Biometrika (2013), **xx**, x, *pp*. 1–15

Supplementary Material for "Multivariate-sign-based high-dimensional tests for sphericity"

BY CHANGLIANG ZOU, LIUHUA PENG, LONG FENG AND ZHAOJUN WANG

School of Mathematical Sciences, Nankai University, Tianjin, China

chlzou@yahoo.com.cn, allenberify@yahoo.com.cn, flnankai@126.com, zjwang@nankai.edu.cn

APPENDIX 1: PROOFS OF THEOREMS 1-2

First of all, note that any random vector X from elliptical density can be decomposed into two parts, $X = RU$, where $R = ||X||$ is the modulus and $U = R^{-1}X$ is the direction vector. If X is spherically symmetric around the origin, its modulus R and direction U are independent, U is uniformly distributed on the unit sphere and the density of R is proportional to $R^{p-1}g_p(R)$. For a sequence of p-dimensional random vectors X_n and a sequence $a_{n,p}$, $X_n = o_p(a_{n,p})$ if and only if $||X_n|| = o_p(a_{n,p})$. $X_n = O_p(a_{n,p})$ can be understood similarly.

Before proving the two main theorems in Section 2, we present several necessary lemmas. The first lemma provides an asymptotic expansion for $\widehat{\theta}_{n,p}$. Recall that $\widehat{\theta}_{n,p}$ solves the equation

$$\sum_{i=1}^{n} U(X_i - \theta) = 0. \tag{1}$$

LEMMA 1. *Under the conditions given in Theorem 1, $\widehat{\theta}_{n,p}$ admits the following asymptotic representation*

$$\widehat{\theta}_{n,p} = n^{-1}c_1^{-1}\sum_{i=1}^{n} U_i + o_p(b_{n,p}),$$

where $c_1 = (p-1)E(R_i^{-1})/p$ and $b_{n,p} = c_1^{-1}n^{-1/2}$.

Proof. We first show that $||\widehat{\theta}_{n,p}|| = O_p(b_{n,p})$. Note that the objective function $L(\theta)$ is a strictly convex function in θ. Thus as long as we can show that it has a $b_{n,p}^{-1}$-consistent local minimizer, it must be $b_{n,p}^{-1}$-consistent global minimizer. The existence of a $b_{n,p}^{-1}$-consistent local minimizer is implied by that fact that for an arbitrarily small $\epsilon > 0$, there exists a sufficiently large constant C, which does not depend on n or p, such that

$$\liminf_{n} Pr\left\{\inf_{u\in\mathbb{R}^p:||u||=C} L(b_{n,p}u) > L(0)\right\} > 1-\epsilon. \tag{2}$$

Next, we prove (2). Consider the expansion of $||X_i - b_{n,p}u||$,

$$||X_i - b_{n,p}u|| = ||X_i||\left\{1 - 2b_{n,p}R_i^{-1}u^TU_i + b_{n,p}^2R_i^{-2}||u||^2\right\}^{1/2}.$$

Since $b_{n,p}R_i^{-1}u^TU_i = O_p(n^{-1/2})$ and $b_{n,p}^2R_i^{-2}||u||^2 = O_p(n^{-1})$, we can see that

$$\begin{aligned}||X_i - b_{n,p}u|| =& ||X_i|| - b_{n,p}u^TU_i + 2^{-1}b_{n,p}^2R_i^{-1}||u||^2 - 2^{-1}b_{n,p}^2R_i^{-1}u^TU_iU_i^Tu + O_p(c_1^{-1}n^{-3/2})\\ =& ||X_i|| - b_{n,p}u^TU_i + b_{n,p}^2u^T(2R_i)^{-1}(I_p - U_iU_i^T)u + O_p(c_1^{-1}n^{-3/2}).\end{aligned}$$

So, it can be easily seen

$$
\begin{aligned}
&c_1\{L(b_{n,p}u)-L(0)\}\\
&\qquad\qquad =c_1\sum_{i=1}^n\{||X_i-b_{n,p}u||-||X_i||\}\\
&\qquad\qquad =-n^{-1/2}u^T\sum_{i=1}^n U_i+c_1^{-1}n^{-1}u^T\left\{\sum_{i=1}^n(2R_i)^{-1}(I_p-U_iU_i^T)\right\}u+O_p(n^{-1/2}). \qquad (3)
\end{aligned}
$$

Firstly, as $E\left(||n^{-1/2}\sum_{i=1}^n U_i||^2\right)=1$ and $\mathrm{var}\left(||n^{-1/2}\sum_{i=1}^n U_i||^2\right)=O(1)$, we know that $||n^{-1/2}\sum_{i=1}^n U_i||=O_p(1)$, and accordingly $|-n^{-1/2}u^T\sum_{i=1}^n U_i|\le||u||||n^{-1/2}\sum_{i=1}^n U_i||=O_p(1)$. In addition, let $A=n^{-1}\sum_{i=1}^n R_i^{-1}U_iU_i^T-n^{-1}p^{-1}\sum_{i=1}^n R_i^{-1}I_p$. Since

$$E\left\{\mathrm{tr}\left(A^2\right)\right\}=\frac{p-1}{pn}E(R_i^{-2})=O(c_1^2n^{-1}),$$

then

$$E(u^TAu)^2\le E\left\{\mathrm{tr}(uu^T)^2\mathrm{tr}(A^2)\right\}=O(c_1^2n^{-1}),$$

which leads to $u^TAu=O_p(c_1n^{-1/2})$. Thus, we have

$$
\begin{aligned}
n^{-1}u^T\left\{\sum_{i=1}^n R_i^{-1}(I_p-U_iU_i^T)\right\}u=&u^T\left\{(p-1)p^{-1}n^{-1}\sum_{i=1}^n R_i^{-1}\right\}u+O_p(c_1n^{-1/2})\\
=&c_1||u||^2+O_p(c_1n^{-1/2}),
\end{aligned}
$$

where we use the fact that R_i is independent of U_i and $n^{-1}\sum_{i=1}^n R_i^{-1}=E(R_i^{-1})+O_p(c_1n^{-1/2})$. Thus, by choosing a sufficiently large C, the second terms in (3) dominates the first term uniformly in $||u||=C$. Hence, (2) holds and accordingly $||\widehat{\theta}_{n,p}||=O_p(b_{n,p})$.

Finally, by a first-order Taylor expansion of (1) at 0, we have

$$\sum_{i=1}^n(U_i-R_i^{-1}\widehat{\theta}_{n,p})\left\{1+R_i^{-1}U_i^T\widehat{\theta}_{n,p}+O_p(n^{-1})\right\}=0.$$

By simple algebras, we get the equation that

$$\left\{n^{-1}\sum_{i=1}^n R_i^{-1}+O_p(c_in^{-1/2})\right\}\widehat{\theta}_{n,p}-\left(n^{-1}\sum_{i=1}^n R_i^{-1}U_iU_i^T\right)\widehat{\theta}_{n,p}=\left(n^{-1}\sum_{i=1}^n U_i\right)\left\{1+O_p(n^{-1})\right\}.$$

By noting $E\left(||A\widehat{\theta}_{n,p}||^2\right)\le E\left\{\mathrm{tr}(\widehat{\theta}_{n,p}\widehat{\theta}_{n,p}^T)\mathrm{tr}(A^2)\right\}=O(n^{-2})$, $\left(n^{-1}\sum_{i=1}^n R_i^{-1}U_iU_i^T\right)\widehat{\theta}_{n,p}=\left\{n^{-1}p^{-1}\sum_{i=1}^n R_i^{-1}+O_p(c_1n^{-1/2})\right\}\widehat{\theta}_{n,p}$. Using again $n^{-1}\sum_{i=1}^n R_i^{-1}=E(R_i^{-1})+O_p(c_1n^{-1/2})$ leads to the assertion that

$$\widehat{\theta}_{n,p}=\{c_1+O_p(c_1n^{-1/2})\}^{-1}\left(n^{-1}\sum_{i=1}^n U_i\right)\left\{1+O_p(n^{-1})\right\}=n^{-1}c_1^{-1}\sum_{i=1}^n U_i+o_p(b_{n,p}),$$

which completes the proof. □

The next lemma measures the asymptotic difference between $\widehat{U}_i$ and U_i.

LEMMA 2. *Under the conditions given in Theorem 1,*

$$\widehat{U}_i = U_i - R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p} - 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2 U_i + o_p(n^{-1}).$$

Proof. By the definition of $\widehat{U}_i$,

$$\widehat{U}_i = (X_i - \widehat{\theta}_{n,p})/||X_i - \widehat{\theta}_{n,p}|| = (U_i - R_i^{-1}\widehat{\theta}_{n,p})\left(1 - 2R_i^{-1}U_i^T\widehat{\theta}_{n,p} + R_i^{-2}||\widehat{\theta}_{n,p}||^2\right)^{-1/2}.$$

Note that $R_i^{-1}U_i^T\widehat{\theta}_{n,p} = o_p(n^{-1/2})$ and $R_i^{-2}||\widehat{\theta}_{n,p}||^2 = O_p(n^{-1})$. Accordingly, by the first-order Taylor expansion, we can get that

$$\begin{aligned}\widehat{U}_i &= (U_i - R_i^{-1}\widehat{\theta}_{n,p})\left\{1 + R_i^{-1}U_i^T\widehat{\theta}_{n,p} - 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2 + o_p(n^{-1})\right\}\\ &= U_i - R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p} - 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2 U_i + o_p(n^{-1}),\end{aligned}$$

from which we get the assertion. □

Define $\alpha_i = -R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p} - 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2 U_i$, then

$$\widehat{U}_i = U_i + \alpha_i + o_p(n^{-1}).$$

To prove Theorem 1, it is required to show that $Q'_{\mathrm{S}} = \{n(n-1)\}^{-1}\sum_{i\neq j} p(U_i^TU_j)^2 - 1$ is asymptotically normal. Clearly, Q'_{S} is a U-statistic with order two. However, the standard CLT for U-statistics (Serfling 1980) is not directly applicable. This is because $\mathrm{var}[E\{(U_i^TU_j)^2 \mid U_i\}] = \mathrm{var}(p^{-1}I_p) = 0$. However, by using the martingale central limit theorem (Hall and Hyde 1980), we have the following result.

LEMMA 3. *Under H_0, as $p\to\infty$ and $n\to\infty$, $\{\mathrm{var}(Q'_{\mathrm{S}})\}^{-1/2}\{Q'_{\mathrm{S}} - E(Q'_{\mathrm{S}})\} \to N(0,1)$ in distribution, where $E(Q'_{\mathrm{S}}) = 0$ and $\mathrm{var}(Q'_{\mathrm{S}}) = 4(p-1)/\{n(n-1)(p+2)\}$.*

Proof. The expectation of Q'_{S} can be easily verified and thus omitted here. $\mathrm{var}(Q'_{\mathrm{S}})$ can be computed as follows:

$$\begin{aligned}\mathrm{var}(Q'_{\mathrm{S}}) =&\{n(n-1)\}^{-2}p^2E\left\{\sum_{i\neq j}(U_i^TU_j)^2\right\}^2 - 1\\ =&\{n(n-1)\}^{-2}p^2\big[2n(n-1)E(U_i^TU_j)^4 + 4n(n-1)(n-2)E\left\{(U_i^TU_j)^2(U_i^TU_k)^2\right\}\\ &+ n(n-1)(n-2)(n-3)E\left\{(U_i^TU_j)^2(U_k^TU_l)^2\right\}\big] - 1\\ =&\{n(n-1)\}^{-2}p^2\big[6n(n-1)/\{p(p+2)\} + 4n(n-1)(n-2)/p^2\\ &+ n(n-1)(n-2)(n-3)/p^2\big] - 1\\ =&4(p-1)/\{n(n-1)(p+2)\}.\end{aligned}$$

To show this lemma, we need to use the martingale central limit theorem (Billingsley 1995). For this purpose, let $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_k = \sigma\{U_1,\dots,U_k\}$, $k = 1,\dots,n$. Let $E_k(\cdot)$ denote the conditional expectation of given $\mathcal{F}_k$ and $E_0(\cdot) = E(\cdot)$. Write $Q'_{\mathrm{S}} - E(Q'_{\mathrm{S}}) = \sum_{k=1}^n D_{n,k}$, where $D_{n,k} = (E_k - E_{k-1})Q'_{\mathrm{S}}$. Then for every n, $\{D_{n,k}\}_{k=1}^n$ is a martingale difference sequence with respect to the σ-fields $\{\mathcal{F}_k, 1\le k\le n\}$. Let $\sigma_{n,k}^2 = E_{k-1}(D_{n,k}^2)$. It suffices to show that, as $n\to\infty$,

$$\frac{\sum_{k=1}^n \sigma_{n,k}^2}{\mathrm{var}(Q'_{\mathrm{S}})} \to 1 \quad \text{in probability and} \quad \frac{\sum_{k=1}^n E(D_{n,k}^4)}{\mathrm{var}^2(Q'_{\mathrm{S}})} \to 0. \tag{4}$$

First of all, note that

$$D_{n,k} = \frac{2p}{n(n-1)}\left(\sum_{i=1}^{k-1} U_k^T U_i U_i^T U_k - \frac{k-1}{p}\right) = 2\{n(n-1)\}^{-1} p U_k^T \Gamma_{k-1} U_k,$$

where $\Gamma_{k-1} = \sum_{i=1}^{k-1}\left(U_i U_i^T - p^{-1} I_p\right)$. So, $D_{n,k}^2 = 4\{n(n-1)\}^{-2} p^2 \left(U_k^T \Gamma_{k-1} U_k\right)^2$. It follows that

$$\sum_{k=1}^{n} \sigma_{n,k}^2 = \frac{8}{\{n(n-1)\}^2} \sum_{k=1}^{n} \operatorname{tr}(\Gamma_{k-1}^2).$$

As $\sum_{k=1}^{n} \sigma_{n,k}^2 = \operatorname{var}(Q'_{\mathrm{S}})$, to see the first part of (4), we only show $\operatorname{var}(\sum_{k=1}^{n} \sigma_{n,k}^2) = o\{\operatorname{var}^2(Q'_{\mathrm{S}})\}$. By noting that

$$\begin{aligned}
\operatorname{tr}\left(\sum_{k=1}^{n} \Gamma_{k-1}^2\right) &= \sum_{k=1}^{n}\sum_{i=1}^{k-1}\sum_{j=1}^{k-1} \operatorname{tr}\left\{\left(U_i U_i^T - p^{-1} I_p\right)\left(U_j U_j^T - p^{-1} I_p\right)\right\} \\
&= \frac{n(n-1)(p-1)}{2p} + \sum_{i\neq j} 2\left\{n - \max(i,j)\right\} \operatorname{tr}\left\{\left(U_i U_i^T - p^{-1} I_p\right)\left(U_j U_j^T - p^{-1} I_p\right)\right\},
\end{aligned}$$

we can obtain

$$E\left(\sum_{k=1}^{n} \sigma_{n,k}^2\right) = \frac{4(p-1)}{n(n-1)p}, \ \operatorname{var}\left(\sum_{k=1}^{n} \sigma_{n,k}^2\right) = \frac{128(n-2)(p-1)}{3\{n(n-1)\}^3 p^2 (p+2)}.$$

Clearly, $\operatorname{var}(\sum_{k=1}^{n} \sigma_{n,k}^2) = o\{\operatorname{var}^2(Q'_{\mathrm{S}})\}$.

Finally, we verify that the second part of (4). Note that

$$\begin{aligned}
\sum_{k=1}^{n} E(D_{n,k}^4) =& \frac{16p^4}{\{n(n-1)\}^4}\left[\frac{n(n-1)}{2} E\left\{U_k^T\left(U_i U_i^T - p^{-1} I_p\right) U_k\right\}^4\right. \\
&\left. + n(n-1)(n-2) E\left\{\left(U_k^T\left(U_i U_i^T - p^{-1} I_p\right) U_k\right)^2 \left(U_k^T (U_j U_j^T - p^{-1} I_p) U_k\right)^2\right\}\right].
\end{aligned}$$

By the fact that

$$\begin{aligned}
&E\left\{U_k^T (U_i U_i^T - p^{-1} I_p) U_k\right\}^4 = O(p^{-4}), \\
&E\left[\left\{U_k^T (U_i U_i^T - p^{-1} I_p) U_k\right\}^2 \left\{U_k^T (U_j U_j^T - p^{-1} I_p) U_k\right\}^2\right] = O(p^{-4}),
\end{aligned}$$

it is straightforward to see $\sum_{k=1}^{n} E(D_{n,k}^4) = o\{\operatorname{var}^2(Q'_{\mathrm{S}})\}$ which completes the proof of this lemma. □

Proof of Theorem 1. Now, write

$$\begin{aligned}
(\widehat{U}_i^T \widehat{U}_j)^2 &= \left\{(U_i + \alpha_i)^T (U_j + \alpha_j)\right\}^2 \left\{1 + o_p(n^{-1})\right\} \\
&= \left\{(U_i^T U_j)^2 + \Delta\right\} \left\{1 + o_p(n^{-1})\right\}.
\end{aligned}$$

Next, we will compute the expectation of Δ. Note that

$$\begin{aligned}
E(\Delta) =& E(U_i^T \alpha_j + U_j^T \alpha_i)^2 + E(\alpha_i^T \alpha_j)^2 + 2E\left\{(U_i^T \alpha_j + U_j^T \alpha_i)\alpha_i^T \alpha_j\right\} \\
&+ 2E(U_i^T U_j \alpha_i^T \alpha_j) + 2E\left\{(U_i^T \alpha_j + U_j^T \alpha_i) U_i^T U_j\right\} \\
=& \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5.
\end{aligned}$$

By using Lemmas 1 and 2 and tedious algebra, it can be verified that

$$\Delta_1 = \frac{2}{n^2} + \frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\left(\frac{2}{np}+\frac{2}{n^2}\right) + o(n^{-2}p^{-1}),$$

$$\Delta_2 = \left[\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\right]^2\left(\frac{1}{n^2}-\frac{4}{n^3}\right) + \frac{E(R_i^{-2})E(R_i^{-3})}{\{E(R_i^{-1})\}^5}\frac{2}{n^3} + o(n^{-3}),$$

$$\Delta_3 = -\frac{4E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\left(\frac{1}{n^2}-\frac{2}{n^3}\right) - \frac{2E(R_i^{-3})}{\{E(R_i^{-1})\}^3}\frac{1}{n^3} - \left[\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\right]^2\frac{2}{n^3} + o(n^{-3}),$$

$$\Delta_4 = O(n^{-2}p^{-1}),\ \Delta_5 = -\frac{2E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\frac{1}{np} + o(n^{-2}p^{-1}).$$

See more details in Appendix 5 of the supplementary material. Combining all these, we can obtain (6) in the paper.

Similarly, by straightforward but tedious algebra, we can show that $\mathrm{var}(\Delta) = o\{(np)^{-2}\}$. Hence, combing the fact that $(U_i^T U_j)^2 = O_p(p^{-1})$, we have

$$p(\widehat{U}_i^T \widehat{U}_j)^2 = p(U_i^T U_j)^2 + pE(\Delta) + o_p(pn^{-3} + n^{-1}).$$

Taking the equation above and Lemma 3 together, the assertion in the theorem immediately follows from the assumption $p/n^2 = O(1)$. □

In order to prove Theorem 2, we state another necessary lemma. Consider the alternative $H_1 : \Lambda_p = I_p + D_{n,p}$ and $\Sigma_p = \sigma_p \Lambda_p$. Denote $X_i^* = \Sigma_p^{-1/2} X_i$, $U_i^* = X_i^*/\|X_i^*\|$, and $R_i^* = \|X_i^*\|$.

LEMMA 4. *Suppose U_i^*'s are independent identically distributed uniform on the unit p sphere. For any $p \times p$ symmetric matrix A, we have*

$$E(U_i^{*T} A U_i^*)^2 = \{\mathrm{tr}^2(A) + 2\mathrm{tr}(A^2)\}/(p^2+2p),$$
$$E(U_i^{*T} A U_i^*)^4 = \{3\mathrm{tr}^2(A^2) + 6\mathrm{tr}(A^4)\}/\{p(p+2)(p+4)(p+6)\}.$$

Proof of Theorem 2. Now, write $U_i = \{\Lambda_p^{1/2} U_i^*\}/\{1 + U_i^{*T} D_p U_i^*\}^{1/2}$, and then

$$E(U_i^T U_j)^2 = \mathrm{tr}\left(\left[E\left\{\Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2} (1 + U_i^{*T} D_p U_i^*)^{-1}\right\}\right]^2\right)$$
$$= \mathrm{tr}\left[\left\{E\left(\Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2}\right)\right\}^2\right] + \mathrm{tr}\left(\left[E\left\{C_i \Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2}\left(U_i^{*T} D_p U_i^*\right)\right\}\right]^2\right),$$

where C_i is a bounded random variable between -1 and $-(1 + U_i^{*T} D_{n,p} U_i^*)^{-2}$. Obviously, $\mathrm{tr}\left[\left\{E\left(\Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2}\right)\right\}^2\right] = p^{-2}\mathrm{tr}(\Lambda_p^2) = p^{-2}(p + \mathrm{tr}(D_{n,p}^2))$. By the Cauchy inequality and Lemma 4,

$$\mathrm{tr}\left(\left[E\left\{C_i \Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2}\left(U_i^{*T} D_p U_i^*\right)\right\}\right]^2\right)$$
$$\le C\mathrm{tr}\left[\left\{E\left(\Lambda_p^{1/2} U_i^* U_i^{*T} \Lambda_p^{1/2}\right)^2\right\}\right] E\left\{\left(U_i^{*T} D_p U_i^*\right)^2\right\}$$
$$\le Cp^{-4}\mathrm{tr}(\Lambda_p^2)\mathrm{tr}(D_p^2) = Cp^{-4}\{p + \mathrm{tr}(D_p^2)\}\mathrm{tr}(D_p^2) = o(p^{-1}n^{-1})$$

by the condition $\mathrm{tr}(D_p^2) = O(n^{-1}p)$. Consequently, $E(Q'_{\mathrm{S}}) = p\mathrm{tr}(\Lambda_p^2) - 1 + o(n^{-1})$. Taking the same procedure as $E\{(U_i^T U_j)^2\}$, we can obtain that

$$\begin{aligned} E(U_i^T U_j)^4 =& \{3\mathrm{tr}^2(\Lambda_p^2) + 6\mathrm{tr}(\Lambda_p^4)\}/\{p(p+2)(p+4)(p+6)\}[1 + O\{p^{-2}\mathrm{tr}(D_p^2)\}], \\ E\left\{(U_i^T U_j)^2 (U_i^T U_k)^2\right\} =& \{\mathrm{tr}^2(\Lambda_p^2) + 2\mathrm{tr}(\Lambda_p^4)\}/\{p^3(p+2)\}[1 + O\{p^{-2}\mathrm{tr}(D_p^2)\}]. \end{aligned}$$

Combining all above, we can get

$$\mathrm{var}\left\{\frac{1}{n(n-1)}\sum_{i\neq j}(U_i^T U_j)^2\right\} = \left[\frac{4\mathrm{tr}^2(\Lambda_p^2)}{n(n-1)p^4} + \frac{8\{p\mathrm{tr}(\Lambda_p^4) - \mathrm{tr}^2(\Lambda_p^2)\}}{(n-1)p^4}\right]\{1 + o(1)\}.$$

Using similar algebra in the proof of Theorem 1, we can show that $E(\Delta) = \delta_{n,p} + o(n^{-3} + n^{-1}p^{-1})$ and $\mathrm{var}(\Delta) = o\{(np)^{-2}\}$. Thus,

$$\begin{aligned} E(\tilde{Q}) &= \mathrm{tr}(D_p^2)/p + p\delta_{n,p} + o(n^{-1}), \\ \mathrm{var}(\tilde{Q}) &= \left[\frac{4\mathrm{tr}^2(\Lambda_p^2)}{n(n-1)p^2} + \frac{8\{\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2)\}}{(n-1)p^2}\right]\{1 + o(1)\}. \end{aligned}$$

By carrying out similar procedures in the proof of Lemma 3, we can complete the proof of this theorem with rather lengthy algebra, see Appendix 4 in the supplementary material. □

Proof of Corollary 1. From Theorems 1-2,

$$\liminf_n pr\left(\frac{\tilde{Q} - p\delta_{n,p}}{\tilde{\sigma}_0} > z_\alpha\right) \geq 1 - \limsup_n \Phi\left\{\frac{\tilde{\sigma}_0 z_\alpha - p^{-1}\mathrm{tr}(D_{n,p}^2)}{\tilde{\sigma}_1}\right\}.$$

Obviously, $\tilde{\sigma}_0/\tilde{\sigma}_1 = O(1)$ due to $\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2) \geq 0$. Denote

$$\begin{aligned} \gamma_{1n} &= \frac{8\left\{\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2)\right\}}{p^2}, \\ \gamma_{2n} &= \frac{8\left\{\mathrm{tr}(\Lambda_p^4)\mathrm{tr}^2(\Lambda_p) + \mathrm{tr}^3(\Lambda_p^2) - 2\mathrm{tr}(\Lambda_p)\mathrm{tr}(\Lambda_p^2)\mathrm{tr}(\Lambda_p^3)\right\}}{\mathrm{tr}^2(\Lambda_p^2)p^2}. \end{aligned}$$

Firstly, consider the case $p/\mathrm{tr}(D_{n,p}^2) = o(1)$. The condition $n\mathrm{tr}(D_{n,p}^2)/p \to \infty$ leads to

$$\begin{aligned} \frac{\tilde{\sigma}_1^2}{p^{-2}\mathrm{tr}^2(D_{n,p}^2)} &= O\left\{\frac{p^2}{n^2\mathrm{tr}^2(D_{n,p}^2)}\right\} + O\left\{\frac{\mathrm{tr}(\Lambda_p^4)}{n\mathrm{tr}^2(D_{n,p}^2)}\right\} \\ &= O\left\{\frac{\mathrm{tr}^2(D_{n,p}^2)}{n\mathrm{tr}^2(D_{n,p}^2)}\right\} + o(1) \to 0, \end{aligned}$$

which implies the assertion of Corollary 1. For the case $p/\mathrm{tr}(D_{n,p}^2) = O(1)$, it can be seen that $\gamma_{2n}/\gamma_{1n} = O(1)$. By Theorem 4-(i) in Chen et al. (2010), we have $\gamma_{2n}/\{np^{-2}\mathrm{tr}^2(D_{n,p}^2)\} \to 0$ from which the corollary follows immediately. □

Proof of Corollary 2. By Theorem 1 in Chen et al. (2010),

$$\frac{C_n - \mathrm{tr}(D_{n,p}^2)/p}{(4n^{-2} + \gamma_{2n}n^{-1})} \to N(0,1)$$

in distribution, where C_n is the test statistic proposed by Chen et al. (2010). It follows from Theorem 2 that the limiting distributions of two test statistics are of the same type. Given the condition $C_1 \leq$

$n\mathrm{tr}(D_{n,p}^2)/p \leq C_2$, both of the powers of U_n and $\tilde{Q}$ are nontrivial and converge to some constants smaller than 1. Thus, the efficiency comparisons may be based on asymptotic variances. The result follows by straightforward calculation. □

APPENDIX 2: PROOF OF PROPORTIONS 1 AND DISCUSSIONS ABOUT THE ESTIMATION OF $E(R_i^{-2})/\{E(R_i^{-1})\}^2$ AND $E(R_i^{-3})/\{E(R_i^{-1})\}^3$

Proof of Proposition 1. Consider $Z_i = aOX_i + c$ where O is an orthogonal matrix. We use the subscripts "Z_i" or "Z" to distinguish the corresponding statistics or parameters based on the sample Z. Firstly, note that the spatial median $\widehat{\theta}_{n,p}$ is orthogonal equivariant, say $\widehat{\theta}_{YO^T} = O\widehat{\theta}_{n,p}$. Furthermore, $\widehat{\theta}_Z = aO\widehat{\theta}_{n,p} + c$. Thus,

$$\widehat{U}_{Z_i} = U\{Z_i - \widehat{\theta}(Z)\} = OU(Y_i - \widehat{\theta}_{n,p}) = O\widehat{U}_i,$$

which leads to $\tilde{Q}_Z = \frac{p}{n(n-1)}\sum_{i\neq j}(\widehat{U}_i^T O^T O\widehat{U}_j)^2 - 1 = \tilde{Q}$. □

Next, we provide verifications of the claims given in the paper about the estimation of $E(R_i^{-k})/\{E(R_i^{-1})\}^k$ by using $n^{k-1}\sum_{i=1}^n \widehat{R}_{i*}^{-k}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^k$, $k = 2$ and 3.

Claim 1: Under H_0 and Assumption 1, $n^{-1}\sum_{i=1}^n \widehat{R}_{i*}^{-k} = n^{-1}\sum_{i=1}^n R_i^{-k}\{1 + o_p(n^{-1})\}$, $k = 1, 2, 3$.

Proof. According to Lemmas 1-2, we have

$$\widehat{R}_i = ||X_i - \widehat{\theta}_{n,p}|| = ||X_i|| - \widehat{\theta}_{n,p}^T U_i + 2^{-1}R_i^{-1}||\widehat{\theta}_{n,p}||^2 + 2^{-1}R_i^{-1}(\widehat{\theta}_{n,p}^T U_i)^2\{1 + o_p(1)\}$$

and

$$\widehat{U}_i = U_i - R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p} - 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2 U_i + o_p(n^{-1}).$$

Then we can obtain

$$\widehat{R}_{i*}^{-1} - R_i^{-1} = -\frac{3}{2}\widehat{R}_{i*}^{-1}R_i^{-2}(\widehat{\theta}_{n,p}^T U_i)^2\{1 + o_p(1)\}.$$

By calculating the mean and variance of $R_i^{-2}(\widehat{\theta}_{n,p}^T U_i)^2$, we can claim that $\widehat{R}_{i*}^{-1} = R_i^{-1}\{1 + o_p(n^{-1})\}$. Similarly, we have $n^{-1}\sum_{i=1}^n \widehat{R}_{i*}^{-k} = n^{-1}\sum_{i=1}^n R_i^{-k}\{1 + o_p(n^{-1})\}$. □

Claim 2: Under H_0 and Assumption 1, we have
(1) $n\sum_{i=1}^n \widehat{R}_{i*}^{-2}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^2 = E(R_i^{-2})/\{E(R_i^{-1})\}^2\{1 + O_p(n^{-1/2})\}$, and
(2) $n\sum_{i=1}^n \widehat{R}_{i*}^{-3}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^3 = E(R_i^{-3})/\{E(R_i^{-1})\}^3\{1 + o_p(1)\}$.

Proof. (1) It is obviously that under Assumption 1, $\mathrm{var}(R_i^{-k})/\{E(R_i^{-k})\}^2 = O(1)$ for $k = 1$ and 2. Then, by noting $E(n^{-1}\sum_{i=1}^n R_i^{-1}) = E(R_i^{-1})$ and $\mathrm{var}(n^{-1}\sum_{i=1}^n R_i^{-1}) = n^{-1}[E(R_i^{-2}) - \{E(R_i^{-1})\}^2] = O(\{E(R_i^{-1})\}^2 n^{-1})$, we know that

$$n^{-1}\sum_{i=1}^n R_i^{-1} - E(R_i^{-1}) = O_p[\{E(R_i^{-1})\}n^{-1/2}],$$

which leads to $n^{-1}\sum_{i=1}^n R_i^{-1} = E(R_i^{-1})\left\{1 + O_p\left(n^{-1/2}\right)\right\}$. Similarly, we can prove that $n^{-1}\sum_{i=1}^n R_i^{-2} = E(R_i^{-2})\left\{1 + O_p\left(n^{-1/2}\right)\right\}$. Combining this with Claim 1 we can get the result that $n\sum_{i=1}^n \widehat{R}_{i*}^{-2}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^2 = E(R_i^{-2})/\{E(R_i^{-1})\}^2\{1 + O_p(n^{-1/2})\}$.

(2) As $R_1,\dots,R_n$ are independent and identically distributed random variables, it is easy to know that $n^{-1}\sum_{i=1}^n R_i^{-3}=E(R_i^{-3})\left\{1+o_p\left(1\right)\right\}$ by using Laws of Large Numbers. Combining this with Claim 1 we can get the result that $n\sum_{i=1}^n \widehat{R}_{i*}^{-3}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^3=E(R_i^{-3})/\{E(R_i^{-1})\}^3\{1+o_p(1)\}$. □

Claim 3: Suppose $\mathrm{var}(R_i^{-k})/\{E(R_i^{-k})\}^2=o(p^{-1/2})$ for $k=1$ and 2 hold. Under H_0 and Assumption 1, $n\sum_{i=1}^n \widehat{R}_{i*}^{-2}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^2=E(R_i^{-2})/\{E(R_i^{-1})\}^2\{1+o_p(n^{-1}+n^{-1/2}p^{-1/4})\}$.

Proof. Under the condition $\mathrm{var}(R_i^{-k})/\{E(R_i^{-k})\}^2=o(p^{-1/2})$ for $k=1$ and 2, by noting again that $E(n^{-1}\sum_{i=1}^n R_i^{-1})=E(R_i^{-1})$ and $\mathrm{var}(n^{-1}\sum_{i=1}^n R_i^{-1})=n^{-1}[E(R_i^{-2})-\{E(R_i^{-1})\}^2]=o(\{E(R_i^{-1})\}^2n^{-1}p^{-1/2})$, we get that

$$(np)^{1/2}\left\{n^{-1}\sum_{i=1}^n R_i^{-1}-E(R_i^{-1})\right\}=o_p\{p^{1/4}E(R_i^{-1})\}.$$

Then we have $n^{-1}\sum_{i=1}^n R_i^{-1}=E(R_i^{-1})\left\{1+o_p\left(n^{-1/2}p^{-1/4}\right)\right\}$. Similarly, we can prove $n^{-1}\sum_{i=1}^n R_i^{-2}=E(R_i^{-2})\left\{1+o_p\left(n^{-1/2}p^{-1/4}\right)\right\}$. Combining this with Claim 1 we can get that $n\sum_{i=1}^n \widehat{R}_{i*}^{-2}/(\sum_{i=1}^n \widehat{R}_{i*}^{-1})^2=E(R_i^{-2})/\{E(R_i^{-1})\}^2\{1+o_p(n^{-1}+n^{-1/2}p^{-1/4})\}$. □

Claim 4: When $X_i\sim N_p(0,I_p)$, $\mathrm{var}(R_i^{-k})/\{E(R_i^{-k})\}^2=O(p^{-1})$ for $k=1$ and 2.

Proof. First, we can calculate that $E(R_i^{-1})=2^{-1/2}\Gamma\{(p-1)/2\}/\Gamma(p/2)$, $E(R_i^{-2})=(p-2)^{-1}$ and $E(R_i^{-4})=\{(p-2)(p-4)\}^{-1}$. By Stirling formula, we know that

$$\Gamma\{(p-1)/2\}\sim\left\{(2e)^{-1}(p-3)\right\}^{(p-3)/2}\{\pi(p-3)\}^{1/2},\Gamma(p/2)\sim\left\{(2e)^{-1}(p-2)\right\}^{(p-2)/2}\{\pi(p-2)\}^{1/2}$$

as $p\to\infty$. Then $\left\{E(R_i^{-1})\right\}^2=2^{-1}\left\{\Gamma\{(p-1)/2\}\right\}^2/\left\{\Gamma(p/2)\right\}^2\sim(p-3)^{p-2}/(p-2)^{p-1}$. So $\mathrm{var}(R_i^{-k})=E(R_i^{-2})-\left\{E(R_i^{-1})\right\}^2\sim(p-2)^{-1}-(p-3)^{p-2}/(p-2)^{p-1}=O(p^{-2})$. This leads to $\mathrm{var}(R_i^{-1})/\{E(R_i^{-1})\}^2=O(p^{-1})$. The case for $k=2$ is obvious and we omit here. □

APPENDIX 3: VERIFICATION OF ASSUMPTION 1 FOR MULTIVARIATE NORMAL, t AND MIXTURES OF MULTIVARIATE NORMAL DISTRIBUTIONS

When $X_i\sim N_p(0,I_p)$, we have

$$E(R_i^{-1})=\frac{\Gamma\{(p-1)/2\}}{2^{1/2}\Gamma(p/2)},E(R_i^{-2})=\frac{\Gamma(p/2-1)}{2\Gamma(p/2)}=\frac{1}{p-2},$$
$$E(R_i^{-3})=\frac{\Gamma\{(p-3)/2\}}{2^{3/2}\Gamma(p/2)},E(R_i^{-4})=\frac{\Gamma(p/2-2)}{4\Gamma(p/2)}=\frac{1}{(p-2)(p-4)},$$

where $\Gamma(\cdot)$ are the usual Gamma functions.

By Stirling formula

$$\lim_{x\to\infty}\frac{\Gamma(x+1)}{(x/e)^x(2\pi x)^{1/2}}=1,$$

it is straightforward to see

$$\lim_{p\to\infty}\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}=\lim_{p\to\infty}\frac{e(p-2)(p-4)}{(p-3)^2}\left\{\left(1+\frac{1}{p-4}\right)^{p-4}\right\}^{-1}=1,$$
$$\lim_{p\to\infty}\frac{E(R_i^{-3})}{\{E(R_i^{-1})\}^3}=\lim_{p\to\infty}\frac{(p-2)^2}{e(p-3)^2}\left(1+\frac{1}{p-3}\right)^{p-3}=1,$$
$$\lim_{p\to\infty}\frac{E(R_i^{-4})}{\{E(R_i^{-1})\}^4}=\lim_{p\to\infty}\frac{e^2(p-2)^3(p-4)}{(p-3)^4}\left\{\left(1+\frac{1}{p-4}\right)^{p-4}\right\}^{-2}=1.$$

When $X_i\sim t_p(0,I_p,v)$, where $t_p(0,I_p,v)$ is the standard p-dimensional multivariate t distribution with v degrees of freedom. By noting $R_i^2/p\sim F_{p,v}$, where $F_{p,v}$ is the F distribution with parameters p and v, then we can calculate that

$$E(R_i^{-1})=\frac{\Gamma\{(v+1)/2\}}{v^{1/2}\Gamma(v/2)}\frac{\Gamma\{(p-1)/2\}}{\Gamma(p/2)},$$
$$E(R_i^{-2})=\frac{\Gamma\{(v+2)/2\}}{v\Gamma(v/2)}\frac{\Gamma\{(p-2)/2\})}{\Gamma(p/2)}=\frac{1}{p-2},$$
$$E(R_i^{-3})=\frac{\Gamma\{(v+3)/2\}}{v^{3/2}\Gamma(v/2)}\frac{\Gamma\{(p-3)/2\}}{\Gamma(p/2)},$$
$$E(R_i^{-4})=\frac{\Gamma\{(v+4)/2\}}{v^{2}\Gamma(v/2)}\frac{\Gamma\{(p-4)/2\}}{\Gamma(p/2)}=\frac{v+2}{v(p-2)(p-4)},$$

and accordingly

$$\lim_{p\to\infty}\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}=\frac{v\Gamma^2(v/2)}{2\Gamma^2\{(v+1)/2\}},\lim_{p\to\infty}\frac{E(R_i^{-3})}{\{E(R_i^{-1})\}^3}=\frac{(v+1)\Gamma^2(v/2)}{2\Gamma^2\{(v+1)/2\}},$$
$$\lim_{p\to\infty}\frac{E(R_i^{-4})}{\{E(R_i^{-1})\}^4}=\frac{v(v+2)\Gamma^4(v/2)}{4\Gamma^4\{(v+1)/2\}}.$$

When X_i is from the mixtures of two multivariate normal distributions with density function $\kappa f_p(0,I_p)+(1-\kappa)f_p(0,\sigma^2I_p)$, where $f_p(;)$ is the density function of p-variate multivariate normal distribution, we have

$$E(R_i^{-1})=\frac{\{\kappa+(1-\kappa)/\sigma\}\{\kappa+(1-\kappa)\sigma^2\}^{1/2}}{2^{1/2}}\frac{\Gamma\{(p-1)/2\}}{\Gamma(p/2)},$$
$$E(R_i^{-2})=\frac{\{\kappa+(1-\kappa)/\sigma^2\}\{\kappa+(1-\kappa)\sigma^2\}}{2}\frac{\Gamma\{(p-2)/2\}}{\Gamma(p/2)}=\frac{\{\kappa+(1-\kappa)/\sigma^2\}\{\kappa+(1-\kappa)/\sigma^2\}}{p-2},$$
$$E(R_i^{-3})=\frac{\{\kappa+(1-\kappa)/\sigma^3\}\{\kappa+(1-\kappa)\sigma^2\}^{3/2}}{2^{3/2}}\frac{\Gamma\{(p-3)/2\}}{\Gamma(p/2)},$$
$$E(R_i^{-4})=\frac{\{\kappa+(1-\kappa)/\sigma^4\}\{\kappa+(1-\kappa)\sigma^2\}^2}{4}\frac{\Gamma\{(p-4)/2\}}{\Gamma(p/2)}=\frac{\{\kappa+(1-\kappa)/\sigma^4\}\{\kappa+(1-\kappa)/\sigma^2\}^2}{(p-2)(p-4)},$$

and as a consequence we have

$$\lim_{p\to\infty}\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}=\frac{\{\kappa+(1-\kappa)/\sigma^2\}}{\{\kappa+(1-\kappa)/\sigma\}^2},\ \lim_{p\to\infty}\frac{E(R_i^{-3})}{\{E(R_i^{-1})\}^3}=\frac{\{\kappa+(1-\kappa)/\sigma^3\}}{\{\kappa+(1-\kappa)/\sigma\}^3},$$
$$\lim_{p\to\infty}\frac{E(R_i^{-4})}{\{E(R_i^{-1})\}^4}=\frac{\{\kappa+(1-\kappa)/\sigma^4\}}{\{\kappa+(1-\kappa)/\sigma\}^4}.$$

For general situation, consider X_i comes from a elliptical distribution with density function $\det(\Sigma_p)^{-1/2}g_p(||\Sigma_p^{-1/2}(X-\theta_p)||)$ and $\Sigma_p=I_p$. First, we can verify that the distribution function of R_i is

$$f_R(t)=\frac{2\pi^{p/2}}{\Gamma(p/2)}t^{p-1}g_p(t),$$

then it is obviously that

$$E(R_i^{-k})=\frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty t^{p-k-1}g_p(t)dt.$$

So, to ensure the validity of Assumption 1, the following equations must hold:

$$\frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty t^{p-2}g_p(t)dt=O(1),$$

and

$$\frac{\frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty t^{p-k-1}g_p(t)dt}{\left\{\frac{2\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty t^{p-2}g_p(t)dt\right\}^k}\to d_k\in[1,\infty),k=2,3,4.$$

APPENDIX 4: DETAILS OF THE PROOF OF THEOREM 2

It suffices to show that $T_n=\{n(n-1)\}^{-1}\sum_{i\neq j}p(U_i^TU_j)^2$ is asymptotically normal. First, we compute $\mathrm{var}(T_n)$ as follows:

$$\begin{aligned}\mathrm{var}(T_n)=&p^2\mathrm{var}\left\{\frac{1}{n(n-1)}\sum_{i\neq j}(U_i^TU_j)^2\right\}\\=&p^2\left[\frac{4\mathrm{tr}^2(\Lambda_p^2)}{n(n-1)p^4}+\frac{8\{\mathrm{tr}(\Lambda_p^4)-p^{-1}\mathrm{tr}^2(\Lambda_p^2)\}}{(n-1)p^4}\right]\{1+o(1)\}.\end{aligned}$$

It follows that

$$\mathrm{var}^2(T_n)\geq K\max\left\{\frac{\{\mathrm{tr}(\Lambda_p^4)-p^{-1}\mathrm{tr}^2(\Lambda_p^2)\}\mathrm{tr}^2(\Lambda_p^2)}{n(n-1)^2p^4},\frac{\mathrm{tr}^4(\Lambda_p^2)}{\{n(n-1)\}^2p^4}\right\}$$

for sufficiently large n, where K is some constant.

Then we use the martingale central limit theorem (Hall and Hyde 1980) to prove the asymptotical normality. For this purpose, let $\mathcal{F}_0=\{\varnothing,\Omega\}$, $\mathcal{F}_k=\sigma\{U_1,\ldots,U_k\},k=1,\ldots,n$. Let $E_k(\cdot)$ denote the conditional expectation of given $\mathcal{F}_k$ and $E_0(\cdot)=E(\cdot)$. Write $T_n-E(T_n)=\sum_{k=1}^nD_{n,k}$, where $D_{n,k}=(E_k-E_{k-1})T_n$. Then for every n, $\{D_{n,k}\}_{k=1}^n$ is a martingale difference sequence with respect to the

σ-fields $\{\mathcal{F}_k, 1 \leq k \leq n\}$. Let $\sigma_{n,k}^2 = E_{k-1}(D_{n,k}^2)$. It suffices to show that, as $n \to \infty$,

$$\frac{\sum_{k=1}^n \sigma_{n,k}^2}{\mathrm{var}(T_n)} \to 1 \quad \text{in probability and} \quad \frac{\sum_{k=1}^n E(D_{n,k}^4)}{\mathrm{var}^2(T_n)} \to 0. \tag{5}$$

As $E(\sum_{k=1}^n \sigma_{n,k}^2) = \mathrm{var}(T_n)$, to see the first part of (5), we only show $\mathrm{var}(\sum_{k=1}^n \sigma_{n,k}^2) = o\{\mathrm{var}^2(T_n)\}$.

Define $E(U_iU_i^T) = \Gamma_p$ and note that

$$\begin{aligned} D_{n,k} &= (E_k - E_{k-1})T_n \\ &= \frac{2p}{n(n-1)}\left\{U_k^T\Gamma_{k-1}U_k - \mathrm{tr}(\Gamma_{k-1}\Gamma_p)\right\} + \frac{2p}{n}\left\{U_k^T\Gamma_pU_k - \mathrm{tr}(\Gamma_p^2)\right\}, \end{aligned}$$

where $\Gamma_{k-1} = \sum_{i=1}^{k-1}\left(U_iU_i^T - \Gamma_p\right)$. So, by the same procedure as $E\{(U_i^TU_j)^2\}$,

$$\begin{aligned} \sigma_{n,k}^2 =& E_{k-1}(D_{n,k}^2) \\ =& \Bigg[\frac{8p^2}{\{n(n-1)\}^2}\frac{\left\{\mathrm{tr}(\Gamma_{k-1}\Lambda_p)^2\mathrm{tr}^2(\Lambda_p) - \mathrm{tr}^2(\Gamma_{k-1}\Lambda_p)\mathrm{tr}(\Lambda_p^2)\right\}}{\mathrm{tr}^4(\Lambda_p)} \\ &+ \frac{16p^2}{n^2(n-1)}\frac{\left\{\mathrm{tr}(\Gamma_{k-1}\Lambda_p^3)\mathrm{tr}^2(\Lambda_p) - \mathrm{tr}(\Gamma_{k-1}\Lambda_p)\mathrm{tr}^2(\Lambda_p^2)\right\}}{\mathrm{tr}^5(\Lambda_p)} \\ &+ \frac{8p^2}{n^2}\frac{\{\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2)\}}{\mathrm{tr}^4(\Lambda_p)}\Bigg]\left[1 + o\{p^{-2}\mathrm{tr}(D_p^2)\}\right]. \end{aligned}$$

Then

$$\sum_{k=1}^n \sigma_{n,k}^2 = (R_{1,n} + R_{2,n} + R_{3,n} + R_{4,n} + R_{5,n} + C)\{1 + o(1)\},$$

where C is a constant, and

$$\begin{aligned} R_{1,n} &= \frac{16p^2}{\{n(n-1)\}^2}\frac{\mathrm{tr}^2(\Lambda_p^2)\sum_{k=1}^n(k-1)(\sum_{i=1}^{k-1}U_i^T\Lambda_pU_i)}{\mathrm{tr}^5(\Lambda_p)},\ R_{2,n} = -\frac{16p^2}{\{n(n-1)\}^2}\frac{\sum_{k=1}^n(k-1)(\sum_{i=1}^{k-1}U_i^T\Lambda_p^3U_i)}{\mathrm{tr}^3(\Lambda_p)}, \\ R_{3,n} &= \frac{16p^2}{n^2(n-1)}\frac{(\sum_{k=1}^n\sum_{i=1}^{k-1}U_i^T\Lambda_p^3U_i)}{\mathrm{tr}^3(\Lambda_p)},\ R_{4,n} = -\frac{16p^2}{n^2(n-1)}\frac{\mathrm{tr}^2(\Lambda_p^2)(\sum_{k=1}^n\sum_{i=1}^{k-1}U_i^T\Lambda_pU_i)}{\mathrm{tr}^5(\Lambda_p)}, \\ R_{5,n} &= \frac{8p^2}{\{n(n-1)\}^2}\frac{\sum_{k=1}^n\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(U_i^T\Lambda_pU_j)^2}{\mathrm{tr}^2(\Lambda_p)}. \end{aligned}$$

We need to show $\mathrm{var}(R_{i,n}) = o\{\mathrm{var}^2(T_n)\}$ for $i = 1, \dots, 6$. Using

$$\begin{aligned} \mathrm{var}\left\{\sum_{k=1}^n(k-1)\left(\sum_{i=1}^{k-1}U_i^T\Lambda_pU_i\right)\right\} &= \left\{\sum_{i=1}^n\frac{(n-i)^2(n+i-1)^2}{4}\right\}\left[E(U_i^T\Lambda_pU_i)^2 - \left\{E(U_i^T\Lambda_pU_i)\right\}^2\right] \\ &= \left\{\sum_{i=1}^n\frac{(n-i)^2(n+i-1)^2}{2}\right\}\frac{\left\{\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2)\right\}}{p^2}\{1 + o(1)\}, \end{aligned}$$

we have

$$\frac{\mathrm{var}(R_{1,n})}{\mathrm{var}^2(T_n)} \leq K\frac{\mathrm{tr}^2(\Lambda_p^2)}{\mathrm{tr}^4(\Lambda_p)} \to 0.$$

By carrying out similar procedures we can show that $\mathrm{var}(R_{i,n}) = o\{\mathrm{var}^2(T_n)\}$ for $i = 1, \dots, 6$, and hence complete the proof for the first part of (5).

To show the second part of (5),

$$\sum_{k=1}^{n} E(D_{n,k}^4) \leq \frac{128p^4}{n^3} E\Big\{U_k^T \Gamma_p U_k - \mathrm{tr}(\Gamma_p^2)\Big\}^4 + \frac{128p^4}{\{n(n-1)\}^4} \sum_{k=1}^{n} E\Big\{U_k^T \Gamma_{k-1} U_k - \mathrm{tr}(\Gamma_{k-1}\Gamma_p)\Big\}^4.$$

By some algebra, we get

$$E\Big\{U_k^T \Gamma_p U_k - \mathrm{tr}(\Gamma_p^2)\Big\}^4 \leq K \frac{\mathrm{tr}(\Lambda_p^4)\left\{\mathrm{tr}(\Lambda_p^4) - p^{-1}\mathrm{tr}^2(\Lambda_p^2)\right\}}{\mathrm{tr}^8(\Lambda_p)},$$

which leads to

$$\frac{\frac{128p^4}{n^3} E\Big\{U_k^T \Gamma_p U_k - \mathrm{tr}(\Gamma_p^2)\Big\}^4}{\mathrm{var}^2(T_n)} \leq K \frac{\mathrm{tr}(\Lambda_p^4)}{\mathrm{tr}^2(\Lambda_p^2)}.$$

By the Cauchy inequality, $\mathrm{tr}(D_{n,p}^4) \leq \mathrm{tr}^2(D_{n,p}^2)$ and $\mathrm{tr}^2(D_{n,p}^3) \leq \mathrm{tr}(D_{n,p}^4)\mathrm{tr}(D_{n,p}^2)$, so $\mathrm{tr}(\Lambda_p^4) = o(p^2) = o(\mathrm{tr}^2(\Lambda_p^2))$ by the condition $\mathrm{tr}(D_{n,p}^2) = O(n^{-1}p)$. Thus, $\frac{128p^4}{n^3} E\Big\{U_k^T \Gamma_p U_k - \mathrm{tr}(\Gamma_p^2)\Big\}^4 = o(\mathrm{var}^2(T_n))$. Similarly, we can get

$$\frac{128p^4}{\{n(n-1)\}^4} \sum_{k=1}^{n} E\Big\{U_k^T \Gamma_{k-1} U_k - \mathrm{tr}(\Gamma_{k-1}\Gamma_p)\Big\}^4 = o\{\mathrm{var}^2(T_n)\},$$

from which we can complete the proof for the second part of (5)

APPENDIX 5: DETAILED CALCULATION OF Δ_i IN THE PROOF OF THEOREM 1

By using Lemmas 1 and 2, we have

$$\begin{aligned}
\Delta_1 &= 2E(U_i^T \alpha_j)^2 + 2E(U_i^T \alpha_j U_j^T \alpha_i) \\
&= \frac{2}{n^2}\left[\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2} E\left\{U_i^T (I_p - U_j U_j)(U_i + \sum_{k\neq i} U_k)\right\}^2 + E\left(U_i^T \sum_k U_k \sum_k U_k^T U_j\right)\right] + o(n^{-2}p^{-1}) \\
&= \frac{2}{n^2}\left[\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\left\{1 + (n-1)E(U_i^T U_k)^2\right\} + 1\right] + o(n^{-2}p^{-1}) \\
&= \frac{2}{n^2} + \frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\left(\frac{2}{np} + \frac{2}{n^2}\right) + o(n^{-2}p^{-1}),
\end{aligned}$$

$$
\begin{aligned}
\Delta_2 =& E\left[\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p} + 2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2U_i\right\}^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p} + 2^{-1}R_j^{-2}||\widehat{\theta}_{n,p}||^2U_j\right\}\right]^2\\
=& E\left\{R_i^{-1}R_j^{-1}\widehat{\theta}_{n,p}^T(I_p - U_iU_i^T)(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}^2\\
&+ 2E\left[\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p}\right\}^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p}\right\}^T\left(2^{-1}R_j^{-2}||\widehat{\theta}_{n,p}||^2U_j\right)\right]\\
&+ 2E\left[\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p}\right\}^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}^T\left(2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2U_i\right)\right]\\
&+ o(n^{-3})\\
=&\left\{\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\right\}^2\left(\frac{1}{n^2} - \frac{4}{n^3}\right) + \frac{E(R_i^{-2})E(R_i^{-3})}{\{E(R_i^{-1})\}^5}\frac{2}{n^3} + o(n^{-3}),\\
\Delta_3 =& -4E\left[U_i^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p}\right\}^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\right]\\
&- 4E\left[U_i^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\left\{R_i^{-1}(I_p - U_iU_i^T)\widehat{\theta}_{n,p}\right\}^T\left(2^{-1}R_j^{-2}||\widehat{\theta}_{n,p}||^2U_j\right)\right]\\
&- 4E\left[U_i^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\left\{2^{-1}R_i^{-2}||\widehat{\theta}_{n,p}||^2U_i\right\}^T\left\{R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p}\right\}\right] + o(n^{-3})\\
=& -\frac{4E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\left(\frac{1}{n^2} - \frac{2}{n^3}\right) - \frac{2E(R_i^{-3})}{\{E(R_i^{-1})\}^3}\frac{1}{n^3} - \left[\frac{E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\right]^2\frac{2}{n^3} + o(n^{-3}),\\
\Delta_4 =& 2\mathrm{tr}\{E(\alpha_iU_i^T)\}^2 = O(n^{-2}p^{-1}),\\
\Delta_5 =& 4E(\alpha_j^TU_iU_i^TU_j)\\
=& 4E\left[\left\{-R_j^{-1}(I_p - U_jU_j^T)\widehat{\theta}_{n,p} - 2^{-1}R_j^{-2}||\widehat{\theta}_{n,p}||^2U_j\right\}^TU_iU_i^TU_j\right]\\
=& -2E(R_i^{-2})E\left(||\widehat{\theta}_{n,p}||^2U_j^TU_iU_i^TU_j\right) + o(n^{-2}p^{-1})\\
=& -2n^{-2}E(R_i^{-2})E\left(\sum_k U_k^TU_kU_j^TU_iU_i^TU_j\right) + o(n^{-2}p^{-1})\\
=& -\frac{2E(R_i^{-2})}{n\{E(R_i^{-1})\}^2}\mathrm{tr}\{(EU_iU_i^T)^2\} = -\frac{2E(R_i^{-2})}{\{E(R_i^{-1})\}^2}\frac{1}{np} + o(n^{-2}p^{-1}).
\end{aligned}
$$

APPENDIX 6: MULTIVARIATE t RANDOM VECTORS DO NOT SATISFY THE DATA MODEL (2)

For simplicity, consider $p = 2$ and suppose random vector $(X_1, X_2)^T$ has the standard multivariate t distribution with n degrees of freedom

$$
f(x_1, x_2) = (2\pi)^{-1}\left\{1 + (x_1^2 + x_2^2)/n\right\}^{(n+2)/2}.
$$

Clearly, $X_i \sim t_n$, where t_n is the univariate t-distribution with n degrees of freedom, and $E(X_i) = 0$, $E(X_i^2) = n/(n-2)$, and $E(X_i^4) = 3n^2/\{(n-2)(n-4)\}$. By straightforward calculation, we have

$$
E(X_1^2X_2^2) = n^2/\{(n-2)(n-4)\}, \quad E(X_1^4X_2^4) = 9n^2/\{(n-2)(n-4)(n-6)(n-8)\},
$$

from which we can see that $E(X_1^2X_2^2) \neq E(X_1^2)E(X_2^2)$, and $E(X_1^4X_2^4) \neq E(X_1^4)E(X_2^4)$.

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SERFLING, R. J. (1980).*Approximation theorems of mathematical statistics.* John Wiley & Sons, New York, NY.

APPENDIX 7: ADDITIONAL SIMULATION RESULTS

Tables A.1-A.5 report empirical sizes and power under the scenarios (I)-(V) when $n = 20, 40, 60, 80$ and $p = 38, 55, 89, 181, 331, 642$.

With respect to the performance under alternatives, the bias-corrected sign-based test has quite good power and generally performs similarly to the test proposed by Chen et al. (2010) although its sizes are usually smaller. To see this more clearly, we perform a size-corrected power comparison between the test proposed by Chen et al. (2010) and the bias-corrected sign-based test in the sense that the actual critical values are found through simulations so that both of the two tests have accurate sizes in each case. For each scenario, two settings are chosen: $(p, n) = (89, 40)$ and $(p, n) = (181, 60)$. In this comparison, we fix $v = 0.25$ and set $A = \text{diag}((1 + \delta)^{1/2} 1_{[vp]}, 1_{p-[vp]})$ for $0 \leq \delta \leq 2$. The power curves of the test proposed by Chen et al. (2010) and the bias-corrected sign-based test against the magnitude of δ are plotted in Figure A.1(a)-(d) for scenarios (II)-(V), respectively. Apparently, the bias-corrected sign-based test outperforms the test proposed by Chen et al. (2010) in scenarios (II) and (III) by a quite large margin. For scenario (IV), the difference between two tests is insignificant and the two power curves cannot be distinguished when $(p, n) = (181, 60)$. The bias-corrected sign-based test performs uniformly better than the test proposed by Chen et al. (2010) for scenario (V) although the advantage is not remarkable. This figure together with the results in Tables A.1-A.5 suggest that the bias-corrected sign-based test is quite robust and efficient in testing sphericity, especially for heavy-tailed or skewed distributions.

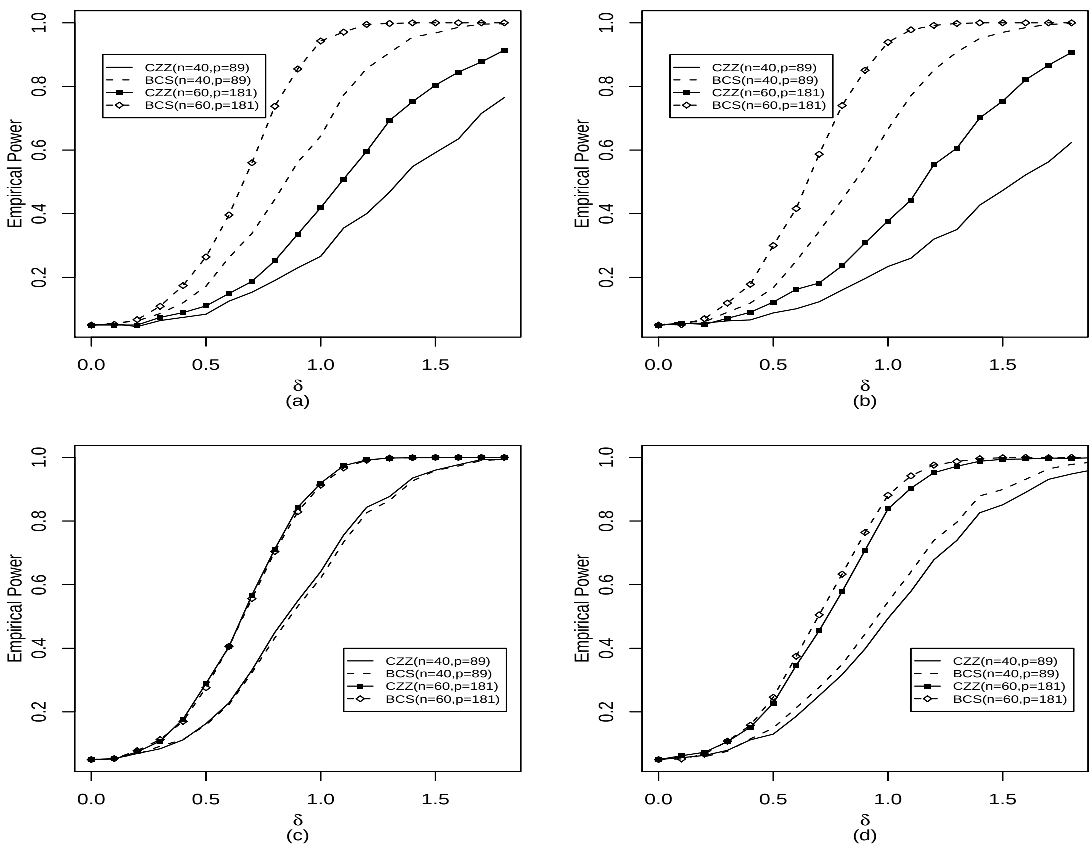

Fig. A.1. Size-corrected power comparison between the test proposed by Chen et al. (2010) and the bias-corrected sign procedure under scenarios: (a) (II); (b) (III); (c) (IV) (d) (V)

Table A.1. *Empirical sizes and power (%) of four tests at 5% significance for multivariate normal random vectors (scenario I); T_{BCS}: the proposed bias-corrected sign test; T_{HP}: the uncorrected sign test studied by Hallin&Paindaveine (2006); T_{LW}: Ledoit & Wolf's (2002) test; T_{CZZ}: Chen et al.'s (2010) test.*

n	v	T_{LW}	T_{CZZ}	T_{HP}	T_{BCS}	T_{LW}	T_{CZZ}	T_{HP}	T_{BCS}	T_{LW}	T_{CZZ}	T_{HP}	T_{BCS}
		$p = 38$				$p = 55$				$p = 89$			
	0.000	4.8	6.5	23	5.1	4.7	7.1	39	4.6	6.1	6.4	72	5.8
	0.125	20	23	45	17	21	23	63	18	21	25	89	19
	0.250	30	34	57	26	30	33	75	27	31	34	95	29
$n = 20$		$p = 181$				$p = 331$				$p = 642$			
	0.000	5.2	6.3	100	5.6	4.7	7.0	100	4.6	5.1	6.4	100	5.8
	0.125	21	24	100	21	22	24	100	21	21	24	100	21
	0.250	32	34	100	30	33	34	100	29	31	34	100	33
		$p = 38$				$p = 55$				$p = 89$			
	0.000	5.3	6.2	13	5.3	4.8	5.6	17	4.9	4.5	6.0	29	4.6
	0.125	43	44	54	38	44	44	65	41	50	50	81	46
	0.250	67	68	77	60	68	69	84	64	71	72	94	67
$n = 40$		$p = 181$				$p = 331$				$p = 642$			
	0.000	48	5.8	74	4.9	5.4	5.4	100	5.6	5.8	6.2	100	5.1
	0.125	50	50	98	47	50	51	100	48	50	52	100	49
	0.250	72	72	99	68	74	73	100	71	74	74	100	72
		$p = 38$				$p = 55$				$p = 89$			
	0.000	4.8	5.6	10	5.1	4.9	5.8	11	5.0	5.4	5.8	19	5.6
	0.125	66	67	69	59	70	70	77	65	76	77	90	72
	0.250	91	91	93	88	93	92	96	90	94	94	99	92
$n = 60$		$p = 181$				$p = 331$				$p = 642$			
	0.000	5.1	6.3	46	5.0	4.8	5.4	87	4.8	5.4	5.6	100	5.4
	0.125	76	77	98	73	78	79	100	77	81	79	100	80
	0.250	95	95	100	94	95	95	100	94	96	96	100	95
		$p = 38$				$p = 55$				$p = 89$			
	0.000	5.6	5.7	8.4	5.1	4.6	5.5	9.0	4.7	5.1	5.6	14	5.0
	0.125	83	84	84	79	87	87	90	84	92	92	96	89
	0.250	99	99	98	98	99	99	99	99	100	100	100	99
$n = 80$		$p = 181$				$p = 331$				$p = 642$			
	0.000	4.9	6.0	31	5.2	5.5	5.0	67	5.2	5.2	5.2	100	5.1
	0.125	93	92	99	91	94	94	100	93	95	95	100	94
	0.250	100	100	100	100	100	100	100	100	100	100	100	100

Table A.2. *Empirical size and power (%) comparisons at 5% significance for multivariate t random vectors (scenario II)*

n	v	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}
		$n = 20$		$n = 40$		$n = 60$		$n = 80$	
	0.000	9.9	6.4	12	5.1	12	5.4	13	5.4
$p = 38$	0.125	20	18	32	36	43	57	56	77
	0.250	26	28	45	62	61	86	75	97
	0.000	9.6	5.6	12	4.7	13	5.3	14	5.1
$p = 55$	0.125	22	20	34	39	49	63	61	84
	0.250	26	27	46	65	64	91	76	98
	0.000	10	6.0	11	5.7	13	5.2	13	6.2
$p = 89$	0.125	21	22	35	45	47	74	60	89
	0.250	27	29	46	68	63	93	76	99
	0.000	10	6.7	12	5.9	13	5.8	14	4.4
$p = 181$	0.125	22	22	36	47	50	75	63	92
	0.250	26	32	47	69	65	95	79	99
	0.000	11	6.7	13	6.0	13	5.7	15	5.5
$p = 331$	0.125	22	24	36	49	49	76	61	93
	0.250	27	35	47	70	64	95	78	100
	0.000	11	8.8	12	6.6	12	5.6	13	5.7
$p = 642$	0.125	22	28	35	50	49	77	61	94
	0.250	27	39	46	71	64	96	77	100

Table A.3. *Empirical size and power (%) comparisons at 5% significance for mixtures of multivariate normal random vectors (scenario III)*

n	v	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}
		$n = 20$		$n = 40$		$n = 60$		$n = 80$	
	0.000	12	5.5	15	5.4	15	5.0	17	4.7
$p = 38$	0.125	19	17	33	35	43	60	54	77
	0.250	24	26	42	62	60	88	76	98
	0.000	13	5.0	16	5.4	15	5.1	16	5.4
$p = 55$	0.125	21	16	37	40	46	64	57	82
	0.250	25	26	44	64	61	90	75	98
	0.000	13	4.9	15	4.6	16	5.4	16	5.0
$p = 89$	0.125	21	19	36	44	49	71	61	90
	0.250	25	29	44	67	63	91	79	99
	0.000	13	5.2	15	5.0	16	5.5	16	4.5
$p = 181$	0.125	20	20	38	46	48	73	60	92
	0.250	27	28	44	69	63	93	79	100
	0.000	13	4.6	16	4.7	16	4.7	16	5.0
$p = 331$	0.125	21	20	36	46	50	75	62	94
	0.250	25	26	45	70	64	94	79	100
	0.000	13	8.2	16	6.2	16	6.1	16	5.4
$p = 642$	0.125	21	28	35	50	50	77	62	94
	0.250	26	36	45	72	64	96	79	100

Table A.4. *Empirical size and power (%) comparisons at 5% significance for marginal Gamma random vectors (scenario IV)*

n	v	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}
		$n=20$		$n=40$		$n=60$		$n=80$	
	0.000	7.3	5.7	6.7	6.1	6.4	6.2	6.4	5.7
$p=38$	0.125	23	17	42	36	63	55	78	72
	0.250	34	27	65	57	88	84	97	95
	0.000	7.4	6.1	6.3	5.9	6.0	5.8	5.5	5.5
$p=55$	0.125	24	19	45	36	65	59	82	80
	0.250	32	27	67	62	89	88	98	97
	0.000	8.2	5.8	7.2	5.4	6.5	6.7	6.4	5.6
$p=89$	0.125	25	20	49	45	73	69	90	88
	0.250	35	28	69	66	92	90	99	99
	0.000	7.5	5.5	5.9	4.8	5.9	5.2	5.1	5.3
$p=181$	0.125	25	21	51	45	76	71	92	90
	0.250	35	27	73	68	94	93	100	99
	0.000	7.4	5.5	5.6	4.6	5.8	5.9	5.6	5.6
$p=331$	0.125	25	22	52	47	78	75	93	92
	0.250	34	31	72	70	95	93	100	100
	0.000	7.0	6.7	6.5	4.9	5.9	5.2	4.8	5.1
$p=642$	0.125	24	23	52	50	80	77	95	93
	0.250	34	34	73	71	95	94	100	100

Table A.5. *Empirical size and power (%) comparisons at 5% significance for marginal t_4 random vectors (scenario V)*

n	v	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}	T_{CZZ}	T_{BCS}
		$n=20$		$n=40$		$n=60$		$n=80$	
	0.000	8.8	5.5	9.8	5.8	11	5.7	11	6.2
$p=38$	0.125	23	16	37	29	51	48	65	63
	0.250	30	24	56	52	75	75	88	91
	0.000	9.6	5.4	11	6.1	10	6.3	9.3	5.0
$p=55$	0.125	23	17	38	33	57	54	70	68
	0.250	30	24	58	54	80	80	90	93
	0.000	8.7	5.7	9.8	6.6	10	6.7	9.4	5.5
$p=89$	0.125	26	19	44	39	65	63	80	80
	0.250	31	26	60	59	84	85	93	96
	0.000	8.8	5.7	8.4	5.6	9.0	6.0	8.3	5.8
$p=181$	0.125	25	21	48	42	68	66	84	83
	0.250	33	29	65	63	87	88	96	98
	0.000	8.1	6.4	7.3	5.5	7.4	5.4	7.1	5.4
$p=331$	0.125	25	20	49	46	72	71	87	88
	0.250	34	28	69	66	91	91	98	99
	0.000	7.5	7.1	6.8	5.5	6.2	4.5	6.2	5.6
$p=642$	0.125	24	22	50	48	75	74	90	91
	0.250	33	29	69	67	92	92	99	99