A linear SEIR epidemic model for contact networks

Sen-Zhong Huang

ZhiYing Research Center for Health Data, Nankai University, and Univ. Rostock

Abstract. For a single species, our linear model has the form

\( F'(t) = r(t)(G(t) - R(t)), \)

\( G(t) = \int_0^t F(t - s)W'(s) \, ds, \)

\( R(t) = \int_0^t G'(s)A(s, t - s) \, ds. \)

It is a SEIR model in the sense that

\( E = F - G, \quad I = G - R. \)

We extend the above model prototype to general contact networks (CN), and give some applications including controllability of epidemic spreading on contact networks.

According to our general result, the realization of a safe CN of the scale-free type (e.g., internet) is theoretically very difficult.
1. Linear SEIR model for one single species

\[
\begin{align*}
S & \xrightarrow{r_I} E \xrightarrow{W} I \xrightarrow{A} R \\
S \to E : \quad F'(t) &= \text{RoI}(t) \quad \text{with} \\
\text{RoI}(t) &:= r(t)(G(t) - R(t)), \\
E \xrightarrow{W} I : \quad G(t) &= \int_0^t F(t-s)W'(s) \, ds, \\
I \xrightarrow{A} R : \quad R(t) &= \int_0^t G'(t-s)A(t-s,s) \, ds. \\
\end{align*}
\]

(RoI = Rate of Infection)

In the above, \( F(t)/G(t)/R(t) \) is the cumulative number of exposures/infectives/recovereds that emerged within the time interval \([0, t]\).

The differences

\[
E(t) := F(t) - G(t), \quad I(t) := G(t) - R(t)
\]

give the number of exposures and active infectives at the time point \( t \).

**Explanation:** The value \( 1 - W(t) \) is the probability that an exposed individual will remain in the exposure state after its emergence of \( t \) units of time. Similar meaning for \( A(t, s) \).
Many known models can be derived from the above model under suitable choices of $W$ and $A$.

**Assumptions:**

(i) $r$ is bounded.

(ii) $W$ is a CDF on $\mathbb{R}_+$ with $W(0) = 0$ and

\[
(1.1a) \quad \tau := \int_{0}^{\infty} (1 - W(t)) \, dt < \infty.
\]

(mean incubation period)

(iii) For each $t \geq 0$, $A(t, \cdot)$ is a CDF on $\mathbb{R}_+$ with

\[
(1.1b) \quad \Sigma(t) := \int_{0}^{\infty} (1 - A(t, s)) \, ds < \infty.
\]

(time-dependent mean infectious period)
Let
\begin{equation}
R_{\text{eff}}(t) := \int_0^\infty r(t + s)(1 - A(t, s)) \, ds \quad (t \geq 0)
\end{equation}
be the effective reproductive number.

**Theorem 1.1. (Controllability and Threshold)** Let $(F, G, R)$ be a solution with $F(0) > 0$, and let $F_\infty = \lim_{t \to \infty} F(t)$. We have the following assertions (i)-(ii).

(i) If there exists some $t_0 \geq 0$ such that
\begin{equation}
(1.3a) \quad R_c := \sup_{t \geq t_0} R_{\text{eff}}(t) < 1,
\end{equation}
then there holds
\begin{equation}
(1.3b) \quad F_\infty / F(t_0) \leq (M + 1 - R_c) / (1 - R_c)
\end{equation}
with $M := \sup_{t \geq 0} R_{\text{eff}}(t)$.

(ii) (Epidemic spreading into infinity) If
\begin{equation}
(1.4a) \quad R_{\text{eff}}(t) \geq 1 \quad \forall t \geq 0,
\end{equation}
then
\begin{equation}
(1.4b) \quad F_\infty = \infty.
\end{equation}

**Basic reproductive number $R_0$:** If $r \equiv \beta$, $\Sigma \equiv \sigma$, then
\begin{equation}
(1.5) \quad R_{\text{eff}}(t) \equiv R_0 := \beta \times \sigma.
\end{equation}
### Figure 1. Modeling of SARS in China 2002-2003

<table>
<thead>
<tr>
<th>Region</th>
<th>τ (95%, 99%)</th>
<th>σ (95%, 99%)</th>
<th>β</th>
<th>θ (%)</th>
<th>$R_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HK</td>
<td>6.945 (16.0, 24.8)</td>
<td>12.191 (37.3, 61.5)</td>
<td>0.298</td>
<td>11.0</td>
<td>3.633</td>
</tr>
<tr>
<td>SNG</td>
<td>5.108 (12.6, 19.7)</td>
<td>11.359 (34.7, 56.9)</td>
<td>0.191</td>
<td>10.0</td>
<td>2.170</td>
</tr>
<tr>
<td>TW</td>
<td>7.089 (16.1, 24.9)</td>
<td>11.779 (35.9, 58.8)</td>
<td>0.260</td>
<td>9.5</td>
<td>3.063</td>
</tr>
<tr>
<td>CHN</td>
<td>5.209 (13.4, 20.9)</td>
<td>9.959 (30.3, 49.5)</td>
<td>0.209</td>
<td>9.0</td>
<td>2.081</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Known</th>
<th>4 - 7</th>
<th>7 - 14</th>
<th>0 - 11</th>
<th>2.0 - 4.2</th>
</tr>
</thead>
</table>
**Problem:** What would happen if the effective reproductive numbers
\[ R_{\text{eff}}(t) = \int_0^\infty r(t + s)(1 - A(t, s)) \, ds \quad (t \geq 0) \]
oscillate along the critical value 1? (For example, if \( R_{\text{eff}}(\cdot) \) is periodic and oscillates along the critical value 1.)

**Remark: Classical SEIR model**

Take
\[ W(t) = 1 - e^{-t/\tau}, \quad A(t, s) \equiv 1 - e^{-s/\sigma}. \]

Using \( E = F - G, I = G - R \) and \( S = N - F \), as a special case of (1.1) with (replacing (1.1a) by \( F'(t) = r(t)I(t)S(t)/N \)):

\[
S'(t) = -\frac{r(t)}{N}I(t)S(t), \\
E'(t) = \frac{r(t)}{N}I(t)S(t) - \frac{1}{\tau}E(t), \\
I'(t) = -\frac{1}{\tau}E(t) - \frac{1}{\sigma}I(t), \\
R'(t) = \frac{1}{\sigma}I(t).
\]

Classical SEIR model with a time-dependent mean infection rate \( r(t) \).
2. **Linear SEIR model for structured populations**

The structured population $\mathcal{G}$ is divided into subgroups as

$$(2.1a) \quad \mathcal{G} = \{ \mathcal{G}_k : k \in \Omega \}, \quad \Omega \subset \mathbb{N}.$$

Let $P$ be a **discrete and unit** measure on $\Omega$ such that

$$(2.1b) \quad \int_{\Omega} dP(k) = 1.$$

**Linear SEIR model**

- $(S \rightarrow E)_k : \quad F'(t, k) = \text{RoI}(t, k)$ with
  $$\text{RoI}(t, k) := \int_{\Omega} T(t, k, l)(G(t, l) - R(t, l)) \, dP(l),$$

- $(E \rightarrow I)_k : \quad G(t, k) = \int_0^t F(t - s, k)W'(s, k) \, ds,$

- $(I \rightarrow R)_k : \quad R(t, k) = \int_0^t G'(t - s, k)A(t - s, s, k) \, ds.$
**Assumption A:** Let $0 < \rho \in L^1(\Omega)$ be such that

\[
(2.2a) \quad \rho(l) > 0 \quad (\forall l \in \Omega) \quad \text{and} \quad \int_{\Omega} \rho(l) \, dP(l) < \infty.
\]

Define

\[
(2.2b) \quad L^\infty_\rho(\Omega) := \left\{ f : \|f\| := \sup_{k \in \Omega} \frac{|f(k)|}{\rho(k)} < \infty \right\}
\]

Non-negative and locally bounded solutions in the state space

\[
(2.2c) \quad \mathcal{Z} := L^\infty_{loc}(\mathbb{R}_+, L^\infty_\rho(\Omega)),
\]

**Assumptions B:**

(i) *(Boundedness conditions on $T$)* There holds

\[
(2.2d) \quad \int_{\Omega} T(\cdot, \cdot, l) \rho(l) \, dP(l) \in \mathcal{Z}.
\]

(ii) *(Conditions on the latency and recovery CDFs)* For each $t \geq 0, k \in \Omega$ the functions $W(\cdot, k)$ and $A(t, \cdot, k)$ are CDFs on $\mathbb{R}_+$. Moreover, $W(0, \cdot) \equiv 0$ and

\[
(2.2e) \quad W'(\cdot, \cdot) \in \mathcal{Z} \quad (W'(t, k) = \partial_t W(t, k)).
\]
Denote

\[(2.3) \quad \langle f \rangle := \int_{\Omega} f(k) \, dP(k) \quad (f \in L^1(dP)).\]

**Theorem 2.1. (Controllability and Threshold)** Let \((F, G, R) \in \mathbb{Z}^3\) be a non-negative solution.

Let \(w : \Omega \to (0, \infty)\) be a strictly positive function such that

\[(2.4a) \quad \int_{\Omega} w(l) \rho(l) \, dP(l) < \infty,\]

and

\[(2.4b) \quad M(t, l) := \frac{1}{w(l)} \int_{\Omega} w(k) R_{\text{eff}}(t, k, l) \, dP(k) < \infty,\]

where

\[(2.4c) \quad R_{\text{eff}}(t, k, l) := \int_0^\infty T(t + s, k, l)(1 - A(t, s, l)) \, ds\]

for all \(t \geq 0, k, l \in \Omega\). We have the following assertions (i)-(ii).

(i) **(Controllability)** If there exist some \(t_0 \geq 0\) and two positive constants \(L\) and \(R_c < 1, R_c \leq L\), such that

\[(2.5a) \quad M(t, l) \leq L \quad (\forall t < t_0), \quad M(t, l) \leq R_c \quad (\forall t \geq t_0)\]

for all \(l \in \Omega\), then there holds \(F_k(\infty) < \infty \quad (\forall k \in \Omega)\) and

\[(2.5b) \quad \langle wF(t, \cdot) \rangle \leq \tilde{L} \times \langle wF(t_0, \cdot) \rangle \quad \forall t \geq t_0\]
with \( \tilde{L} := (L + 1 - R_c)/(1 - R_c) \). As consequence,
\[
(2.5c) \quad \lim_{t \to \infty} \langle wF(t, \cdot) \rangle \leq \tilde{L} \times \langle wF(t_0, \cdot) \rangle.
\]
Moreover,
\[
(2.5d) \quad \lim_{t \to \infty} E(t, k) = 0, \quad \lim_{t \to \infty} I(t, k) = 0
\]
for all \( k \in \Omega \) and
\[
(2.5e) \quad \lim_{t \to \infty} \langle wE(t, \cdot) \rangle = 0, \quad \lim_{t \to \infty} \langle wI(t, \cdot) \rangle = 0.
\]

(ii) (Epidemic spreading into infinity) Assume \( F(0, \cdot) \neq 0 \). If
\[
(2.6a) \quad M(t, l) \geq 1 \quad \forall t \geq 0, \forall l \in \Omega,
\]
then
\[
(2.6b) \quad \lim_{t \to \infty} \langle wF(t, \cdot) \rangle = \infty.
\]
Basic reproductive number $R_0$: existence problem

Assume

$$T(t, k, l) = T_0(k, l), \quad A(t, s, k) = A_k(s)$$

for all $t, s \geq 0$, $k, l \in \Omega$. Let

$$\sigma_k := \int_0^\infty (1 - A_k(s)) \, ds < \infty.$$

Then $M(t, l) = M(l)$ and

$$w(l) M(l) = \sigma_l \times \int_\Omega w(k) T_0(k, l) \, dP(k) \quad (l \in \Omega).$$

Looking for existence of $w > 0$ such that $(\cdot) \equiv \text{const.}$

Let

$$X := L^1(\rho \, dP).$$

Define a bounded kernel operator $K_0 : X \to X$ by

$$(K_0 u)(l) := \sigma_l \times \int_\Omega u(k) T_0(k, l) \, dP(k) \quad (l \in \Omega)$$

for all $u \in X$. Define

$$R_0 := r(K_0) \quad (= \text{spectral radius of } K_0).$$

Needed: Compactness + Irreducibility (Intuitively, irreducibility of $T_0 = \text{connectivity of all subgroups}$).

Application of Jentzsch’s Theorem.
Special case: Bipartite Populations

Assume

\[ T_0(1, 1) = 0, \quad T_0(1, 2) = \beta_1 > 0 \]
\[ T_0(2, 1) = \beta_2 > 0, \quad T_0(2, 2) = 0. \]

Then

\[ R_0 = \sqrt{(\beta_1 \sigma_1) \times (\beta_2 \sigma_2)} \]

(Malaria Model: Ross (1911) and MacDonald (1957))
3. **Linear SEIR model for Contact Networks**

Let $\mathcal{G}$ be a **dynamic contact network** which as a graph is undirected. Define

\[(3.1a) \quad \mathcal{G}_k := \{\text{all nodes from } \mathcal{G} \text{ of degree } k\}.\]

Choose

\[(3.1b) \quad \Omega \subset \{n \in \mathbb{N} : n \geq 1\}\]

as the parametrization space of $\mathcal{G}$ and define a discrete and unit measure $P$ on $\Omega$ as follows. **For each $k \in \Omega$, we define $P(\{k\}) \ (= P(k))$ to be the probability that a randomly chosen node has degree $k$.**

Let $p(l|k)$ be the conditional probability that a node of degree $k$ is connected to a node of degree $l$. We assume that the network $\mathcal{G}$ is totally connected in the sense that

\[(3.1c) \quad \sum_l p(l|k) = 1 \quad \forall k \in \Omega.\]
Our SEIR model governing the dynamics of the disease spreading over the network $\mathcal{G}$ has the form:

(3.2a) \[ F'(t, k) = r(t)\rho(k) \cdot \sum_l p(l|k)(G(t, l) - R(t, l)), \]

(3.2b) \[ G(t, k) = \int_0^t F(t - s, k)W'(s) \, ds, \]

(3.2c) \[ R(t, k) = \int_0^t G'(t - s, k)A(t - s, s) \, ds \]

for all $k \in \Omega$ and $t \geq 0$.

**Note.** $W, A$ are independent of node degrees.

**Conditions:** $r \geq 0$, $\rho$ is strictly positive such that

(3.2d) \[ r \in L^\infty(\mathbb{R}_+), \quad \langle \rho \rangle = \int_\Omega \rho(k) \, dP(k) < \infty, \]

and

(3.2e) \[ \sup_{k \in \Omega} \sum_l \rho(l)p(l|k) < \infty. \]

Model (3.2) is a special case of the general SEIR model in §2 with $T$ given by

(3.3) \[ T(t, k, l) := r(t)T_0(k, l), \quad T_0(k, l) := \rho(k)p(l|k)/P(l), \]

for all $t \geq 0, k, l \in \Omega$. 


Define

\[(3.4) \quad (Hu)(l) = \sum_k u(k)\rho(k)p(k|l) \quad (u \in L^\infty(\Omega), \ l \in \Omega).\]

**Facts:** $H^2$ is compact. Jentzsch’s Theorem implies $r(H) > 0$ with strictly positive eigenvalue $w_H \in X = L^1(\rho dP)$.

Choose $w = w_H$. The function $M(t,l)$ defined by (2.4b) with $w = w_K$ and $T(t,k,l) = r(t)T_0(k,l)$ is independent of $l$:

\[(3.5a) \quad M(t,l) = R_{\text{eff}}(t) \quad \forall t \geq 0, l \in \Omega,\]

where

\[(3.5b) \quad R_{\text{eff}}(t) := r(H) \times \int_0^\infty r(t + s)(1 - A(t,s)) \, ds \quad \forall t \geq 0.\]

We call $R_{\text{eff}}(t)$ the *effective reproductive number* at time $t$.

As consequences of Theorem 2.1:

(a) If there exists some $t_0 \geq 0$ and two positive numbers $L$ and $R_c < 1$ such that $R_{\text{eff}}(t) \leq L$ for all $t \leq t_0$ and $R_{\text{eff}}(t) \leq R_c$ for all $t \geq t_0$, then any epidemic course given by a non-negative solution of (3.2) will be stopped with a finite epidemic size.

(b) If $R_{\text{eff}}(t) \geq 1$ for all $t \geq 0$, then any epidemic course given by a non-trivial and non-negative solution of (3.2) will spread with an infinite epidemic size.
Basic reproductive number $R_0$: Assume \[ r \equiv \beta, A(t, \cdot) = A_0(\cdot). \]

Then

\[(3.5c) \quad R_0 := r(H) \times (\beta \sigma). \]
Application to scale-free (SF) contact networks

Assume
\[ P(k) \propto k^{-\gamma} \quad (2 \leq \gamma \leq 3). \]
Assume that \((\mathcal{G}, p(\cdot|\cdot))\) is pseudo-uncorrelated, i.e.,
\[ c_1 \tilde{p}(l|k) \leq p(l|k) \leq c_2 \tilde{p}(l|k), \quad \tilde{p}(l|k) = \sum_{i=1}^{n} \eta_i(k) \frac{\phi_i(l) P(l)}{\langle \phi_i \rangle} \]
for all \(k, l \in \Omega\), where \(c_1 \leq 1 \leq c_2\) are constants, and \(\{(\eta_i, \phi_i) : i = 1, 2, ..., n\}\) are positive functions such that
\[ \sum_{i=1}^{n} \langle \rho \eta_i \phi_i \rangle < \infty, \quad \sum_{i=1}^{n} \eta_i(k) = 1 \quad \forall k \in \Omega. \]

Assume that the maximum degree \(k_{\text{max}}\) of \(\mathcal{G}\) is finite. Let \(R_0(k_{\text{max}})\) be the resulted basic reproductive number. We have
\[ R_0(k_{\text{max}}) \propto \langle k^2 \rangle / \langle k \rangle \]
and thus
\[ R_0(k_{\text{max}}) \rightarrow \infty \quad \text{as} \quad k_{\text{max}} \rightarrow \infty. \]

Consequence: Any SF network with exponent \(2 \leq \gamma \leq 3\) is not controllable under any imperfect interventions whenever the conditional probability \(p\) is pseudo-uncorrelated.
One of such SF networks is the so-called Barabási-Albert model.
Figure 2. Realization of a Barabási-Albert model

**Problem**: Realization of a safe CN. “Vaccinating” the nodes, how?
Reference:


