A necessary test for complete independence in high dimensions using rank-correlations

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Abstract

We propose a nonparametric necessary test for the complete independence of random variables in high-dimensional environment. The test is constructed based on Spearman’s rank-correlations and is shown to be asymptotically normal by martingale central limit theorem as both the sample size and the dimension of variables go to infinity. Simulation studies show that the proposed test works well in finite-sample situations.

Keywords: Asymptotic normality; Complete independence; High-dimensional problem; Necessary tests; Spearman’s rank-correlation.

1. Introduction

With the rapid development of technology, various types of high-dimensional data have been generated in many areas, such as hyperspectral imagery, internet portals, microarray analysis and DNA. High-dimensional data refers to a data whose dimension $p$ increases to infinity as the number of observations $n \rightarrow \infty$. Traditional statistical methods may not work any more in this situation since they assume that $p$ keeps unchanged as $n$ increases. This challenge calls for new research on properties of traditional methods and new statistical approaches to deal with high-dimensional data, e.g., see Bai and Saranadasa (1996), Ledoit and Wolf (2002), Fan et al. (2005), Wang et al. (2009) and Chen and Qin (2010) and the references therein.

Tests for the independence of the variables play an important role in a number of statistical problems. The practical needs for testing the independence in high-dimensional environment come from several areas of nowadays statistical applications, in particular from microarray analysis (or the associated large-scale multiple testing) which is a typical “large $p$, small $n$” problem. A common assumption made

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when analyzing the microarray data is the so-called gene-wise independence, namely independence in the expression levels among different genes. Hence, it is important to carry out high-dimensional independence tests before statistical procedures are employed.

Let $y_1, \ldots, y_n$ be independent and identically distributed $p$-dimensional random vectors from a continuous multivariate distribution $F(\cdot)$, where $y_k = (y_{k1}, \ldots, y_{kp})^T$. Under multinormal assumption on $y_k$, testing for complete independence means testing the covariance matrix to be diagonal or the population correlation matrix to be identity. The classical likelihood ratio test is not valid for high-dimensional data when $p \geq n$. Schott (2005) proposed a test

$$t_{np} = \sum_{i=2}^{p} \sum_{j=1}^{i-1} \rho_{ij}^2 - \frac{p(p-1)}{2(n-1)}$$

based on the $p(p-1)/2$ Pearson’s sample correlation coefficient $\rho_{ij}$’s (between $i$th and $j$th variables). He showed that under the hypothesis of complete independence and the condition $p/n \to \gamma$ as $p, n \to \infty$, $t_{np}$ converges in distribution to a normal random variable with mean 0 and variance $\gamma^2$. Srivastava (2005, 2006) proposed a test statistic

$$FZ_{np} = \frac{(n-3) \sum_{i<j} z_{ij}^2 - \frac{1}{2} p(p-1)}{\sqrt{p(p-1)}}$$

based on Fisher’s $z$-transformation $z_{ij} = \frac{1}{2} \log \frac{1+\rho_{ij}}{1-\rho_{ij}}$, for $i \neq j$.

It is well recognized that, in many applications, the underlying process distribution is unknown and not multinormal, so that properties of the tests above could potentially be (highly) affected. A test for the diagonality of the covariance matrix (which implies independence of the components under normality) has been proposed by Srivastava (2005), with the test statistic

$$W_{np} = \frac{n-1}{2} \frac{\hat{\gamma} - 1}{[1 - \frac{1}{p} \hat{a}_{20}^2]^{1/2}},$$

where $\hat{\gamma} = \hat{a}_2/\hat{a}_{20}$, $\hat{a}_2 = \frac{(n-1)^2}{(n-2)(n+1)} \frac{1}{p} [\text{tr} S^2 - \frac{1}{n-1} (\text{tr} S)^2]$, $\hat{a}_{20} = n-1 \sum_{i=1}^p s_{ii}^2$ and $\hat{a}_{40} = \frac{1}{p} \sum_{i=1}^p s_{ii}^4$. Srivastava et al. (2011) have shown that this test is asymptotically robust under certain departure from normality. However, as shown by the simulation results in Section 3, such a moment-based method cannot achieve satisfactory sizes under certain non-normal distributions in finite-sample settings. Nonparametric or robust correlations may be useful in such situations. In this spirit, we suggest to replace the Pearson’s correlation coefficient $\rho_{ij}$ in (1) by the Spearman’s rank-correlation
coefficient (Spearman 1904)

\[
r_{ij} = \frac{\sum_{k=1}^{n} (\tilde{R}_{ik} - \frac{n+1}{2})(\tilde{R}_{jk} - \frac{n+1}{2})}{\sqrt{\sum_{k=1}^{n} (\tilde{R}_{ik} - \frac{n+1}{2})^2 \sum_{k=1}^{n} (\tilde{R}_{jk} - \frac{n+1}{2})^2}},
\]

formulating a test statistic \( Q_{np} = \sum_{i=2}^{p} \sum_{j=1}^{i-1} r_{ij}^2 \), where \( \tilde{R}_{jk} \) is the rank of \( y_{kj} \) in \( \{y_{lj}\}_{l=1}^{n} \). The proposed test is shown to be asymptotically normal without any distributional assumption and our simulation studies demonstrate it works well in finite-sample situations.

It should be emphasized that under non-normal distributions, the independence hypothesis would be no longer equivalent to the uncorrelated hypothesis. The test procedure developed here is such that when the calculated test statistic exceeds the critical value, one can conclude that the components are not independent. However, when the test statistic does not exceed the critical value, we cannot conclude that the components are independent, only that they are uncorrelated. A test of this sort is often referred to as a “necessary test” of the null hypothesis (Liang et al. 2008).

2. A necessary test for complete independence using rank-correlations

Let \( \tilde{R}_{i}^{T} = (\tilde{R}_{i1}, \ldots, \tilde{R}_{in}) \). Spearman’s rank-correlation coefficient can be written as

\[
r_{ij} = \frac{(\tilde{R}_{i} - \frac{n+1}{2}1_{n})^{T}(\tilde{R}_{j} - \frac{n+1}{2}1_{n})}{\|\tilde{R}_{i} - \frac{n+1}{2}1_{n}\|_2 \|\tilde{R}_{j} - \frac{n+1}{2}1_{n}\|_2},
\]

where \( \|x\|_2 = \sqrt{x^{T}x} \) and \( 1_{n} \) denotes the \( n \times 1 \) vector with each component being 1. As discussed before, the statistic \( Q_{np} = \sum_{i=2}^{p} \sum_{j=1}^{i-1} r_{ij}^2 \) would be an intuitively plausible measure for the complete independence since a large value of \( Q_{np} \) apparently indicates certain correlations exist among all the \( p \) variables.

Let \( R_{i}^{T} = (R_{i1}, \ldots, R_{in}) \) is the standardized rank vector, say \( R_{i}^{T} = \frac{\tilde{R}_{i} - \frac{n+1}{2}1_{n}}{\|\tilde{R}_{i} - \frac{n+1}{2}1_{n}\|_2} \), for \( i = 1, \ldots, p \). So we have \( r_{ij} = R_{i}^{T}R_{j} \) and the basic properties \( R_{i}^{T}1_{n} = 0 \). By Proposition 1 in Appendix A.1, we can get, under the null hypothesis,

\[
E[Q_{np}] = \frac{p(p-1)}{2(n-1)},
\]

\[
\sigma_{Q_{np}}^2 \equiv \text{Var}[Q_{np}] = \frac{p(p-1)(50n^3 - 114n^2 - 80n + 216)}{50n(n-1)^3(n+1)}.
\]

Following Schott (2005), we consider the set that the sample size \( n \) and the dimension of variables \( p \) go to infinity in such a way that

\[
\lim_{n} \frac{p}{n} = \gamma \in (0, \infty).
\]
Note that the variance of $Q_{np}$ converges to $\gamma^2$ which is in line with Schott’s $t_{np}$. We establish the asymptotic null distribution of $Q_{np}$ in the following theorem.

**Theorem 1.** Suppose that $F(\cdot)$ is continuous and condition (4) holds. Under the null hypothesis that the $p$ variables are of complete independence, 

$$(Q_{np} - E[Q_{np}])/\sigma_{Q_{np}} \xrightarrow{d} N(0,1).$$

The proposed test with $\alpha$ level of significance rejects the null hypothesis if $(Q_{np} - E[Q_{np}])/\sigma_{Q_{np}} > z_\alpha$ where $z_\alpha$ is the upper $\alpha$ quantile of $N(0,1)$. Note that the asymptotic normality of $t_{np}$ requires the existence of the eighth moments of marginal distributions, while the validity of Theorem 1 does not require such conditions due to the use of ranks. By this feature, We would expect that the $Q_{np}$ could has some advantage over $t_{np}$ in controlling false alarm rates under skewed or heavy-tailed distributions (see some evidence in Section 3). In fact, it is easy to see that for every $n$, if $F$ is continuous, then $Q_{np}$ has the remarkable property that its exact distribution is completely independent of $F$ under the null hypothesis, say $Q_{np}$ is exactly distribution-free. Hence, for given $n$ and $p$, the true sampling distribution of $Q_{np}$ can be approximated by simulation with any complete independent random variables.

This theorem is proved by using the martingales CLT (e.g., see Hall and Heyde 1980). Although $Q_{np}$ is of a similar form to Schott’s (2005) $t_{np}$ in (1), theoretical derivation is more cumbersome due to the use of ranks. We shall highlight some key parts here but the details are deferred into the appendix. The main difficulty is the results stated by Lemma 1 which involves the derivation of the variances of quadratic forms $\text{Var}[R_l^TAR_l]$. The components of $R_l$ are correlated, making theoretical derivations a little complicated.

Denote $q_{np} = Q_{np} - E[Q_{np}]$ for simplicity and note $r_{ij} = R_i^T R_j$. Moreover, Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_{l-1} = \sigma\{R_1, \ldots, R_{l-1}\}$, for $l = 2, \ldots, p$. Let $E_{l-1}[\cdot]$ denote the conditional expectation when given $\mathcal{F}_{l-1}$ and $E_0[\cdot] = E[\cdot]$. Let

$$X_{nl} = q_{nl} - q_{n,l-1} = \sum_{i=1}^{l-1} r_{li}^2 - \frac{l - 1}{n - 1},$$

where $q_{n1} = 0$ and thus $q_{np} = \sum_{l=2}^{p} X_{nl}$.

It’s easy to show that $E_{l-1}[X_{nl}] = 0$. So, for each $n$, $q_{nl}$ is a martingale and $X_{n2}, \ldots, X_{np}$ are martingale differences. To prove Theorem 1, by martingales CLT, it suffices to show that, as $n, p \to \infty$,

$$\frac{\sum l E_{l-1}[X_{nl}^2]}{\text{Var}[q_{np}]} \to 1,$$

$$\frac{\sum l E[X_{nl}^4]}{\text{Var}^2[q_{np}]} \to 0.$$  (6)
To show condition (5) we verify that

\[ E\left[ \frac{\sum_t E_{t-1}[X_{nl}^2]}{\text{Var}[q_{np}]} - 1 \right]^2 \to 0. \]

By expanding this and noting the fact that

\[ E\left[ \frac{\sum_t E_{t-1}[X_{nl}^2]}{\text{Var}[q_{np}]} \right] = \frac{\sum_t E[X_{nl}^2]}{\text{Var}[q_{np}]} = \frac{E[(\sum_t X_{nl})^2]}{\text{Var}[q_{np}]} = E[q_{np}^2] = 1, \]

we only need to show that

\[ \sum_t E_{t-1}[X_{nl}^2] \text{Var}[q_{np}] = E\left[ \frac{(\sum_t E_{t-1}[X_{nl}^2])^2}{\text{Var}[q_{np}]} \right] \to 1. \]  

(7)

This involves the evaluation of the terms like \( E_{t-1}[r_{li}^4] \) and \( E_{t-1}[r_{li}^2 r_{lj}^2] \) which are approximated in the following lemma.

**Lemma 1.** Suppose that \( F(\cdot) \) is continuous. Under the null hypothesis, we have

\[ E_{t-1}[r_{li}^4] = \frac{3}{n(n-1)} + o(n^{-2}), \]

\[ E_{t-1}[r_{li}^2 r_{lj}^2] = \frac{1}{n(n-1)} + c_1 \sum_{k=1}^n R_{ik} R_{jk}^2 + c_2 (R_i^T R_j)^2 + c_3 + o(n^{-3}), \]

where \( c_1 = -\frac{6n^2}{5(n-1)^2(n+1)^2}, c_2 = \frac{2}{n(n-1)}, c_3 = -\frac{9n^2}{5(n-1)^3(n+1)^2} + \frac{2}{n(n-1)(n-2)}. \)

A succinct proof is provided in Appendix A.2. Using the lemma above we can verify that (7) holds. For condition (6), by expanding the numerator that involves the joint 3-th or 4-th moments of \( r_{l1}^2, \ldots, r_{l t-1}^2 \), we can further show that

\[ \sum_t E[X_{nl}^4] \text{Var}^2[q_{np}] = O(p^{-1}), \]  

(8)

where the details can be found in Appendix A.3.

3. Simulation Study

We present some simulation results in this section evaluating the performance of our proposed test statistic \( Q_{np} \). Schott’s (2005) \( t_{np} \), Srivastava’s (2005, 2006) \( F_{Z_{np}} \) and \( W_{np} \) are used here as benchmarks for comparison use. The settings of the combinations of \( p \) and \( n \) in Schott (2005) are adopted here. Let \( p \) range over the values 4, 8, 16, 32, 64, 128 and 256 and \( n \) range over the values 5, 9, 17, 33, 65, 129
and 257. All results are obtained from 5,000 simulations and the nominal significant level is 0.05.

The number of variety of distributions and parameters are too large to allow a comprehensive, all-encompassing comparison. We choose certain representative examples for illustration. Under $H_0$, four scenarios are considered: standard multivariate normal distribution; In the other three scenarios, we consider that each component of the $p$-dimension vector $y$ comes from a non-normal distribution independently. Three distributions are considered, $t_3$, $\chi^2_1$ and a standard log-normal distribution. Under $H_1$, for the first scenario, we generate $y \sim N_p(0, \Sigma)$ with $\Sigma = 0.9I_p + 0.11I_p1_p^T$; where $I_p$ is the $p$-dimensional identity matrix; For the other three scenarios, we firstly generate data $y$ as under $H_0$ and then transform them to $My$, where the elements of $M = (m_{ij})_{p \times p}$ are chosen to be $m_{ij} = \tau^{|i-j|}$. Here we set $\tau$ to be 0.08 which brings certain degrees of dependence among the variables.

Table 1: The empirical sizes under normal distribution for $Q_{np}$.

<table>
<thead>
<tr>
<th></th>
<th>$n = 5$</th>
<th>$n = 9$</th>
<th>$n = 17$</th>
<th>$n = 33$</th>
<th>$n = 65$</th>
<th>$n = 129$</th>
<th>$n = 257$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p = 4$</td>
<td>0.062</td>
<td>0.070</td>
<td>0.068</td>
<td>0.069</td>
<td>0.064</td>
<td>0.064</td>
<td>0.065</td>
</tr>
<tr>
<td>$p = 8$</td>
<td>0.065</td>
<td>0.058</td>
<td>0.065</td>
<td>0.061</td>
<td>0.064</td>
<td>0.059</td>
<td>0.065</td>
</tr>
<tr>
<td>$p = 16$</td>
<td>0.061</td>
<td>0.065</td>
<td>0.054</td>
<td>0.059</td>
<td>0.055</td>
<td>0.055</td>
<td>0.053</td>
</tr>
<tr>
<td>$p = 32$</td>
<td>0.067</td>
<td>0.063</td>
<td>0.059</td>
<td>0.055</td>
<td>0.050</td>
<td>0.056</td>
<td>0.055</td>
</tr>
<tr>
<td>$p = 64$</td>
<td>0.069</td>
<td>0.063</td>
<td>0.058</td>
<td>0.053</td>
<td>0.049</td>
<td>0.052</td>
<td>0.048</td>
</tr>
<tr>
<td>$p = 128$</td>
<td>0.073</td>
<td>0.061</td>
<td>0.059</td>
<td>0.062</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p = 256$</td>
<td>0.064</td>
<td>0.060</td>
<td>0.055</td>
<td>0.055</td>
<td>0.046</td>
<td>0.053</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Table 2: The empirical sizes for $t_{np}$, $FZ_{np}$ and $W_{np}$ in the case that $n=33$ fixed while $p$ varies. The symbol "-" in the table means NA (not available)

<table>
<thead>
<tr>
<th></th>
<th>Normal</th>
<th>$t_3$</th>
<th>$\chi^2_1$</th>
<th>Log-Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>$t_{np}$</td>
<td>$FZ_{np}$</td>
<td>$W_{np}$</td>
<td>$t_{np}$</td>
</tr>
<tr>
<td>4</td>
<td>0.072</td>
<td>0.074</td>
<td>0.068</td>
<td>0.073</td>
</tr>
<tr>
<td>8</td>
<td>0.062</td>
<td>0.066</td>
<td>0.066</td>
<td>0.075</td>
</tr>
<tr>
<td>16</td>
<td>0.060</td>
<td>0.064</td>
<td>0.055</td>
<td>0.078</td>
</tr>
<tr>
<td>32</td>
<td>0.055</td>
<td>0.055</td>
<td>0.053</td>
<td>0.068</td>
</tr>
<tr>
<td>64</td>
<td>0.060</td>
<td>0.061</td>
<td>0.052</td>
<td>0.069</td>
</tr>
<tr>
<td>128</td>
<td>0.055</td>
<td>0.057</td>
<td>0.050</td>
<td>0.071</td>
</tr>
<tr>
<td>256</td>
<td>0.056</td>
<td>0.053</td>
<td>0.052</td>
<td>0.066</td>
</tr>
</tbody>
</table>

Noting that under the null hypothesis the exact distribution of our $Q_{np}$ does not depend on the sampling distribution, the empirical sizes under the four scenarios only show different realizations of Monte Carlo errors. So we only tabulate the resulting sizes under normal distribution for $Q_{np}$; see Table 1. For the other three
tests, we choose to present three \((n, p)\)-varying cases to save some space: \(n = 33\) fixed, while \(p\) varies (Table 2); \(p = 128\) fixed, while \(n\) varies (Table 3); both \(n\) and \(p\) vary but \(n = p + 1\) (Table 4). The whole cases as listed in Table 1 are available from the authors upon request. Under the normal distribution, the empirical sizes of all tests are converging to the nominal level. However, under the other three non-normal scenarios, the proposed \(Q_{np}\) can always achieve (approximately) the nominal size but the other tests have considerable biases in size as we would expect. In Tables 2-4, the entries of \(W_{np}\) with the symbol “-” represent the corresponding empirical sizes are not available. The reason of this unavailability is that \(1 - \frac{1}{p} \frac{a_{np}}{a_{20}^2}\) in the denominator of (3) may be negative in some repeats.

Now, let us turn to the power comparison. Since for the non-normal scenarios, \(t_{np}\), \(FZ_{np}\) and \(W_{np}\) have considerable size distortion as shown in Tables 2-4, we perform a size-corrected power comparison in the sense that the actual critical values are found through simulations so that all tests have accurate sizes 0.05 in each scenario. In addition, as shown in Tables 2-4, the empirical sizes of \(W_{np}\) are not available in most of the considered cases, we choose not to include it in this power

### Table 3: The empirical sizes for \(t_{np}\), \(FZ_{np}\) and \(W_{np}\) in the case that \(p = 128\) fixed while \(n\) varies.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
<th>(t_{3})</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
<th>(\chi^2_{1})</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.073</td>
<td>0.000</td>
<td>0.061</td>
<td>0.068</td>
<td>0.001</td>
<td>-</td>
<td>0.073</td>
<td>0.650</td>
<td>-</td>
<td>0.075</td>
<td>0.254</td>
</tr>
<tr>
<td>9</td>
<td>0.058</td>
<td>0.027</td>
<td>0.055</td>
<td>0.061</td>
<td>0.149</td>
<td>-</td>
<td>0.076</td>
<td>0.982</td>
<td>-</td>
<td>0.074</td>
<td>0.979</td>
</tr>
<tr>
<td>17</td>
<td>0.058</td>
<td>0.056</td>
<td>0.052</td>
<td>0.064</td>
<td>0.218</td>
<td>-</td>
<td>0.077</td>
<td>0.826</td>
<td>0.123</td>
<td>0.091</td>
<td>0.960</td>
</tr>
<tr>
<td>33</td>
<td>0.055</td>
<td>0.057</td>
<td>0.050</td>
<td>0.071</td>
<td>0.184</td>
<td>-</td>
<td>0.086</td>
<td>0.422</td>
<td>0.119</td>
<td>0.107</td>
<td>0.790</td>
</tr>
<tr>
<td>65</td>
<td>0.050</td>
<td>0.050</td>
<td>0.046</td>
<td>0.075</td>
<td>0.141</td>
<td>-</td>
<td>0.082</td>
<td>0.187</td>
<td>0.093</td>
<td>0.128</td>
<td>0.520</td>
</tr>
<tr>
<td>129</td>
<td>0.052</td>
<td>0.053</td>
<td>0.053</td>
<td>0.077</td>
<td>0.123</td>
<td>-</td>
<td>0.079</td>
<td>0.115</td>
<td>0.079</td>
<td>0.140</td>
<td>0.325</td>
</tr>
<tr>
<td>257</td>
<td>0.046</td>
<td>0.047</td>
<td>0.049</td>
<td>0.083</td>
<td>0.104</td>
<td>-</td>
<td>0.067</td>
<td>0.076</td>
<td>0.076</td>
<td>0.137</td>
<td>0.225</td>
</tr>
</tbody>
</table>

### Table 4: The empirical sizes for \(t_{np}\), \(FZ_{np}\) and \(W_{np}\) in the case that \(n = p + 1\).

<table>
<thead>
<tr>
<th>((n, p))</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
<th>(t_{3})</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
<th>(\chi^2_{1})</th>
<th>(t_{np})</th>
<th>(FZ_{np})</th>
<th>(W_{np})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5, 4)</td>
<td>0.069</td>
<td>0.066</td>
<td>-</td>
<td>0.068</td>
<td>0.077</td>
<td>-</td>
<td>0.071</td>
<td>0.115</td>
<td>-</td>
<td>0.071</td>
<td>0.096</td>
</tr>
<tr>
<td>(9, 8)</td>
<td>0.055</td>
<td>0.065</td>
<td>-</td>
<td>0.068</td>
<td>0.082</td>
<td>-</td>
<td>0.072</td>
<td>0.152</td>
<td>-</td>
<td>0.081</td>
<td>0.151</td>
</tr>
<tr>
<td>(17, 16)</td>
<td>0.052</td>
<td>0.057</td>
<td>0.060</td>
<td>0.072</td>
<td>0.103</td>
<td>-</td>
<td>0.078</td>
<td>0.167</td>
<td>-</td>
<td>0.087</td>
<td>0.205</td>
</tr>
<tr>
<td>(33, 32)</td>
<td>0.055</td>
<td>0.055</td>
<td>0.053</td>
<td>0.068</td>
<td>0.101</td>
<td>-</td>
<td>0.085</td>
<td>0.157</td>
<td>0.105</td>
<td>0.108</td>
<td>0.260</td>
</tr>
<tr>
<td>(65, 64)</td>
<td>0.054</td>
<td>0.055</td>
<td>0.053</td>
<td>0.082</td>
<td>0.122</td>
<td>-</td>
<td>0.086</td>
<td>0.137</td>
<td>0.102</td>
<td>0.123</td>
<td>0.311</td>
</tr>
<tr>
<td>(129, 128)</td>
<td>0.052</td>
<td>0.053</td>
<td>0.053</td>
<td>0.077</td>
<td>0.123</td>
<td>-</td>
<td>0.079</td>
<td>0.115</td>
<td>0.079</td>
<td>0.140</td>
<td>0.325</td>
</tr>
<tr>
<td>(257, 256)</td>
<td>0.052</td>
<td>0.053</td>
<td>0.048</td>
<td>0.086</td>
<td>0.132</td>
<td>-</td>
<td>0.068</td>
<td>0.086</td>
<td>0.077</td>
<td>0.146</td>
<td>0.342</td>
</tr>
</tbody>
</table>

7
Table 5: Power comparison between $t_{np}$, $Q_{np}$ and $F_{Z_{np}}$ in the case that $n=33$ fixed while $p$ varies.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_3$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$\chi^2_{1}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.143</td>
<td>0.136</td>
<td>0.140</td>
<td>0.161</td>
<td>0.187</td>
<td>0.159</td>
<td>0.162</td>
<td>0.578</td>
<td>0.161</td>
<td>0.149</td>
<td>0.164</td>
<td>0.381</td>
<td>0.149</td>
</tr>
<tr>
<td>8</td>
<td>0.304</td>
<td>0.255</td>
<td>0.300</td>
<td>0.191</td>
<td>0.236</td>
<td>0.182</td>
<td>0.172</td>
<td>0.670</td>
<td>0.161</td>
<td>0.156</td>
<td>0.181</td>
<td>0.493</td>
<td>0.156</td>
</tr>
<tr>
<td>16</td>
<td>0.568</td>
<td>0.515</td>
<td>0.565</td>
<td>0.186</td>
<td>0.248</td>
<td>0.177</td>
<td>0.169</td>
<td>0.750</td>
<td>0.162</td>
<td>0.138</td>
<td>0.171</td>
<td>0.551</td>
<td>0.138</td>
</tr>
<tr>
<td>32</td>
<td>0.879</td>
<td>0.836</td>
<td>0.876</td>
<td>0.193</td>
<td>0.222</td>
<td>0.167</td>
<td>0.179</td>
<td>0.775</td>
<td>0.162</td>
<td>0.146</td>
<td>0.175</td>
<td>0.577</td>
<td>0.146</td>
</tr>
<tr>
<td>64</td>
<td>0.987</td>
<td>0.979</td>
<td>0.986</td>
<td>0.208</td>
<td>0.232</td>
<td>0.202</td>
<td>0.175</td>
<td>0.784</td>
<td>0.155</td>
<td>0.151</td>
<td>0.191</td>
<td>0.593</td>
<td>0.151</td>
</tr>
<tr>
<td>128</td>
<td>1.000</td>
<td>0.999</td>
<td>1.000</td>
<td>0.213</td>
<td>0.233</td>
<td>0.197</td>
<td>0.164</td>
<td>0.805</td>
<td>0.136</td>
<td>0.135</td>
<td>0.189</td>
<td>0.593</td>
<td>0.135</td>
</tr>
<tr>
<td>256</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.223</td>
<td>0.252</td>
<td>0.202</td>
<td>0.181</td>
<td>0.809</td>
<td>0.135</td>
<td>0.103</td>
<td>0.187</td>
<td>0.582</td>
<td>0.103</td>
</tr>
</tbody>
</table>

Table 6: Power comparison between $t_{np}$, $Q_{np}$ and $F_{Z_{np}}$ in the case that $p=128$ fixed while $n$ varies.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_3$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$\chi^2_{1}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.354</td>
<td>0.305</td>
<td>0.339</td>
<td>0.066</td>
<td>0.066</td>
<td>0.062</td>
<td>0.058</td>
<td>0.073</td>
<td>0.009</td>
<td>0.064</td>
<td>0.064</td>
<td>0.072</td>
<td>0.033</td>
</tr>
<tr>
<td>9</td>
<td>0.669</td>
<td>0.603</td>
<td>0.657</td>
<td>0.074</td>
<td>0.083</td>
<td>0.073</td>
<td>0.081</td>
<td>0.140</td>
<td>0.034</td>
<td>0.078</td>
<td>0.078</td>
<td>0.096</td>
<td>0.046</td>
</tr>
<tr>
<td>17</td>
<td>0.941</td>
<td>0.918</td>
<td>0.940</td>
<td>0.119</td>
<td>0.124</td>
<td>0.108</td>
<td>0.093</td>
<td>0.324</td>
<td>0.063</td>
<td>0.113</td>
<td>0.113</td>
<td>0.213</td>
<td>0.071</td>
</tr>
<tr>
<td>33</td>
<td>1.000</td>
<td>0.999</td>
<td>1.000</td>
<td>0.213</td>
<td>0.233</td>
<td>0.197</td>
<td>0.164</td>
<td>0.805</td>
<td>0.136</td>
<td>0.189</td>
<td>0.189</td>
<td>0.593</td>
<td>0.135</td>
</tr>
<tr>
<td>65</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.430</td>
<td>0.602</td>
<td>0.409</td>
<td>0.393</td>
<td>1.000</td>
<td>0.363</td>
<td>0.357</td>
<td>0.357</td>
<td>0.987</td>
<td>0.266</td>
</tr>
<tr>
<td>129</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.897</td>
<td>0.896</td>
<td>0.886</td>
<td>0.866</td>
<td>1.000</td>
<td>0.864</td>
<td>0.756</td>
<td>0.756</td>
<td>1.000</td>
<td>0.670</td>
</tr>
<tr>
<td>257</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.999</td>
<td>0.999</td>
<td>1.000</td>
<td>0.998</td>
</tr>
</tbody>
</table>

Comparison (some numerical results have also shown that $W_{np}$ has similar performance to that of $t_{np}$ and $F_{Z_{np}}$ in the cases that we can obtain its empirical sizes). Again, the three $(n, p)$-varying cases as used in the size comparison are considered. The results are reported in Tables 5-7. Under the normal distribution, $t_{np}$ and $F_{Z_{np}}$ generally outperform $Q_{np}$ but the difference between them is insignificant in most cases. In contrast, for the other three non-normal scenarios, $Q_{np}$ performs better than $t_{np}$ and $F_{Z_{np}}$ by a considerable margin. We conducted some other simulations with various alternative hypotheses, $p$ and nominal size, to check whether the above conclusions would change in other cases. These simulation results, not reported here but available from the authors upon request, show that the $Q_{np}$ test works well for other cases as well in terms of its sizes, and its good power performance still holds for other choices of alternatives.

Acknowledgement

The authors thank the editor, associate editor, and three anonymous referees for their many helpful comments that have resulted in significant improvements in the article.

8
Table 7: Power comparison between $t_{np}$, $Q_{np}$ and $F_{Z_{np}}$ in the case that $n = p + 1$.

<table>
<thead>
<tr>
<th>$(n, p)$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
<th>$t_{np}$</th>
<th>$Q_{np}$</th>
<th>$F_{Z_{np}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5,4)</td>
<td>0.066</td>
<td>0.057</td>
<td>0.058</td>
<td>0.060</td>
<td>0.062</td>
<td>0.051</td>
<td>0.054</td>
<td>0.066</td>
<td>0.067</td>
</tr>
<tr>
<td>(9,8)</td>
<td>0.091</td>
<td>0.088</td>
<td>0.089</td>
<td>0.082</td>
<td>0.071</td>
<td>0.080</td>
<td>0.080</td>
<td>0.113</td>
<td>0.079</td>
</tr>
<tr>
<td>(17,16)</td>
<td>0.291</td>
<td>0.252</td>
<td>0.279</td>
<td>0.112</td>
<td>0.116</td>
<td>0.104</td>
<td>0.112</td>
<td>0.321</td>
<td>0.107</td>
</tr>
<tr>
<td>(33,32)</td>
<td>0.879</td>
<td>0.836</td>
<td>0.876</td>
<td>0.193</td>
<td>0.222</td>
<td>0.167</td>
<td>0.179</td>
<td>0.775</td>
<td>0.162</td>
</tr>
<tr>
<td>(65,64)</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.445</td>
<td>0.607</td>
<td>0.423</td>
<td>0.392</td>
<td>0.113</td>
<td>0.079</td>
</tr>
<tr>
<td>(129,128)</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>0.897</td>
<td>0.986</td>
<td>0.886</td>
<td>0.866</td>
<td>1.000</td>
<td>0.864</td>
</tr>
<tr>
<td>(257,256)</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Appendix

A.1 Proposition 1

We give a proposition which plays an important role in the following derivation.

**Proposition 1.** Suppose that $F(\cdot)$ is a continuous multivariate distribution.

(i) $E[R_i] = 0$ and $E[R_i R_j^T] = n^{-1}(\lambda I_n + (1 - \lambda)I_n)$, where $\lambda = -\frac{1}{n-1}$.

(ii) Under the null hypothesis, any two of the components $r_{12}^2, \ldots, r_{ip}^2, \ldots, r_{p-1,p}^2$ in $Q_{np}$ are uncorrelated.

**Proof.** (i) Since $\sum_{k=1}^n R_{ik} = 0$, $\sum_{k=1}^n R_{ik}^2 = 1$, easy to get $E[R_{ik}] = 0$, $E[R_{ik}^2] = n^{-1}$, $E[R_{ik} R_{ik}] = -\frac{n(n-1)}{n}$ for $k_1 \neq k_2$; (ii) Noting that the complete independence means that $R_1, \ldots, R_p$ are independent. So

$$E[r_{12}^2 r_{34}^2] = E[(R_1^T R_2)^2(R_3^T R_4)^2] = E[(R_1^T R_2)^2]E[(R_3^T R_4)^2] = E[r_{12}^2]E[r_{34}^2]$$

and using Proposition 1-(i) we get

$$E[r_{12}^4 r_{13}^2] = E[R_1^T R_2 R_2^T R_1 R_3^T R_4 R_1] = E[E[R_1^T R_2 R_2^T R_1 R_3^T R_4 R_1 | R_1]] = \frac{(1 - \lambda)^2}{n^2} = \frac{1}{(n-1)^2} = E[r_{12}^2]E[r_{13}^2],$$

from which we obtain the result. □

By this proposition, we have

$$E[r_{ij}^2] = E[R_i^T R_j R_j^T R_i] = E[E[R_i^T R_j R_j^T R_i | R_i]] = n^{-1}(1 - \lambda) = (n-1)^{-1}$$

and accordingly we get $E[Q_{np}]$. David et al. (1951) calculated the moments of Spearman’s correlation coefficient $r_{ij}$. Use the result that $E[r_{ij}^4] = \frac{3(25n^3 - 35n^2 - 35n + 72)}{25n(n+1)(n-1)^3}$ and Proposition 1-(ii) we can easily get $\text{Var}[Q_{np}]$.  

9
A.2 Verification of (7)

**Proof of Lemma 1**

Note that

\[
E_{l-1}[r_i^4] = E[(R_i^T A_1 R_i)^2 | A_1],
\]

\[
E_{l-1}[r_i^2 r_j^2] = E[(R_i^T A_2 R_j)^2 | A_2],
\]

where \( A_1 = R_i R_i^T \) and \( A_2 = R_i R_j^T \).

For a fixed matrix \( A = (a_{ij})_{n \times n} \), we expand

\[
E[(R_i^T A R_i)^2] = \sum_u \sum_v \sum_s \sum_t a_{us} a_{st} E[R_{iu} R_{iv} R_{is} R_{it}].
\]

It is easy to get

\[
\begin{align*}
\mu_4 &\equiv E[R_{11}^4] = \frac{9n^2}{5(n-1)^2(n+1)^2} - \frac{6}{(n-1)^2(n+1)^2} + \frac{21}{5n^2(n-1)^2(n+1)^2}; \\
\mu_{22} &\equiv E[R_{11}^2 R_{12}^2] = \frac{1}{n(n-1)} - \frac{1}{n-1} \mu_4; \\
\mu_{31} &\equiv E[R_{11}^3 R_{13}] = -\frac{1}{n-1} \mu_4; \\
\mu_{211} &\equiv E[R_{12}^2 R_{13} R_{14}] = -\frac{1}{n(n-1)(n-2)} + \frac{2}{(n-1)(n-2)} \mu_4; \\
\mu_{1111} &\equiv E[R_{1i} R_{1j} R_{1k} R_{1l}] = -\frac{3}{n(n-1)(n-2)(n-3)} + \frac{6}{(n-1)(n-2)(n-3)} \mu_4.
\end{align*}
\]

Then, we rewrite \( E[(R_i^T A R_i)^2] = \mu_4 \sum_4 + \mu_{22} \sum_{22} + \mu_{31} \sum_{31} + \mu_{211} \sum_{211} + \mu_{1111} \sum_{1111} \), where \( \sum_s \) is the summation of \( a_{us} a_{st} \) corresponding to \( \mu_s \).

Let \( A = A_1 \), taking the main terms of \( O(n^{-2}) \), we have

\[
\begin{align*}
\sum_4 &= \sum_{k=1}^n a_{kk}^2 = \sum_{k=1}^n R_{kk}^4 = n \mu_4 = o(1), \\
\sum_{22} &= \sum_{u \neq v} (a_{uu}^2 + a_{uv} a_{vu} + a_{vv} a_{vu}) = (\text{tr} A_1)^2 + 2 \text{tr} A_1^2 = 3
\end{align*}
\]

and \( \sum_{31} = \sum_{211} = \sum_{1111} = O(1). \)

Now, let \( A = A_2 \), considering the main terms of \( O(n^{-3}) \) and noting that \( \sum_{k=1}^n R_{ik} = 0 \) and \( \sum_{k=1}^n R_{ik}^2 = 1 \), we have

\[
\begin{align*}
\sum_4 &= \sum_{k=1}^n R_{ik}^2 R_{jk}^2, \quad \sum_{22} = 1 + 2(R_i^T R_j)^2 - 3 \sum_{k=1}^n R_{ik}^2 R_{jk}^2; \\
\sum_{31} &= -4 \sum_{k=1}^n R_{ik}^2 R_{jk}^2, \quad \sum_{211} = 12 \sum_{k=1}^n R_{ik}^2 R_{jk}^2 - 4(R_i^T R_j)^2 - 2.
\end{align*}
\]
Also noting that \( \sum \sum a_{ul}a_{st} = (1^T A_2 1)^2 = 0 \), so we get

\[
\sum_{1111} = -6 \sum_{k=1}^n R_{ik}^2 R_{jk}^2 + 2(R_i^T R_j)^2 + 1.
\]

Using the fact that \( E[\sum_{k=1}^n R_{ik}^2 R_{jk}^2] = n^{-1} \) and \( E[(R_i^T R_j)^2] = (n - 1)^{-1} \), we get the assertion.

Now, let us complete the proof of (7). Denoting \( c_1 \sum_{k=1}^n R_{ik}^2 R_{jk}^2 + c_2 (R_i^T R_j)^2 + c_3 \) as \( C_{ij} \) and noting \( p/n \to \gamma \), we have

\[
\sum_l X_{nl}^2 = \sum_{l=2}^p \left[ \frac{3(l-1)}{n(n-1)} + \frac{(l-1)(l-2)}{n(n-1)} - \frac{(l-1)^2}{(n-1)^2} \right] + \sum_{l=2}^p \sum_{i=1}^{l-1} \sum_{j=1,j\neq i}^{l-1} C_{ij} + o(1)
\]

\[
= \frac{p(p-1)}{n(n-1)} - \frac{(p-1)p(2p-1)}{6n(n-1)^2} + \sum_{l=2}^p \sum_{i=1}^{l-1} \sum_{j=1,j\neq i}^{l-1} C_{ij} + o(1)
\]

\[
= \delta_{pn} + \sum_{l=2}^p \sum_{i=1}^{l-1} \sum_{j=1,j\neq i}^{l-1} C_{ij} + o(1),
\]

where \( \delta_{pn} \equiv \frac{p(p-1)}{n(n-1)} - \frac{(p-1)p(2p-1)}{6n(n-1)^2} \).

We next show that

\[
E[(\sum_{l=2}^p \sum_{i=1}^{l-1} \sum_{j=1,j\neq i}^{l-1} C_{ij})^2] \to \frac{1}{9} \gamma^6. \tag{A.1}
\]

On one hand, easy to calculate that for distinct \( i, j, m, \)

\[
E[(\sum_{k=1}^n R_{ik}^2 R_{jk}^2)(\sum_{k=1}^n R_{ik}^2 R_{mk}^2)] = n(\frac{\mu_4}{n^2} + \frac{\mu_22}{n^2}(n-1)) = \frac{1}{n^2},
\]

\[
E[(R_i^T R_j)^2(\sum_{k=1}^n R_{ik}^2 R_{mk}^2)] = \frac{1}{n^2} + n(\mu_{31}(-\frac{1}{n(n-1)})\frac{1}{2} + \mu_{211}(\frac{1}{n(n-1)})) \frac{1}{n}(n-2)
\]

\[
= \frac{1}{n^2} + \frac{1}{n^2(n-1)^2}
\]

and for distinct \( i, j, m, h, \)

\[
E[(\sum_{k=1}^n R_{ik}^2 R_{jk}^2)(\sum_{k=1}^n R_{mk}^2 R_{hk}^2)] = n^2 \frac{1}{n^4} = \frac{1}{n^2},
\]

\[
E[(R_i^T R_j)^2(\sum_{k=1}^n R_{mk}^2 R_{hk}^2)] = \frac{1}{n^2} + n((-\frac{1}{n(n-1)})^2) \frac{1}{n^2(n-1)^2}
\]

\[
= \frac{1}{n^2} + \frac{1}{n^2(n-1)^2}.
\]
And we already have

\[ E[(R_i^T R_j)^2(R_i^T R_m)^2] = E[(R_i^T R_j)^2(R_m^T R_n)^2] = \frac{1}{(n-1)^2}. \]

So we have

\[ E[C_{ij} C_{im}] = E[C_{ij} C_{mh}] = \frac{1}{n^6}(1 + o(1)). \]

On another hand,

\[ E[(R^T_i R_j)^4] = E[r^4_{ij}], \]

\[ E[\left( \sum_{k=1}^{n} R_{ik}^2 R_{jk}^2 \right)^2] = n \mu_4^2 + (n^2 - n) \mu_{22}^2, \]

\[ E[(R^T_i R_j)^2(\sum_{k=1}^{n} R_{ik}^2 R_{jk}^2)] = n \mu_4^2 + (n^2 - n) \mu_{22}^2 + (2 \mu_{31}^2 + (n - 2) \mu_{211}^2) n. \]

So we have \( E[C_{ij}^2] = O(n^{-6}). \)

Finally, note that terms like \( E[C_{ij}^2] \) in (A.1) add up to \( \frac{1}{3} (p - 2)(p - 1)^2 p \) and they can be neglected compared to \( O(n^{-6}) \), while the total terms in (A.1) is \( \frac{1}{7} p^2 (p - 1)^2 (p - 2)^2 \).

Thus, (A.1) is proved.

Since \( \delta_{pn} \to \gamma^2 - \frac{1}{3} \gamma^3 \) and

\[ E[\sum_{l=2}^{p} \sum_{i=1}^{l-1} \sum_{j=1, j \neq i}^{l-1} C_{ij}] = \frac{1}{3} p(p - 1)(p - 2)(\frac{c_1}{n} + \frac{c_2}{n - 1} + c_3) \to \frac{1}{3} \gamma^3, \]

we have

\[ E[\left( \sum_{l} X_{nl}^2 \right)^2] = \delta_{pn}^2 + 2 \delta_{pn} E[\sum_{l=2}^{p} \sum_{i=1}^{l-1} \sum_{j=1, j \neq i}^{l-1} C_{ij}] + E[\sum_{l=2}^{p} \sum_{i=1}^{l-1} \sum_{j=1, j \neq i}^{l-1} C_{ij}] + o(1) \]

\[ \to \gamma^4. \]

Using the fact that \( \text{Var}[g_{np}] \to \gamma^2 \), (7) is verified. \( \square \)

A.3 Verification of (8)

Expand

\[ X_{nl}^4 = \left( \sum_{i=1}^{l-1} R_{il}^2 \right)^4 - 4 \sum_{i=1}^{l-1} R_{il}^2 \left( \sum_{i=1}^{l-1} R_{il}^2 \right)^3 \frac{l - 1}{n - 1} + 6 \sum_{i=1}^{l-1} R_{il}^2 \left( \sum_{i=1}^{l-1} R_{il}^2 \right)^2 \left( \frac{l - 1}{n - 1} \right)^2 - 4 \sum_{i=1}^{l-1} R_{il}^2 \left( \frac{l - 1}{n - 1} \right)^3 + \left( \frac{l - 1}{n - 1} \right)^4. \]

Note that \( E[(\sum_{i=1}^{l-1} R_{il}^2)^4] \) contains \((l - 1)\) terms of \( E[R_{i1}^8] \), \( 4(l - 1)(l - 2) \) terms of \( E[R_{i1}^6 R_{i2}^2] = \frac{1}{n-1} E[R_{i1}^6], \) \( 3(l - 1)(l - 2) \) terms of \( E[R_{i1}^4 R_{i2}^2] \), \( 6(l - 1)(l - 2)(l - 3) \) terms of
\[ E[R_{1l}^4 R_{2l}^2 R_{3l}^2 R_{4l}^2] = \frac{1}{(n-1)^2} E[R_{1l}^4] \text{ and } (l-1)(l-2)(l-3)(l-4) \text{ terms of } E[R_{1l}^4 R_{2l}^2 R_{3l}^2 R_{4l}^2] = \frac{1}{(n-1)^3}. \]

Similar to this procedure, we can have

\[ E[\left(\sum_{i=1}^{l-1} R_{il}^2\right)^3] = (l-1)E[R_{1l}^6] + 3(l-1)(l-2)\frac{1}{n-1} E[R_{1l}^4] + (l-1)(l-2)(l-3)\frac{1}{(n-1)^3}, \]

\[ E[\left(\sum_{i=1}^{l-1} R_{il}^2\right)^2] = (l-1)E[R_{1l}^4] + (l-1)(l-2)\frac{1}{(n-1)^2}, \]

\[ E[\left(\sum_{i=1}^{l-1} R_{il}^2\right)] = \frac{l-1}{n-1}. \]

From David et al. (1951), we know \( E[R_{1l}^6] = O(n^{-3}) \) and \( E[R_{1l}^4] = O(n^{-4}) \). Moreover, \( (E[R_{1l}^4 R_{2l}^2])^2 \leq E[R_{1l}^4 R_{2l}^2] \leq \sqrt{E[R_{1l}^4]E[R_{2l}^2]} \), we have \( E[R_{1l}^4 R_{2l}^2] = O(n^{-4}) \). Tedious but straightforward calculations yield (8).

□

References:


S.T. David, M.G. Kendall, A. Stuart, Some questions of distribution in the theory of rank correlation, Biometrika 38 (1951) 131–140.


O. Ledoit, M. Wolf, Some hypothesis tests for the covariance matrix when the dimension is large compared to the sample size, Ann. Statist. 30 (2002) 1081–1102.


