Generalized variable resolution designs

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Abstract In this paper, the concept of generalized variable resolution is proposed for designs with nonnegligible interactions between groups. The conditions for the existence of generalized variable resolution designs are discussed. Connections between different generalized variable resolution designs and compromise plans, clear compromise plans and designs containing partially clear two-factor interactions are explored. A general construction method for the proposed designs is also discussed.

Keywords Clear effects · Compromise plans · Partially clear effects · Variable resolution

Mathematics Subject Classification 62K05

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1 Introduction

Models containing a specified set of effects of interest have got great attention by many researchers. In particular, consider two groups of factors (columns) G_1 and G_2 in a single array

$$D = (G_1, G_2).$$

Different types of the specified sets to be estimated have been discussed in the literature, for example, four classes of compromise plans (Addelman 1962; Sun 1993), clear compromise plans (Ke et al. 2005), robust parameter designs (Taguchi 1986; Wu and Zhu 2003; Yang et al. 2013), robust designs with clear two-factor interactions (2fi's) (Tang 2006) and robust designs with partially clear 2fi's (Lekivetz and Tang 2011), among others. All these plans can be used for estimating all main effects and a particular set of 2fi's under certain assumptions. A compromise plan allows the estimation of all main effects and some specified 2fi's in $G_i \times G_j, i, j = 1, 2$ while assuming other effects are negligible. Throughout the paper, we use $G_1 \times G_1$ to denote the set of 2fi's with both factors in G_1 and use $G_1 \times G_2$ to denote the set of 2fi's with one factor from G_1 and the other from G_2. Under the weak assumption that all three-factor and higher order interactions are assumed to be negligible, a clear compromise plan allows the estimation of all main effects and the specified 2fi's. If some of the remaining 2fi's (not to be estimated) can be assumed negligible, and some of them are not, then designs containing partially clear 2fi's can be used (Lekivetz and Tang 2011). In conclusion, for all these cases, the overall strength or resolution of a design D seems to be a rough description for the confounding structure of a single array. To overcome this problem, two or three strength can be used in a single array. For example, a compound orthogonal array (Hedayat and Stufken 1999) can use three strength t_1, t_2 and t_3 to denote the strength of control factors, noise factors, and the overall array, respectively.

Recently, Lin (2012) also used more than one resolution to study designs that potentially important interactions arise only within G_1 and G_2. Then a design of variable resolution consisting of groups of factors of higher resolution than the overall resolution of the design is proposed. For two groups of factors, three resolutions are considered according to the definition of variable resolution designs in the next section. However, variable resolution designs in Lin (2012) can not differentiate confounding structure between groups of factors. Then, a fourth resolution is suggested in designs of variable resolution, which leads to the definition of generalized variable resolution designs. By the fourth resolution, it is shown that generalized variable resolution designs can also be used to investigate interactions between groups of factors.

The paper proceeds as follows. Section 2 introduces the motivation of generalized variable resolution designs and discusses the conditions for the existence. The relationships between variable resolution designs and generalized variable resolution designs are also explored in this section. In Sect. 3, the connection with compromise plans, clear compromise plans, designs containing partially clear 2fi's will be presented. In Sect. 4, an application is used to illustrate the proposed designs of generalized variable resolution. Conclusions will be drawn in Sect. 5.

2 Generalized variable resolution designs

2.1 Motivation

We first give some notation and background. A two-level factorial design of n runs and m factors is denoted by an $n \times m$ matrix $D = (d_{ij}) = (d_1, \ldots, d_m)$ with entries $+1$ and -1. Deng and Tang (1999) defined the J-characteristic as follows. For any k-subset $s = \{d_{j_1}, \ldots, d_{j_k}\}$ of D, $J(s) = J(d_{j_1}, \ldots, d_{j_k}) = \left|\sum_{i=1}^{n} d_{ij_1} \cdots d_{ij_k}\right|$, where $1 \leq j_1 < \cdots < j_k \leq m$ and $k \leq m$. Let r be the smallest integer such that $\max_{|s|=r} J(s) > 0$, where $|s| = r$ means the number of columns in the subset s. Then the generalized resolution of D is defined to be

$$R(D) = r + \left(1 - \max_{|s|=r} \frac{J(s)}{n}\right).$$

Note that $r = \lfloor R(D) \rfloor$, where $\lfloor R \rfloor$ denotes the largest integer not exceeding R. Then design D is of resolution r if and only if $J(d_{j_1}, \ldots, d_{j_k}) = 0$ for all $1 \leq j_1 < \cdots < j_k \leq m$ and $k \leq r - 1$. Following Lin (2012), we use $D(n, m, r)$ to represent a two-level design of n runs, m factors and resolution r. Since Lin (2012) considered the situation that potentially significant interactions arise only within groups of factors, it leads to designs consisting groups of factors with higher resolution than the overall resolution of the design. We now give the definition of designs of variable resolution when D is divided into two groups of factors G_1 and G_2.

Definition 2.1 A design $D(n, m, r)$ is said to be a design of variable resolution $D(n, (m_1, m_2), (r_1, r_2), r)$ if its columns can be divided into two groups with the ith group G_i being a $D(n, m_i, r_i)$ that satisfies (i) $r < r_1 \leq m_1 + 1$; and (ii) either $r < r_2 \leq m_2 + 1$ or $r_2 = r$.

Example 2.1 Lin (2012) used three resolutions in a single array to reflect the resolutions of G_1, G_2 and D, respectively. Consider the following two designs

(i) $D_1 = (G_{11}, G_{12})$ with $G_{11} = (1, 4, 5, 1245)$, $G_{12} = (2, 123, 1235)$ and
(ii) $D_2 = (G_{21}, G_{22})$ with $G_{21} = (1, 4, 5, 1245)$ and $G_{22} = (2, 3, 23)$,

where the numbers 1, 2, 3, 4, 5 in G_{ij} denote the five independent columns of designs of 32 runs. According to Lin (2012), D_1 is a $D(32, (4, 3), (5, 4), 3)$ and D_2 is a $D(32, (4, 3), (5, 3), 3)$. Hence D_1 is superior to D_2 in terms of variable resolution.

However, the confounding structure between groups of factors of D_1 and D_2 is quite different. For D_2, it can be verified that (i) any set of three columns with two columns from G_{21} and one column from G_{22} has resolution 4; (ii) any set of three columns with one column from G_{21} and two columns from G_{22} has resolution 4; (iii) any set of four columns with two columns from G_{21} and two columns from G_{22} has resolution 5; and (iv) any set of four columns with three columns from G_{21} and one column from G_{22} has resolution 5. Hence, any set of six columns with four columns from G_{21} and two columns from G_{22} has resolution 5. Note that this property cannot be found in D_1. For D_1, the main effect {5} in G_{11} is aliased with the 2fi of {123} and {1235} in G_{12}.

That is, D_1 and D_2 have different confounding structures between groups of factors, which leads to the following definition of generalized variable resolution designs.

Definition 2.2 A design $D = (G_1, G_2) = D(n, m, r)$ is said to be a design of generalized variable resolution $D(n, (m_1, m_2), (r_1, r_2)_{r_3}, r)$ if

(i) G_1 is a $D(n, m_1, r_1)$;
(ii) G_2 is a $D(n, m_2, r_2)$; and
(iii) Any (G_1', G_2') has resolution at least r_3 for $G_i' \subseteq G_i$ and $|G_i'| = r_i - 1$, where $|G_i'|$ means the number of columns in G_i'.

Remark 2.1 From Definition 2.2, there are four resolutions in a design of generalized variable resolution. Resolutions r_1 and r_2 describe the confounding information within groups of factors G_1 and G_2, resolution r_3 describes the confounding of factors between G_1 and G_2 and resolution r is the overall resolution of the design. Note that design D_1 in Example 2.1 is a $D(n, (4, 3), (5, 4)_3, 3)$ and design D_2 is a $D(n, (4, 3), (5, 3)_5, 3)$. Thus, D_2 is superior to D_1 in terms of r_3.

The following lemma from Tang (2006) is useful for later development.

Lemma 2.1 *We have that* $J(a_1 \otimes b_1, \ldots, a_k \otimes b_k) = J(a_1, \ldots, a_k)J(b_1, \ldots, b_k)$*, where* $a_i = (a_{1i}, \ldots, a_{n_1 i})^T$, $b_j = (b_{1j}, \ldots, b_{n_2 j})^T$ *and* $a \otimes b$ *denotes the Kronecker product of* a *and* b.

2.2 Connection with variable resolution designs

We now give a relationship between variable resolution designs and their generalized version.

Proposition 2.1 *A generalized variable resolution design* $D(n, (m_1, m_2), (r_1, r_2)_{r_3}, r)$ *is a design of variable resolution* $D(n, (m_1, m_2), (r_1, r_2), r_0)$*, where* $r_0 = \min\{r_1, r_2, r_3\}$.

Since $r_0 = \min\{r_1, r_2, r_3\}$, it can be easily verified that $J(d_{j_1}, \ldots, d_{j_k}) = 0$ for all $j_1, \ldots, j_k$ such that $1 \le j_1 < \cdots < j_k \le m$ and $k \le r_0 - 1$.

Remark 2.2 Compared with the variable resolution design in Lin (2012), the generalized variable resolution design considers one more resolution between groups of factors, hence contains more information. From now on, we omit the overall resolution r and denote a generalized variable resolution design by $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$. In Example 2.1, the design D_1 is $D(n, (4, 3), (5, 4)_3)$, and the design D_2 is $D(n, (4, 3), (5, 3)_5)$.

2.3 Existence of generalized variable resolution designs

For two groups of factors $G_1 = (a_1, \ldots, a_{m_1})$ and $G_2 = (b_1, \ldots, b_{m_2})$, various designs of generalized variable resolution can be obtained from Definition 2.2, such as $D(n, (m_1, m_2), (3, 3)_5)$, $D(n, (m_1, m_2), (4, 4)_5)$, $D(n, (m_1, m_2), (5, 3)_5)$, $D(n, (m_1, m_2), (5, 3)_7)$, and so on. We now give a general condition for the existence of generalized variable resolution designs.

Proposition 2.2 *There exist generalized variable resolution designs $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ if and only if $r_3 \leq r_1 + r_2 - 1$.*

Proof We first prove that there do not exist generalized variable resolution designs $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ with $r_3 \geq r_1 + r_2$. Otherwise according to the definition of generalized variable resolution designs, the resolution of G_1 is at least $r_1^* = r_3 - (r_2 - 1) \geq r_1 + 1$ by noting that $J(a_{j_1}, \ldots, a_{j_k}) = 0$ for all $1 \leq j_1 < \cdots < j_k \leq m_1$ and $k \leq r_3 - (r_2 - 1) = r_1^*$. Similar conclusion can be obtained for G_2. It makes contradiction that G_i is $D(n, m_i, r_i)$. Hence, we only need to consider generalized variable resolution $(r_1, r_2)_{r_3}$ with $r_3 \leq r_1 + r_2 - 1$. This also explains why we only considered $|G_i'| = r_i - 1$ in condition (iii) of Definition 2.2.

By Definition 2.2, a $D(n, (m_1, m_2), (r_1, r_2)_{r_3^1})$ must be a $D(n, (m_1, m_2), (r_1, r_2)_{r_3^2})$ if $t_3^1 \geq t_3^2$. Thus, we need only to consider $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ with $r_3 = r_1 + r_2 - 1$, which will be constructed below. The construction method of cross array in robust parameter designs (Taguchi 1986; Tang 2006) can be generalized to construct $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ with $r_3 = r_1 + r_2 - 1$. Let $L_{n_1} = (c_1, \ldots, c_{m_1})$ be a $D(n_1, m_1, r_1)$ and $L_{n_2} = (d_1, \ldots, d_{m_2})$ be a $D(n_2, m_2, r_2)$. Consider the following design

$$D = (G_1, G_2) \quad \text{with } G_1 = L_{n_1} \otimes 1_{n_2} \text{ and } G_2 = 1_{n_1} \otimes L_{n_2}, \tag{1}$$

where 1_n is a column vector of $n \times 1$ with all elements 1's. By Lemma 2.1, it can be verified that $J(c_{j_1} \otimes 1_{n_2}, \ldots, c_{j_k} \otimes 1_{n_2}, 1_{n_1} \otimes d_{l_1}, \ldots, 1_{n_1} \otimes d_{l_g}) = 0$ for all $j_1, \ldots, j_k, l_1, \ldots, l_g$ such that $1 \leq j_1 < \cdots < j_k \leq m_1$, $1 \leq l_1 < \cdots < l_g \leq m_2$ and $k \leq r_1 - 1$, $g \leq r_2 - 1$. Thus $r_3 = r_1 + r_2 - 1$. This completes the proof.

Example 2.2 Let L_{n_1} and L_{n_2} be obtained from two Hadamard matrices of orders n_1 and n_2 by deleting the first column of all unities, respectively. Then the design D constructed in (1) is a $D(n_1 n_2, (n_1 - 1, n_2 - 1), (3, 3)_5)$.

3 Connection with compromise plans

We give a brief introduction for compromise plans, clear compromise plans and robust designs through partially clear 2fi's. A compromise plan allows estimation of all main effects and some specified 2fi's, assuming that all other effects are negligible. There are four classes of compromise plans (Addelman 1962; Sun 1993). For $D = (G_1, G_2)$, let $G_1 \times G_1$ denote the set of 2fi's within G_1 and $G_1 \times G_2$ denote the set of 2fi's between G_1 and G_2 as before. Thus the set of 2fi's to be estimated in compromise plans of classes one to four are given by: (1) $G_1 \times G_1$; (2) $G_1 \times G_1$ and $G_2 \times G_2$; (3) $G_1 \times G_1$ and $G_1 \times G_2$; (4) $G_1 \times G_2$. For some applications, the assumption that all other 2fi's are negligible in compromise plans may be too strong. Then, Ke et al. (2005) proposed clear compromise plans with resolution IV, where the specified 2fi's are clear. An effect is said to be clear if it is orthogonal to all main effects and all other

2fi's. If it is orthogonal to all main effects, but aliased with some 2fi's, then it is said to be eligible. For more details, please refer to Chen and Hedayat (1998), Tang et al. (2002), Ai and He (2006), Ai and Zhang (2004), Yang et al. (2006), Zhao and Zhang (2008, 2010), Zhao et al. (2013) and Zi et al. (2007). More explicitly, Lekivetz and Tang (2011) divided all 2fi's into three mutually exclusive and exhaustive sets, S_1, S_2 and S_3, where S_1 denotes the set of 2fi's to be estimated, S_2 the set of nonnegligible 2fi's and S_3 the set of negligible 2fi's. Three types of designs containing partially clear 2fi's are given by:

(1) $S_1 = G_1 \times G_1, S_2 = G_1 \times G_2, S_3 = G_2 \times G_2$;
(2) $S_1 = G_1 \times G_1, S_2 = G_2 \times G_2, S_3 = G_1 \times G_2$;
(3) $S_1 = G_1 \times G_2, S_2 = G_1 \times G_1, S_3 = G_2 \times G_2$.

We now derive some relationships between generalized variable resolution designs $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ and compromise plans, clear compromise plans and robust designs through partially clear 2fi's.

3.1 Connection with compromise plans of class one

Proposition 3.1 *(i) A $D(n, (m_1, m_2), (5, 3)_4)$ is a compromise plan of class one. (ii) A $D(n, (m_1, m_2), (5, 3)_5)$ is a clear compromise plan of class one.*

By definition of generalized variable resolution design $D(n, (m_1, m_2), (5, 3)_4)$, it can be verified that $J(a_{i_1}, a_{i_2}, b_{j_1}) = 0$ for any a_{i_1}, a_{i_2} in G_1 and b_{j_1} in G_2. We note that a compromise plan of class one is not necessarily a $D(n, (m_1, m_2), (5, 3)_4)$ since $J(a_i, b_{j_1}, b_{j_2}) = 0$ cannot be guaranteed for a compromise plan of class one. Part (ii) can be obtained similarly.

Remark 3.1 Proposition 3.1 indicates that a $D(n, (m_1, m_2), (5, 3)_5)$ is a clear compromise plan of class one. Then a lower bound of the number of clear 2fi's in a $D(n, (m_1, m_2), (5, 3)_5)$ is $m_1(m_1 - 1)/2$.

For the existence of generalized variable resolution designs, an interesting problem is to consider the maximum values of m_1, m_2, or m for given run size n and resolutions $(r_1, r_2)_{r_3}$. Let D be a regular 2^{m-p} design, $k = m - p$ and $m = m_1 + m_2$. For the upper bound of m, we have the following proposition.

Proposition 3.2 *If D is a $D(n, (m_1, m_2), (5, 3)_5)$, then $m \leq 2^{k-2} + 1$.*

Proof We first note that $m \leq 2^{k-2} + 2$ for $r_3 = 5$ (See Lemma 2 of Lekivetz and Tang (2011)). Next, it can be proved that there is no such design with $m = 2^{k-2} + 2$. By Definition 2.2, all main effects and 2fi's in $G_1 \times G_1$ and $G_1 \times G_2$ are mutually orthogonal, which implies $m + m_1(m_1 - 1)/2 + m_1 m_2 \leq 2^k - 1$. Let $f(m_1) = m_1^2 - (2m - 1)m_1 + 2^{k+1} - 2m - 2$. We have $f'(m_1) = 2m_1 - (2m - 1) < 0$. Note that, for $m = 2^{k-2} + 2$, we have $f(2) = 2^{k-1} - 8 \geq 0$ for $k \geq 4$ and $f(3) = -6 < 0$. Hence, $m_1 \leq 2$ for $f(m_1) \geq 0$.

For $m_1 = 2$, then $m_2 = 2^{k-2} + 2 - 2 = 2^{k-2}$. Let $G_1 = (a_1, a_2)$ and $G_2 = (b_1, \ldots, b_{m_2})$. Since $D = (G_1, G_2)$ is a $D(n, (m_1, m_2), (5, 3)_5)$, then

$$J(a_1, a_2) = J(b_i, b_j) = 0 (1 \leq i, j \leq m_2),$$

Table 1 Upper bounds of m_1 for $D(2^k, (m_1, m_2), (5, 3)_5)$

k	4	5	6	7	8
Upper bounds of m_1	2	4	6	9	15

$$J(a_1, a_2, b_j) = 0 (1 \le j \le m_2),$$
$$J(a_i, b_{j_1}, b_{j_2}) = 0 (1 \le j_1, j_2 \le m_2),$$
$$J(a_1, a_2, b_{j_1}, b_{j_2}) = 0 (1 \le j_1, j_2 \le m_2).$$

Thus the following effects $a_1, a_2, b_1, \ldots, b_{m_2}$, $a_1 b_1, \ldots, a_1 b_{m_2}$, $a_2 b_1, \ldots, a_2 b_{m_2}$, $a_1 a_2 b_1, \ldots, a_1 a_2 b_{m_2}$ are mutually orthogonal. Therefore, we have that $(2^{k-2} + 2) + 3 \times 2^{k-2} = 2^k + 2 > 2^k - 1$. It makes a contradiction.

For the upper bound of m_1 in $D(n, (m_1, m_2), (5, 3)_5)$, we have the following result.

Proposition 3.3 *If D is a* $D(n, (m_1, m_2), (5, 3)_5)$ *and* $m_2 \ge 3$*, then*

$$m_1 \le \min\left\{\lfloor(\sqrt{2^{k+3} + 17} - 7)/2\rfloor, M(k) - 2\right\}, \tag{2}$$

where $\lfloor x \rfloor$ *means the largest integer not exceeding x and* $M(k)$ *is the maximum value of m for a* 2^{m-p} *design with resolution at least 5.*

Proof By the definition of generalized variable resolution designs $D(n, (m_1, m_2), (5, 3)_5)$, all the main effects, 2fi's $a_{i_1} a_{i_2}, i_1, i_2 = 1, \ldots, m_1$, and 2fi's $a_i b_j, i = 1, \ldots, m_1, j = 1, \ldots, m_2$, are mutually orthogonal. Hence, we have $m_1 + m_2 + m_1(m_1 - 1)/2 + m_1 m_2 \le 2^k - 1$. Solving for m_2, we have $m_2 \le (2^k - 1)/(m_1 + 1) - m_1/2$. Note that $m_2 \ge 3$, then $m_1^2 + 7m_1 - 2^{k+1} + 8 \le 0$, which leads to the conclusion $m_1 \le \lfloor(\sqrt{2^{k+3} + 17} - 7)/2\rfloor$.

On the other hand, it is easy to see that (G_1, b_i, b_j) is a design of resolution V, where b_i and b_j are any two distinct columns from G_2. Hence, $m_1 + 2 \le M(k)$. This completes the proof.

Table 1 gives the performance of the upper bound (2) of m_1. For $k = 4, 5, 6, 7$ and 8, the value of $M(k)$ is 5, 6, 8, 11, 17, and the upper bounds of m_1 is 2, 4, 6, 9, 15, respectively. Note that a $D(n, (m_1, m_2), (5, 3)_5)$ is not necessarily a resolution V plan. So it is expected that the number of columns of a $D(n, (m_1, m_2), (5, 3)_5)$ can be larger than $M(k)$.

We now give some constructions of $D(2^k, (m_1, m_2), (5, 3)_5)$ for $k = 6, 7$ and 8. Let H_k be a saturated design with $2^k - 1$ columns by taking all the products of the k independent columns. Similarly, let H_j be the subset of H_k generated by any j independent columns, H_{k-j} be the subset of H_k generated by the remaining $k - j$ independent columns and H_j^V be a resolution V plan in H_j.

For $k = 6$, let $D = H_3^V \cup H_{6-3}$, hence $m_1 = 3, m_2 = 7$ and $m = 10 > M(6)$.

For $k = 7$, both the numbers of columns of the following two designs D_1 and D_2 are larger than $M(7)$. (i) Let $D_1 = H_3^V \cup H_{7-3}$, hence $m_1 = 3, m_2 = 15$ and

$m = 18 > M(7)$. (ii) Let $D_2 = H_4^V \cup H_{7-4}$. We have $m_1 = M(4) = 5, m_2 = 7$ and $m = 12 > M(7)$.

For $k = 8$, there also exist designs with the numbers of columns larger than $M(8)$. (i) Let $D_1 = H_3^V \cup H_{8-3}$, hence $m_1 = 3, m_2 = 31$ and $m = 34 > M(8)$. (ii) Let $D_2 = H_4^V \cup H_{8-4}$, hence $m_1 = M(4) = 5, m_2 = 15$ and $m = 20 > M(8)$.

3.2 Connection with compromise plans of class three

Note that a clear compromise design of class two is equivalent to a $D(n, (m_1, m_2), (5, 5)_5)$, so there exists no clear compromise plan of class two with $r \leq 4$. The same point is also illustrated by Ke et al. (2005). For compromise plans of class three, we have the following proposition.

Proposition 3.4 *A* $D(n, (m_1, m_2), (5, 4)_5)$ *is a clear compromise plan of class three.*

Proof By Definition 2.2, it can be verified that

$$J(a_{i_1}, a_{i_2}, b_{j_1}) = J(a_{i_1}, b_{j_1}, b_{j_2}) = J(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = 0,$$
$$J(a_{i_1}, a_{i_2}, a_{i_3}, b_{j_1}) = J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2}) = J(a_{i_1}, b_{j_1}, b_{j_2}, b_{j_3}) = 0,$$

for any $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ in G_1 and $b_{j_1}, b_{j_2}, b_{j_3}$ in G_2. Hence, $D(n, (m_1, m_2), (5, 4)_5)$ is a clear compromise plan of class three.

The method of Construction 3 in Tang (2006) can be used to construct $D(n, (m_1, m_2), (5, 4)_5)$. Let $G_1 = (a_1, \ldots, a_{m_1})$ be a $D(n_1, m_1, 5)$ and $G_2 = (b_1, \ldots, b_{m_2})$ be a $D(n_2, m_2, 3)$. Consider the design

$$D = (c_1, \ldots, c_{m_1}, d_1, \ldots, d_{m_2}), \tag{3}$$

where $c_i = a_i \otimes 1_{n_2}, d_j = (a_1 a_2) \otimes b_j$. Then D is a $D(n_1 n_2, (m_1, m_2), (5, 4)_5)$.

Example 3.1 Let G_1 be a two-level design with 16 runs and 5 columns of resolution V and G_2 be a $D(n_2, n_2 - 1, 3)$ obtained from a Hadamard matrix of order n_2. Then the design D constructed in (3) is a $D(16n_2, (5, n_2 - 1), (5, 4)_5)$.

Remark 3.2 By Proposition 3.4, a lower bound of the number of clear 2fi's in a $D(n, (m_1, m_2), (5, 4)_5)$ is $m_1(m_1 + 2m_2 - 1)/2$.

3.3 Connection with compromise plans of class four

Proposition 3.5 *A* $D(n, (m_1, m_2), (3, 3)_5)$ *is a compromise plan of class four.*

By Definition 2.2, it can be verified that $J(a_i, b_{j_1}, b_{j_2}) = J(a_{i_1}, a_{i_2}, b_{j_1}) = J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2}) = 0$ for any a_{i_1}, a_{i_2} in G_1 and b_{j_1}, b_{j_2} in G_2. Hence, all main effects and 2fi's $a_i b_j$ are distinct.

If we let $L_{n_1} = (c_1, \ldots, c_{m_1})$ and $L_{n_2} = (d_1, \ldots, d_{m_2})$ be $D(n_1, m_1, 3)$ and $D(n_2, m_2, 3)$, respectively, then the design D in construction (1) is a $D(n, (m_1, m_2),$

$(3, 3)_5)$. It can also be constructed by the method in Sect. 3.1. Let H_j and H_{k-j} be defined as before. Consider

$$D_j = H_j \cup H_{k-j}. \tag{4}$$

It is easy to see that D_j is a $D(n, (m_1, m_2), (3, 3)_5)$, where $m_1 = 2^j - 1$ and $m_2 = 2^{k-j} - 1$. All 2fi's in $H_j \times H_{k-j}$ are clear. Note that construction (4) is used only for regular designs and construction (1) can also be used for nonregular ones.

Remark 3.3 We now give some discussion for the upper bounds of m_1 and m for given k. By the definition of generalized variable resolution designs $D(n, (m_1, m_2), (3, 3)_5)$, all the main effects, 2fi's $a_i b_j$ for any a_i in G_1 and any b_j in G_2, are mutually orthogonal. Hence, we have $m_1 + m_2 + m_1 m_2 \leq 2^k - 1$. Solving for m_2, we have $m_2 \leq \frac{2^k - 1 - m_1}{1 + m_1}$. Note that $m_2 \geq 3$, then $m_1 \leq 2^{k-2} - 1$. For the number of columns in D, it can be shown that a $D(n, (m_1, m_2), (3, 3)_5)$ satisfies $m \leq 2^{k-2} + 2$ (see the appendix of Lekivetz and Tang (2011)). Hence, the design D_2 constructed by (4) has the maximum number of clear 2fi's in $D(n, (m_1, m_2), (3, 3)_5)$'s.

For clear compromise plans of class four, we have

Proposition 3.6 *A $D(n, (m_1, m_2), (4, 4)_5)$ is a clear compromise plan of class four.*

It can be verified that $J(a_{i_1}, a_{i_2}, b_{j_1}) = J(a_{i_1}, b_{j_1}, b_{j_2}) = 0$ and $J(a_{i_1}, a_{i_2}, a_{i_3}, b_{j_2}) = J(a_{i_1}, b_{j_1}, b_{j_2}, b_{j_3}) = J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2}) = 0$ for any $a_{i_1}, a_{i_2}, a_{i_3}$ in G_1 and $b_{j_1}, b_{j_2}, b_{j_3}$ in G_2.

A method modified from Tang (2006) can be used to construct $D(n, (m_1, m_2), (4, 4)_5)$. Let $G_1 = (a_1, \ldots, a_{m_1})$ be a $D(n_1, m_1, 3)$ and $G_2 = (b_1, \ldots, b_{m_2})$ be a $D(n_2, m_2, 3)$. Consider the design

$$D = (c_2, \ldots, c_{m_1}, d_2, \ldots, d_{m_2}), \tag{5}$$

where $c_i = a_i \otimes b_1$, $d_j = a_1 \otimes b_j$. Then the design D is a $D(n_1 n_2, (m_1 - 1, m_2 - 1), (4, 4)_5)$.

Example 3.2 Let G_1 and G_2 be $D(n_1, n_1 - 1, 3)$ and $D(n_2, n_2 - 1, 3)$ obtained from two Hadamard matrices of orders n_1 and n_2, respectively. Then the design D constructed in (5) is a $D(n_1 n_2, (n_1 - 1, n_2 - 1), (4, 4)_5)$.

3.4 Connection with partially clear 2fi's plans

Proposition 3.7 *A $D(n, (m_1, m_2), (4, 3)_5)$ is a partially clear 2fi's plan of type three.*

Proof By Definition 2.2, it can be verified that

$$J(a_{i_1}, a_{i_2}, a_{i_3}) = J(a_{i_1}, a_{i_2}, b_{j_1}) = J(a_{i_1}, b_{j_1}, b_{j_2}) = 0,$$
$$J(a_{i_1}, a_{i_2}, a_{i_3}, b_{j_1}) = J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2}) = 0,$$

for any $a_{i_1}, a_{i_2}, a_{i_3}$ in G_1 and b_{j_1}, b_{j_2} in G_2. Hence, it is a partially clear 2fi's plan of type three.

Remark 3.4 Note that a $D(n, (m_1, m_2), (5, 3)_5)$ can be a partially clear 2fi's plan of type one or two. However, a partially clear 2fi's plan of type one or two is not necessarily a $D(n, (m_1, m_2), (5, 3)_5)$, since a partially clear 2fi's plan of type one or two only satisfies two of the following three conditions: (i) $J(a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}) = 0$; (ii) $J(a_{i_1}, a_{i_2}, a_{i_3}, b_{j_1}) = 0$; (iii) $J(a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2}) = 0$ for any $a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}$ in G_1 and b_{j_1}, b_{j_2} in G_2.

4 An application in robust parameter designs

In robust parameter designs, Wu and Hamada (2009) arranged the importance of effects according to the effect ordering principle in the following descending order:

(i) control-by-noise interactions, control main effects, and noise main effects;
(ii) control-by-control interactions and control-by-control-by-noise interactions;
(iii) noise-by-noise interactions.

If all the effects in (i) and (ii) need to be estimated by experimenter while assuming other effects negligible, a $D(n, (m_1, m_2), (5, 3)_7)$ is a natural choice. From Table 2, all main effects, control-by-noise interactions, control-by-control interactions and control-by-control-by-noise interactions are distinct. Such a design can be constructed by (1) in Proposition 2.2. Let $L_{n_1} = (c_1, \ldots, c_{m_1})$ be a $D(n_1, m_1, 5)$ and $L_{n_2} = (d_1, \ldots, d_{m_2})$ be a $D(n_2, m_2, 3)$. Then the following design

$$D = (G_1, G_2) \quad \text{with } G_1 = L_{n_1} \otimes 1_{n_2} \text{ and } G_2 = 1_{n_1} \otimes L_{n_2},$$

is a $D(n, (m_1, m_2), (5, 3)_7)$.

Some useful relationships between clear effects, eligible effects and generalized variable resolution designs are summarized below. To save space, $D(n, (m_1, m_2),$

Table 2 Connection with robust parameter designs

	G_1	G_2	$G_1 \times G_1$	$G_1 \times G_2$	$G_2 \times G_2$	$G_1 \times G_1 \times G_2$
$(3, 3)_3$	∘	∘				
$(3, 3)_4$	∘	∘		∘		
$(4, 3)_4$	•	∘	∘	∘		
$(4, 4)_4$	•	•	∘	∘	∘	
$(3, 3)_5$	∘	∘		∘/•		
$(4, 3)_5$	•	∘	∘	∘/•		
$(4, 4)_5$	•	•	∘	•	∘	
$(5, 3)_5$	•	∘	•	∘/•		
$(5, 4)_5$	•	•	•	•	∘	
$(5, 5)_5$	•	•	•	•	•	
$(5, 3)_7$	•	∘	•	•		•
$(5, 4)_7$	•	•	•	•	∘	•
$(5, 5)_7$	•	•	•	•	•	•

$(r_1, r_2)_{r_3}$) is denoted by $(r_1, r_2)_{r_3}$, clear effect is denoted by "•" and eligible effect is denoted by "∘". Notation "∘/•" means both effects are available.

Table 2 shows different generalized variable resolution designs can be used for different purposes of experimenters. For instance, the sets G_1, G_2 and $G_1 \times G_2$ in a generalized variable resolution designs $(3, 3)_4$ are denoted by ∘, which shows all main effects and two-factor interactions in $G_1 \times G_2$ are eligible.

5 Conclusion and discussion

In this paper, designs of variable resolution have been generalized to a more flexible version, designs of generalized variable resolution. It can be seen that the generalized variable resolution designs can also be used to estimate the interactions between groups of factors. The conditions for the existence of generalized variable resolution designs were discussed. Some connections between compromise plans and generalized variable resolution designs were also studied. The upper bounds of m_1 for designs $D(n, (m_1, m_2), (5, 3)_5)$ have also been obtained. By the relationships between generalized variable resolution designs and clear compromise plans, the lower bounds of the number of clear 2fi's in some special generalized variable resolution designs can be obtained.

The construction method of $D(n, (m_1, m_2), (5, 3)_5)$ and $D(n, (m_1, m_2), (3, 3)_5)$ can be generalized for the general case $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$. Let H_j^{r1} be a $D(n, m_1, r_1)$ in H_j and H_{k-j}^{r2} be a $D(n, m_2, r_2)$ in H_{k-j}. Then it can be verified that the following design

$$D = H_j^{r1} \cup H_{k-j}^{r2}$$

is a $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$.

From a practical standpoint, the design of generalized variable resolution is appealing because prior information can be used to different parts of the design $D = (G_1, G_2)$ directly. If two-factor interactions in $G_1 \times G_1$ and $G_2 \times G_2$ are assumed to be negligible, then a $D(n, (m_1, m_2), (3, 3)_5)$ can be used for the estimation of both main effects and two-factor interaction in $G_1 \times G_2$. Otherwise, a $D(n, (m_1, m_2), (3, 3)_4)$ may be used for the estimation of all main effects regardless of interactions in $G_1 \times G_2$. From Table 2, various designs of generalized variable resolution can be used for different requirement of experimenters. In this paper, some construction methods for regular $D(n, (m_1, m_2), (r_1, r_2)_{r_3})$ are provided. However, more constructions are still needed, especially for nonregular plans. Such work is under progress.

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